Off the Energy Shell $T$ Matrix. II

---Triplet State---

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The coupled equations of the off-shell Lippman-Schwinger equation for triplet states are solved. From the outgoing wave parts of the solutions, the expressions of the off-shell $T$ matrices for triplet-state interactions are derived. It is shown that the present $T$ matrices on the energy shell agree with the usual ones and the off-shell $T$ matrices have the symmetry property,

$$t_{ij}^{(1)}(p, k; z) = t_{ji}^{(1)}(k, p; z).$$

§ 1. Introduction

The importance of the investigation of the off-shell $T$ matrix consists in the fact that the elements of the $T$ matrix are measurable quantities and if all of them will be known in the future from the experimental data, it will be useful to construct the interaction potential. It is only the on-shell elements that can be obtained from the analysis of the free two-body scattering data, but the off-shell elements must be obtained from the analysis of the $p$-$p$ bremsstrahlung, nuclear matter, three-body bound and scattering problems, etc.

Unfortunately, the knowledge of the off-shell $T$ matrix has been scarecely obtained from the experimental data up to the present day. Therefore it seems to be the second best that the off-shell $T$ matrix can be obtained from the realistic potential which reproduces the free two-body scattering data.

Once a realistic potential is assumed, the $T$ matrix should be uniquely determined, because it is defined through the integral equation

$$T = V + V(E - H_0 + iz)^{-1}T.$$  

Wong and Zambotti solved the above integral equation for the Yukawa potential by use of the matrix inversion method. However this method is of no use for the potential with hard cores.

We have to look for the method to obtain the off-shell $T$ matrix, which is valid for the potential with or without hard cores.

Previously, we presented a simple method to obtain the off-shell $T$ matrix for the singlet interaction potentials with hard cores. Although the method was applied to the singlet-state interaction of the Hamada-Johnston potential, it is considerably general and is applicable to any finite-range local potential with or without hard cores even if the potential may have energy dependences.
For the applications of off-shell $T$ matrices to three-body scattering problems, we need to know the off-shell $T$ matrices of the spin triplet state interactions as well as the spin singlet ones. For example, in the nucleon-deuteron scattering, the $L$-$S$ and tensor potentials play an important role for the polarization of scattered nucleon. In this paper, the previous method is extended to the case of the spin triplet state interaction potentials. The substantial parts of the off-shell $T$ matrix have already been shown in Ref. 2), i.e., the reduced off-shell $T$ matrix (hereafter called off-shell $T$ matrix) for the total angular momentum state $J$,

$$t_{l'l''}^{(s)}(p, k; z) = \frac{1}{pk} \int_{-\infty}^{\infty} \mathcal{J}_0(pr) \left[ \frac{d^2}{dr^2} - \frac{l'(l' + 1)}{r^2} + z \right] \chi_{l'l''}^{(s)}(z, k, r) dr ,$$

(1)

where we follow Ref. 2) in the meaning of every notation.

In the case of triplet states ($s=1$), the quantum numbers $l$ and $l'$ of the orbital angular momentum states are permitted to take the values $J+1$, $J$ and $J-1$. The problem is to obtain the function $\chi_{l'l''}^{(s)}(z, k, r)$ for all $r$. For this aim, it is convenient for us to classify the functions $\chi_{l'l''}^{(s)}(z, k, r)$ ($l, l' = J+1, J, J-1$) with respect to the states $l$ and $l'$ as follows:

(i) $l = l' = J$,

(ii) $l = J+1$, $l' = J+1$ or $J-1$,

(iii) $l = J-1$, $l' = J+1$ or $J-1$.

In the case of class (i), the function $\chi_{l'l''}^{(s)}(z, k, r)$ satisfies the uncoupled equation, so that the off-shell $T$ matrix is calculated by use of the same method as Ref. 2). In order to avoid same discussions, we do not treat this case here. We only consider the cases of classes (ii) and (iii) throughout this paper. In these cases, the outgoing radial wave functions $\chi_{l'l''}^{(s)}(z, k, r)$ must be obtained as the solutions of coupled equations because of the tensor force potential in the triplet state interactions. Then the functions $\chi_{l'l''}^{(s)}(z, k, r)$ will be represented in somewhat complicated forms. However the off-shell $T$ matrix is not so complicated.

In the next section, the outgoing radial wave functions $\chi_{l'l''}^{(s)}(z, k, r)$ are discussed and the expression of the off-shell $T$ matrix element is derived. In § 3 it is shown that the on-shell elements of the present $T$ matrix agree with usual ones which are related to the phase shifts and coupling parameter. In connection with the off-shell $T$ matrix, the expression of the off-shell $K$ matrix is given. In § 4 numerical examples of the off-shell $T$ matrix for the Hamada-Johnston potential are shown for the total angular momentum state $J=1$ at the positive energy parameter 75 MeV. In the last section, the main points of the present method and some remarks are mentioned. The Coulomb potential is not taken into account again as well as Ref. 2).
§ 2. Function $\chi^{(j)}_{J_l}(z, k, r)$ and off-shell $T$ matrix

There are four outgoing radial wave functions $\chi^{(j)}_{J_{l_1}}(z, k, r)$, in classes (ii) and (iii), corresponding to $l = J \pm 1$ and $l' = J \pm 1$,

$$
\chi^{(j)}_{J_{l_1} J_{l_1} + 1}(z, k, r), \quad \chi^{(j)}_{J_{l_1} J_{l_1} - 1}(z, k, r), \quad \chi^{(j)}_{J_{l_1} J_{l_1} + 1}(z, k, r) \quad \text{and} \quad \chi^{(j)}_{J_{l_1} J_{l_1} - 1}(z, k, r).
$$

(2)

According to Appendix A in Ref. 2), the first two functions which belong to class (ii) satisfy the following type of coupled equations:

$$
\begin{align*}
\left[ \frac{d^2}{dr^2} \right] - \frac{(J+1)(J+2)}{r^2} - U_1(r) - U_1'(r) &= \chi^{(j)}_{J_{l_1} J_{l_1} + 1}(z, k, r), \\
- U_1(r) \chi^{(j)}_{J_{l_1} J_{l_1} - 1}(z, k, r) &= U_1(r) \mathcal{J}_{J_{l_1} + 1}(kr), \\
\left[ \frac{d^2}{dr^2} \right] - \frac{(J-1)J}{r^2} - U_1(r) - U_1'(r) &= \chi^{(j)}_{J_{l_1} J_{l_1} - 1}(z, k, r), \\
- U_1(r) \chi^{(j)}_{J_{l_1} J_{l_1} + 1}(z, k, r) &= U_1(r) \mathcal{J}_{J_{l_1} - 1}(kr).
\end{align*}
$$

(3)

If we assume the potential

$$
V = V_0 + V_T S_{12} + V_{LS} (LS) + V_{LL} L_{12},
$$

the potentials $U_1(r)$, $U_2(r)$ and $U_{12}(r)$ are given as follows:

$$
\begin{align*}
U_1(r) &= \frac{M}{\hbar^2} \left[ V_0 - (J+2) \left( V_{LS} + \frac{2}{2J+1} V_T + V_{LL} \right) \right], \\
U_2(r) &= \frac{M}{\hbar^2} \left[ V_0 + (J-1) \left( V_{LS} - \frac{2}{2J+1} V_T + V_{LL} \right) \right], \\
U_{12}(r) &= \frac{M}{\hbar^2} \frac{\sqrt{J(J+1)}}{2J+1} V_T, \quad r > r_c,
\end{align*}
$$

(4)

where $r_c$ means the hard core radius. The last two functions $\chi^{(j)}_{J_{l_1} J_{l_1} \pm 1}(z, k, r)$ which belong to class (iii) satisfy the following type of coupled equations:

$$
\begin{align*}
\left[ \frac{d^2}{dr^2} \right] - \frac{(J+1)(J+2)}{r^2} - U_1(r) - U_1'(r) &= \chi^{(j)}_{J_{l_1} J_{l_1} \pm 1}(z, k, r), \\
- U_1(r) \chi^{(j)}_{J_{l_1} J_{l_1} \mp 1}(z, k, r) &= U_1(r) \mathcal{J}_{J_{l_1} \mp 1}(kr), \\
\left[ \frac{d^2}{dr^2} \right] - \frac{(J-1)J}{r^2} - U_1(r) - U_1'(r) &= \chi^{(j)}_{J_{l_1} J_{l_1} \mp 1}(z, k, r), \\
- U_1(r) \chi^{(j)}_{J_{l_1} J_{l_1} \pm 1}(z, k, r) &= U_1(r) \mathcal{J}_{J_{l_1} \pm 1}(kr),
\end{align*}
$$

(5)

where the potentials $U_1(r)$, $U_2(r)$ and $U_{12}(r)$ are still given by Eq. (4).

Equation (5) can be solved by use of the method similar to Eq. (3). Therefore, for a time, we consider only the solutions of Eq. (3). If the potentials $U_1(r)$, $U_2(r)$ and $U_{12}(r)$ have hard cores the radii of which are same, the solu-
tions of Eq. (3) should be obtained separately for the regions \( r<r_c \) and \( r>r_c \). In order to obtain the solutions for the region \( r<r_c \), as was done in Ref. 2), the hard cores are temporarily replaced by finite repulsive square barriers,

\[
U_1 \rightarrow \bar{U}_1 > z, \quad U_2 \rightarrow \bar{U}_2 > z \quad \text{and} \quad U_{13} \rightarrow \bar{U}_{13}. \quad \text{(finite)}
\]

(6)

The constants \( \bar{U}_1, \bar{U}_2 \) and \( \bar{U}_{13} \) should be obtained from Eq. (4) replacing \( V_\sigma, V_{LS}, \)

\( V_T \) and \( V_{LL} \) by appropriate constants \( \bar{V}_\sigma, \bar{V}_{LS}, \bar{V}_T \) and \( \bar{V}_{LL} \) which satisfy the above conditions. The above conditions must be kept whenever passing the limits \( \bar{U}_1, \bar{U}_2 \) and \( \bar{U}_{13} \rightarrow \infty \). This means that the order of singularity of constants \( \bar{V}_\sigma \) should be higher than those of the other constants \( \bar{V}_{LS}, \bar{V}_T \) and \( \bar{V}_{LL} \).

Each solution of Eq. (3) which is zero at \( r=0 \) is then given by the sum of solutions of the homogeneous equations and a special solution. The solutions of the homogeneous equations, however, diverge at the limits \( \bar{U}_1, \bar{U}_2 \) and \( \bar{U}_{13} \rightarrow \infty \), so that the multiplication constants on the homogeneous solutions should initially be zero. Only the special solutions \( \phi_{J+1,J+1}(z, k, r) \) are retained. These special solutions can be obtained by setting

\[
\phi_{J+1,J+1}(z, k, r) = C \bar{J}_{J+1}(kr),
\]

\[
\phi_{J+1,J-1}(z, k, r) = D \bar{J}_{J-1}(kr).
\]

(7)

Substitution of Eq. (7) into Eq. (3) yields

\[
C (z - \bar{U}_1 - k^2) \bar{J}_{J+1}(kr) - D \bar{U}_{13} \bar{J}_{J-1}(kr) = \bar{U}_1 \bar{J}_{J+1}(kr),
\]

\[
D (z - \bar{U}_1 - k^2) \bar{J}_{J-1}(kr) - C \bar{U}_{13} \bar{J}_{J+1}(kr) = \bar{U}_{13} \bar{J}_{J+1}(kr).
\]

(8)

(9)

The left-hand side of Eq. (8) should be equal to the right-hand side at all \( r \) in the region \( 0<r<r_c \), so that the constants \( C \) and \( D \) are given by

\[
C = \frac{1}{(z - \bar{U}_1 - k^2)},
\]

\[
D = 0.
\]

(10)

(11)

Letting the potential \( \bar{U}_1 \rightarrow \infty \) in Eq. (10), we have

\[
C = -1.
\]

(12)

Then Eqs. (11) and (12) simultaneously satisfy Eq. (9). Consequently, the solutions of Eq. (3) inside the hard core are given by

\[
\chi_{J+1,J+1}^{(1)}(z, k, r) = - \bar{J}_{J+1}(kr),
\]

\[
\chi_{J+1,J-1}^{(1)}(z, k, r) = 0, \quad r<r_c.
\]

(13)

(14)

Next, let us consider the solutions for the region outside the hard core. In this region, the general solutions can be described as follows:

\[
\chi_{J+1,J+1}^{(2)}(z, k, r) = A^{(1)} \xi_{J+1}^{(0)}(qr) + A^{(2)} \xi_{J+1}^{(1)}(qr)
\]

\[
+ B^{(1)} \eta_{J+1}^{(0)}(qr) + B^{(2)} \eta_{J+1}^{(1)}(qr) + \phi_{J+1,J+1}(z, k, r),
\]

(15)

* The results of Eqs. (11) and (12) are easily derived if \( V_T \) vanishes inside the hard core.
\[ \chi^J_{J_1,J-1}(z, k, r) = A^{(1)} \xi^{(1)}_{J_1}(qr) + A^{(2)} \xi^{(2)}_{J_1}(qr) \]
\[ + B^{(1)} \eta^{(1)}_{J_1}(qr) + B^{(2)} \eta^{(2)}_{J_1}(qr) + \varphi^{J_1}_{J_1,J-1}(z, k, r), \]  
(16)

where \( q \) is \( \sqrt{x} \). The functions \( \xi^{(1)}_{J_1}(qr) \), \( \xi^{(2)}_{J_1}(qr) \), \( \eta^{(1)}_{J_1}(qr) \) and \( \eta^{(2)}_{J_1}(qr) \) are the linear independent solutions of homogeneous equations of Eq. (3) and the functions \( \varphi^{J_1}_{J_1,J_1}(z, k, r) \) are the special solutions of Eq. (3).

It is convenient to choose the functions \( \xi^{(1)}_{J_1}(qr) \) which satisfy the following boundary conditions,
\[ \xi^{(1)}_{J_1}(qr) = \xi^{(1)}_{J_1}(qr) = 0, \]
(17)
\[ \xi^{(1)}_{J_1}(qr) \rightarrow \cos \xi_J \cos(qr - \frac{1}{2}(J + 1 + 1) \pi + \delta_{J_1}), \]
(18)
\[ \xi^{(1)}_{J_1}(qr) \rightarrow - \sin \xi_J \sin(qr - \frac{1}{2}(J - 1 + 1) \pi + \delta_{J_1}), \]
(19)

where \( \delta_{J_1} \) and \( \xi_J \) are respectively the bar phase shifts and the coupling parameter introduced by Stapp et al. 6. In order to treat the angular momentum states \( J+1 \) and \( J-1 \) in a symmetric way, functions \( \xi^{(2)}_{J_1}(qr) \) are chosen to satisfy the following conditions:
\[ \xi^{(2)}_{J_1}(qr) = \xi^{(1)}_{J_1}(qr) \]
(20)
\[ \xi^{(2)}_{J_1}(qr) \rightarrow - \sin \xi_J \sin(qr - \frac{1}{2}(J + 1 + 1) \pi + \delta_{J_1}), \]
(21)
\[ \xi^{(2)}_{J_1}(qr) \rightarrow \cos \xi_J \cos(qr - \frac{1}{2}(J - 1 + 1) \pi + \delta_{J_1}). \]
(22)

On the other hand, the functions \( \eta^{(1)}_{J_1}(qr) \) and \( \eta^{(2)}_{J_1}(qr) \) can be taken to satisfy the Wronskians
\[ \eta^{(1)}_{J_1}(qr) \frac{d}{dr} \xi^{(1)}_{J_1}(qr) - \xi^{(1)}_{J_1}(qr) \frac{d}{dr} \eta^{(1)}_{J_1}(qr) \]
\[ + \eta^{(2)}_{J_1}(qr) \frac{d}{dr} \xi^{(2)}_{J_1}(qr) - \xi^{(2)}_{J_1}(qr) \frac{d}{dr} \eta^{(2)}_{J_1}(qr) = q, \]
(23)
\[ \eta^{(1)}_{J_1}(qr) \frac{d}{dr} \xi^{(1)}_{J_1}(qr) - \xi^{(1)}_{J_1}(qr) \frac{d}{dr} \eta^{(1)}_{J_1}(qr) \]
\[ + \eta^{(2)}_{J_1}(qr) \frac{d}{dr} \xi^{(2)}_{J_1}(qr) - \xi^{(2)}_{J_1}(qr) \frac{d}{dr} \eta^{(2)}_{J_1}(qr) = q. \]
(24)

Such functions as \( \eta^{(1)}_{J_1}(qr) \) and \( \eta^{(2)}_{J_1}(qr) \) behave at large \( r \) as follows:
\[ \eta^{(1)}_{J_1}(qr) \rightarrow - \cos \xi_J \sin(qr - \frac{1}{2}(J + 1 + 1) \pi + \delta_{J_1}), \]
(25)
\[ \eta^{(2)}_{J_1}(qr) \rightarrow - \sin \xi_J \cos(qr - \frac{1}{2}(J - 1 + 1) \pi + \delta_{J_1}), \]
(26)
\[ \eta^{(1)}_{J_1}(qr) \rightarrow - \sin \xi_J \cos(qr - \frac{1}{2}(J + 1 + 1) \pi + \delta_{J_1}), \]
(27)
\[ \eta^{(2)}_{J_1}(qr) \rightarrow - \cos \xi_J \sin(qr - \frac{1}{2}(J - 1 + 1) \pi + \delta_{J_1}). \]
(28)

The special solutions \( \varphi^{J_1}_{J_1,J_1}(z, k, r) \) can be determined under the conditions
that the special solutions must vanish outside the potential range $r_\delta$ defined in Ref. 2).

The arbitrary constants $A^{(1)}$, $A^{(2)}$, $B^{(1)}$ and $B^{(2)}$ in Eqs. (15) and (16) are determined under the conditions that the functions $\chi_{+1,1+1}^{(+)}(z, k, r)$ for the region $r > r_\delta$ should be continuous respectively to the solutions (13) and (14) for the region $r < r_\delta$, and should represent the outgoing waves at large $r$. These conditions are described by

$$B^{(1)}\eta_{+1}^{(1)}(qr) + B^{(2)}\eta_{+1}^{(2)}(qr) + \varphi_{+1,1+1}^{(1)}(z, k, r) = -J_{+1}(kr), \quad r \to \infty$$

$$A^{(1)}\xi_{+1}^{(1)}(qr) + A^{(2)}\xi_{+1}^{(2)}(qr) = 0, \quad r \to \infty$$

$$A^{(1)}\xi_{+1}^{(1)}(qr) + A^{(2)}\xi_{+1}^{(2)}(qr) + B^{(1)}\eta_{+1}^{(1)}(qr) + B^{(2)}\eta_{+1}^{(2)}(qr)$$

$$\to f_{+1,1+1}(q, k) \exp(iqr), \quad r \to \infty$$

Equations (29) and (30) give us the values of constants $B^{(1)}$ and $B^{(2)}$ as a function of $q$ and $k$,

$$B^{(1)}_{+1}(q, k) = -\frac{1}{A} \begin{bmatrix} J_{+1}(kr) + \varphi_{+1,1+1}^{(1)}(z, k, r), \eta_{+1}^{(1)}(qr) \\ \varphi_{+1,1+1}^{(1)}(z, k, r), \eta_{+1}^{(2)}(qr) \end{bmatrix}, \quad r \to \infty$$

$$B^{(2)}_{+1}(q, k) = \frac{1}{A} \begin{bmatrix} J_{+1}(kr) + \varphi_{+1,1+1}^{(1)}(z, k, r), \eta_{+1}^{(1)}(qr) \\ \varphi_{+1,1+1}^{(1)}(z, k, r), \eta_{+1}^{(2)}(qr) \end{bmatrix}, \quad r \to \infty$$

where

$$A = \begin{bmatrix} \eta_{+1}^{(1)}(qr), \eta_{+1}^{(2)}(qr) \\ \eta_{+1}^{(2)}(qr), \eta_{+1}^{(1)}(qr) \end{bmatrix}$$

The constants $A^{(1)}$ and $A^{(2)}$ are then given, in terms of Eqs. (31) and (32), by

$$A^{(1)}_{+1}(q, k) = iB^{(1)}_{+1}(q, k),$$

$$A^{(2)}_{+1}(q, k) = iB^{(2)}_{+1}(q, k)$$

respectively. The functions $\chi_{+1,1+1}^{(+)}(z, k, r)$ are completely determined if Eqs. (33), (34), (36) and (37) are substituted into Eqs. (15) and (16).

Now, let us consider the solutions $\chi_{-1,1+1}^{(+)}(z, k, r)$ of Eq. (5). After the same discussions as the above, we find that in the region inside the hard core

$$\chi_{-1,1+1}^{(+)}(z, k, r) = 0,$$

$$\chi_{-1,1+1}^{(+)}(z, k, r) = -J_{-1}(kr), \quad r < r_\delta.$$

The general solutions $\chi_{+1,1+1}^{(+)}(z, k, r)$ of Eq. (5) for $r > r_\delta$ can be described by...
where the constants $B_{0}^{(1)}(q, k)$, $B_{1}^{(1)}(q, k)$, $A_{0}^{(1)}(q, k)$ and $A_{1}^{(1)}(q, k)$ are given by

\begin{align}
B_{0}^{(1)}(q, k) &= -\frac{1}{4} \left| \begin{array}{cc}
\varphi_{1,1}^{(1)}(z, k, r) + \varphi_{1,2}^{(1)}(z, k, r),
\psi_{2,1}^{(1)}(qr) + \psi_{2,2}^{(1)}(qr)
\end{array} \right. \\
B_{1}^{(1)}(q, k) &= \frac{1}{4} \left| \begin{array}{cc}
\varphi_{1,1}^{(1)}(z, k, r),
\psi_{2,1}^{(1)}(qr)
\end{array} \right. \\
A_{0}^{(1)}(q, k) &= iB_{0}^{(1)}(q, k), \\
A_{1}^{(1)}(q, k) &= iB_{1}^{(1)}(q, k).
\end{align}

In either case of Eqs. (3) or (5), the functions $\chi_{l, l'}^{(1)}(z, k, r)$ can be described in the following forms:

\begin{align}
\chi_{l, l'}^{(1)}(z, k, r) &= A_{l}^{(1)} \varphi_{l, l'}^{(1)}(qr) + A_{l}^{(2)} \varphi_{l, l'}^{(2)}(qr) + B_{l}^{(1)} \psi_{l, l'}^{(1)}(qr) \\
&\quad + B_{l}^{(2)} \psi_{l, l'}^{(2)}(qr) + \varphi_{l, l'}^{(3)}(z, k, r), \quad r > r_{e},
\end{align}

\begin{align}
\chi_{l, l'}^{(1)}(z, k, r) &= -\delta_{l, l'} G_{l}(kr), \quad r < r_{e}.
\end{align}

The above equation (47) leads to the result that by means of Eq. (13) in Ref. 2), the off-shell wave function $u_{l, l'}^{(1, 1)}(z, k, r)$ of the two-body scattering state vanishes in the region $0 < r < r_{e}$.

Although the function $\chi_{l, l'}^{(1)}(z, k, r)$ for each $l$ and $l'$ is continuous at all $r$, its derivative is not so at the hard-core edge $r_{e}$. Therefore the second derivative of $\chi_{l, l'}^{(1)}(z, k, r)$ in the neighbourhood of the hard-core edge should be defined by

\begin{align}
\frac{d^{2}}{dr^{2}} \chi_{l, l'}^{(1)}(z, k, r) &= \delta (r - r_{e}) \left[ \frac{d}{dr} \chi_{l, l'}^{(1)}(z, k, r) \right]_{r_{e} + \epsilon} - \frac{d}{dr} \chi_{l, l'}^{(1)}(z, k, r) \bigg|_{r_{e} - \epsilon} \\
&\quad \text{for } r_{e} - \epsilon < r < r_{e} + \epsilon,
\end{align}

where $r_{e} - \epsilon$ and $r_{e} + \epsilon$ mean the left- and right-hand limits to the hard-core edge, respectively.

Finally we are now in the position to calculate the off-shell $T$ matrix element $t_{l, l'}^{(1, 1)}(p, k; z)$ from Eq. (1). In order to carry out the integration of Eq. (1), as was done in Ref. 2), we divide the range of the integration into three parts:

\begin{align}
0 < r < r_{e} - \epsilon, \quad r_{e} - \epsilon < r < r_{e} + \epsilon \quad \text{and} \quad r_{e} + \epsilon < r < R,
\end{align}

and take a limit $\epsilon \rightarrow 0$ after the integrations of each part. The result is then
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Given by

\[ t^{j_1, j_2}_{i_1, i_2}(p, k; z) = \frac{1}{\rho k} \left[ \mathcal{J}_\nu(pr) \frac{d}{dr} \chi^{j_1, j_2}_{i_1, i_2}(z, k, r) - \chi^{j_1, j_2}_{i_1, i_2}(z, k, r) \frac{d}{dr} \mathcal{J}_\nu(pr) \right]_{r=R} \]

This result, Eq. (50), is independent of distance \( R \), if \( R \) is taken larger than the potential range \( r_a \). The factor \( \delta_{i_1, \nu} \) of the second term in Eq. (50) comes from Eq. (47). In the case of \( l = l' \), the off-shell \( T \) matrix has the same form as one of the uncoupled state. The terms similar to the last two integrations in Eq. (50) have been used by Akaishi et al. and Bethe et al. when the reaction matrix of negative energy parameters was obtained.

§ 3. On-shell elements

Let us consider here the on-shell elements of the \( T \) matrix. Letting the momenta \( p \) and \( k \) in Eq. (50) equal to \( q \), we have

\[ t^{j_1, j_2}_{i_1, i_2}(q, q; z) = \frac{1}{q^2} \left[ \mathcal{J}_\nu(qr) \frac{d}{dr} \chi^{j_1, j_2}_{i_1, i_2}(z, q, r) - \chi^{j_1, j_2}_{i_1, i_2}(z, q, r) \frac{d}{dr} \mathcal{J}_\nu(qr) \right]_{r=R} \]  

(51)

The functions \( \chi^{j_1, j_2}_{i_1, i_2}(z, q, r) \) behave outside the potential range \( r_a \) as follows:

\[ \chi^{j_1, j_2}_{i_1, i_2}(z, q, r) = (iB^{(1)}(q, q) \cos \xi_J - B^{(0)}(q, q) \sin \xi_J) \exp(i\delta_{J+1}) \mathcal{J}_{J+1}(qr) \]

(52)

\[ \chi^{j_1, j_2}_{i_1, i_2}(z, q, r) = (iB^{(1)}(q, q) \cos \xi_J + iB^{(0)}(q, q) \sin \xi_J) \exp(i\delta_{J-1}) \mathcal{J}_{J-1}(qr) \]

(53)

Substituting Eqs. (52) and (53) respectively into Eq. (51), we have

\[ t^{j_1, j_2}_{i_1, i_2}(q, q; z) = -\frac{1}{q} (B^{(1)}(q, q) \cos \xi_J + iB^{(0)}(q, q) \sin \xi_J) \exp(i\delta_{J+1}) \]  

(54)

and

\[ t^{j_1, j_2}_{i_1, i_2}(q, q; z) = -\frac{1}{q} (B^{(1)}(q, q) \cos \xi_J + iB^{(0)}(q, q) \sin \xi_J) \exp(i\delta_{J-1}) \]  

(55)

where the Wronskian as to \( \mathcal{J}_i(qr) \) and \( \mathcal{N}_i(qr) \) has been used.

On the other hand, the present on-shell elements of the \( T \) matrix should be related to the elements of the usual \( S \) matrix as follows:

\[ t^{j_1, j_2}_{i_1, i_2}(q, q; z) = \frac{1}{2\rho q} (S_{i_1, \nu} - \delta_{i_1, \nu}) \]  

(56)
In order to confirm Eq. (56), it is sufficient to check the following equations:

\begin{align}
B_{y+1}^{(1)}(q, q) &= -\cos \varepsilon \sin \delta_{j+1}, \\
B_{y+1}^{(2)}(q, q) &= -\sin \varepsilon \cos \delta_{j+1}, \\
B_{y-1}^{(1)}(q, q) &= -\sin \varepsilon \cos \delta_{j-1}, \\
B_{y-1}^{(2)}(q, q) &= -\cos \varepsilon \sin \delta_{j-1}.
\end{align}

Substituting Eqs. (58) ~ (61) into Eq. (55), we have the elements of the second column of Eq. (57). The constants \( B_{y+1}^{(1)}(q, q) \) and \( B_{y+1}^{(2)}(q, q) \) have been calculated from Eqs. (33), (34), (42) and (43), and Eqs. (58) ~ (61) have been confirmed by numerical calculations. By using Eqs. (58) ~ (61) into Eqs. (46) and (47), the functions \( u_{i,\nu}^{(1)}(z, q, r) = \varphi_i^{(1)}(z, q, r) + \delta_{i,\nu} J_1(qr) \) agree with the usual radial wave functions of the coupled equations.

In connection with the off-shell \( T \) matrix, let us derive the elements of the off-shell \( K \) matrix. If we choose \( A_{i,\nu}^{(1)}(q, k) \) and \( A_{i,\nu}^{(2)}(q, k) \), instead of \( A_{i}^{(1)} = iB_{i}^{(1)} \) and \( A_{i}^{(2)} = iB_{i}^{(2)} \),

\begin{align}
A_{i,\nu}^{(1)}(q, k) &= \frac{1}{f} [B_{i}^{(1)}(q, k) \sin \delta_{j+1} \cos \delta_{j-1} \\
&\quad + B_{i}^{(2)}(q, k) \sin \varepsilon \cos \delta_{j+1}] \\
A_{i,\nu}^{(2)}(q, k) &= \frac{1}{f} [B_{i}^{(1)}(q, k) \sin \varepsilon \cos \delta_{j-1} \\
&\quad + B_{i}^{(2)}(q, k) \cos \delta_{j+1} \sin \delta_{j-1}],
\end{align}

where

\begin{align}
f = \frac{1}{2} [\cos(\delta_{j+1} + \delta_{j-1}) + \cos \varepsilon \cos(\delta_{j+1} - \delta_{j-1})],
\end{align}

the elements of real off-shell \( K \) matrix are then given by

\begin{align}
K_{i,\nu}^{(1)}(p, k; z) &= \frac{1}{pk} \left[ J_{\nu}(pr) \frac{d}{dr} \chi_i^{(1)}(z, k, r) - \chi_i^{(1)}(z, k, r) \frac{d}{dr} J_{\nu}(pr) \right]_{r=R} \\
&\quad - \frac{1}{pk} (z - p^2) \delta_{i,\nu} \int_0^R J_{\nu}(pr) J_1(kr) dr \\
&\quad + \frac{1}{pk} (z - p^2) \int_R^\infty J_{\nu}(pr) \varphi_i^{(1)}(z, k, r) dr, \quad R > r_4,
\end{align}

where

\begin{align}
\varphi_i^{(1)}(z, k, r) = A_{i,\nu}^{(1)}(q, k) \xi^{(1)}(qr) + A_{i,\nu}^{(2)}(q, k) \xi^{(2)}(qr) \\
&\quad + B_{i}^{(1)}(q, k) \eta^{(1)}(qr) + B_{i}^{(2)}(q, k) \eta^{(2)}(qr) + \varphi_i^{(1)}(z, k, r), \quad r > r_4.
\end{align}
Letting the momenta $p$ and $k$ equal to $q$, we have

$$K_{i,J+1}(q, q; z) = \frac{1}{qf}[B_i^{(1)}(q, q) \cos \bar{\epsilon}_J \cos \bar{\delta}_{J-1} + B_i^{(2)}(q, q) \sin \bar{\epsilon}_J \sin \bar{\delta}_{J-1}]$$  \hspace{1cm} (67)

and

$$K_{i,J-1}(q, q; z) = \frac{1}{qf}[B_i^{(1)}(q, q) \sin \bar{\epsilon}_J \sin \bar{\delta}_{J+1} + B_i^{(2)}(q, q) \cos \bar{\epsilon}_J \cos \bar{\delta}_{J+1}]$$  \hspace{1cm} (68)

By use of Eqs. (58)～(61), the on-shell elements of the $K$ matrix are given by

$$K_{J+1,J+1}(q, q; z) = -\frac{1}{2fq} \sin (\bar{\delta}_{J+1} + \bar{\delta}_{J-1}) + \cos 2\bar{\epsilon}_J \sin (\bar{\delta}_{J+1} - \bar{\delta}_{J-1})$$  \hspace{1cm} (69)

$$K_{J+1,J-1}(q, q; z) = K_{J-1,J+1}(q, q; z) = -\frac{1}{2fq} \sin 2\bar{\epsilon}_J,$$  \hspace{1cm} (70)

$$K_{J-1,J-1}(q, q; z) = -\frac{1}{2fq} \sin (\bar{\delta}_{J+1} + \bar{\delta}_{J-1}) - \cos 2\bar{\epsilon}_J \sin (\bar{\delta}_{J+1} - \bar{\delta}_{J-1}).$$  \hspace{1cm} (71)

This on-shell $K$ matrix satisfies the following relation:

$$T + \frac{1}{2}iqKT = -iqK,$$  \hspace{1cm} (72)

where $T$ is the matrix $(2iq \mu^{(l+l')}_{i,l}(q, q; z))$.

§ 4. Numerical examples of $\mu^{(l+l')}_{i,l}(p, k; z)$

We shall show here numerical examples of the off-shell $T$ matrices of the triplet-state interaction for the total angular momentum state $J=1$ and discuss the properties of off-shell $T$ matrices. The energy parameter used is 75 MeV. The Hamada-Johnston potential is assumed in Eq. (4).

For the practical calculations, the same unit as the Hamada-Johnston's has been used, i.e., $x = \mu r$ ($\mu = (1.415 \text{ fm})^{-1}$). The potential range $x_d = \mu r_d$ in this unit has been chosen to be $x_d = 9.0$. When the differential equations are solved by the Runge-Kutta-Gill method, the mesh distance is $0.1875 \times 10^{-2}$ from $x_c = 0.343$ to $x_m = 1.5$ and $0.9375 \times 10^{-2}$ from $x_m$ to $x_d$. The last integration in Eq. (50) is done by Simpson's 1/3 rule. Numerical values of the off-shell $T$ matrices are obtained with an accuracy of order $10^{-8}$ and are shown in Tables I, II, III and IV for the pairs $(l, l')$. With the present unit, all of the momenta are given by the usual momenta divided by $\mu$. Therefore, practical values of the off-shell $T$ matrices should be obtained from all of the values in the tables divided by $\mu$.

As is seen from the tables, the off-shell $T$ matrices have the symmetry property

$$\mu^{(l+l')}_{i,l}(p, k; z) = \mu^{(l+l')}_{i,l}(k, p; z).$$  \hspace{1cm} (73)

This property will be understood from the following considerations. For the fixed energy parameter $z$, the $k$-dependence of functions $\chi^{(l+l')}_{i,l}(z, k, r)$ for $l = J \pm 1$
Table I. The off-shell $T$ matrix $t_{ab\uparrow}^{1}(p,k;z)$ for the state $J=1$. The energy parameter $E = \hbar^2 k^2 / 2m$ is 75 MeV. The bar phase shifts and coupling parameter $\tilde{\eta}$ are $\delta_0=0.54176, \delta_2 = -0.30606$ and $\tilde{\eta}=0.06642$, respectively. All momenta are represented in dimensionless forms. See the text for the practical momenta.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$0.5$</th>
<th>$1.0$</th>
<th>$1.903$</th>
<th>$3.0$</th>
<th>$4.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re</td>
<td>-1.0681</td>
<td>-1.0018</td>
<td>-0.8811</td>
<td>-0.7909</td>
<td>-0.4572</td>
</tr>
<tr>
<td>Im</td>
<td>-1.0018</td>
<td>-1.0681</td>
<td>-0.7909</td>
<td>-0.8811</td>
<td>-0.3115</td>
</tr>
</tbody>
</table>

Table II. The off-shell $T$ matrix $t_{ab\uparrow}^{1}(p,k;z)$ for the state $J=1$. For the parameters and momenta, see Table I and the text.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$0.5$</th>
<th>$1.0$</th>
<th>$1.903$</th>
<th>$3.0$</th>
<th>$4.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re</td>
<td>-0.1596</td>
<td>0.0232</td>
<td>-0.0458</td>
<td>0.0166</td>
<td>0.0035</td>
</tr>
<tr>
<td>Im</td>
<td>0.0232</td>
<td>-0.1596</td>
<td>0.0166</td>
<td>-0.0458</td>
<td>0.0036</td>
</tr>
</tbody>
</table>

Table III. The off-shell $T$ matrix $t_{ab\uparrow}^{1}(p,k;z)$ for the state $J=1$. For the parameters and momenta, see Table I and the text.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$0.5$</th>
<th>$1.0$</th>
<th>$1.903$</th>
<th>$3.0$</th>
<th>$4.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re</td>
<td>-0.1596</td>
<td>0.0232</td>
<td>-0.0458</td>
<td>0.0166</td>
<td>0.0035</td>
</tr>
<tr>
<td>Im</td>
<td>0.0232</td>
<td>-0.1596</td>
<td>0.0166</td>
<td>-0.0458</td>
<td>0.0036</td>
</tr>
</tbody>
</table>

Table IV. The off-shell $T$ matrix $t_{ab\uparrow}^{1}(p,k;z)$ for the state $J=1$. For the parameters and momenta, see Table I and the text.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$0.5$</th>
<th>$1.0$</th>
<th>$1.903$</th>
<th>$3.0$</th>
<th>$4.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re</td>
<td>-0.1596</td>
<td>0.0232</td>
<td>-0.0458</td>
<td>0.0166</td>
<td>0.0035</td>
</tr>
</tbody>
</table>
are the same as that of \( g_i(kr) \). Therefore, Eq. (50) can be described by

\[
\tau_{i}^{(l)}(p, k; \omega) = p^{l} k^{l} \sum_{l', J} a_{ij}(l, l'; J, \omega) p^{2(l-1)} k^{2(l-1)},
\]

(74)

where \( a_{ij}(l, l'; J, \omega) \) is a complex quantity independent of the momenta \( p \) and \( k \). Considering the on-shell elements, we have

\[
\tau_{i}^{(l)}(q, q; \omega) = q^{l} \sum_{l', J} a_{ij}(l', l; J, \omega) q^{2(l-1)}.
\]

On the other hand, the on-shell elements of the \( T \) matrix must be symmetric in \( l \) and \( l' \), i.e.,

\[
\tau_{i}^{(l)}(q, q; \omega) = \tau_{i}^{(l')}(q, q; \omega) = q^{l+l'} \sum_{l', J} a_{ij}(l', l; J, \omega) q^{2(l-1)}.
\]

From the above equations, we find that \( a_{ij}(l, l'; J, \omega) \) should be symmetric not only in \( l \) and \( l' \), but also in \( i \) and \( j \). This property of \( a_{ij}(l, l'; J, \omega) \) is retained for all of \( \omega \). Applying this property to Eq. (74), we can immediately verify Eq. (73).

The real parts of off-shell \( T \) matrices for small \( p \) and \( k \) show a characteristic behaviour: For example, the real parts of \( \tau_{0}^{(l)}(p, k; \omega) \) and \( \tau_{1}^{(l)}(p, k; \omega) \) are negative, but that of \( \tau_{2}^{(l)}(p, k; \omega) \) is positive. These properties are related to the signs of the potentials \( U_1(r), U_2(r) \) and \( U_{12}(r) \). Roughly speaking, the off-shell \( T \) matrices are approximately given by

\[
\tau_{i}^{(l)}(p, k; \omega) \approx \frac{1}{pk} \int_{0}^{\infty} g_{J+1}(pr) U_{1}(r) g_{J+1}(kr) dr,
\]

\[
\tau_{i}^{(l')}_{J+1}(p, k; \omega) \approx \frac{1}{pk} \int_{0}^{\infty} g_{J-1}(pr) U_{15}(r) g_{J-1}(kr) dr,
\]

\[
\tau_{i}^{(l)}_{J-1}(p, k; \omega) \approx \frac{1}{pk} \int_{0}^{\infty} g_{J+1}(pr) U_{15}(r) g_{J-1}(kr) dr,
\]

\[
\tau_{i}^{(l')}_{J-1}(p, k; \omega) \approx \frac{1}{pk} \int_{0}^{\infty} g_{J-1}(pr) U_{1}(r) g_{J-1}(kr) dr.
\]

For the triplet even state, potentials \( U_5(r) \) and \( U_{15}(r) \) are attractive and \( U_1(r) \) repulsive. Therefore, we can arrive at the above result. Generally speaking, the signs of the real parts of off-shell \( T \) matrices for small momenta \( p \) and \( k \) coincide with those of the potentials. This conclusion can be true not only for the off-shell \( T \) matrices, but also for the off-shell \( K \) matrices.

Lastly, it takes 82 seconds for all of the values in Tables I, II, III and IV on the FACOM 230-60, Computer Centre, at Kyushu University.

§ 5. Summary and remarks

We shall summarize here the main points of the discussions in the previous sec-

\[
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\]
tions and mention some remarks in the following. The function \( \chi^{l, i}_{i, l'}(z, k, r) \) is uniquely determined under the boundary conditions that the function \( \chi^{l, i}_{i, l'}(z, k, r) \) for \( r > r_c \) should be continuous to the function \(-\delta_{i, l'} \mathcal{J}_i(kr)\) for \( r < r_c \) and should represent the outgoing wave at large \( r \). It should be noted that the solution \( u^{l, i}_{i, l'}(z, k, r) \) of the off-shell Lippman-Schwinger equation, in terms of Eq. (13) in Ref. 2), vanishes for \( r < r_c \), though the function \( \chi^{l, i}_{i, l'}(z, k, r) \) for \( l = l' \) does not, for \( r < r_c \). The advantage of the present method exists in the fact that each element of the off-shell \( T \) matrix of the triplet state interaction is independently calculated by use of the function \( \chi^{l, i}_{i, l'}(z, k, r) \) and the usual on-shell elements are automatically reproduced. It is emphasized that the present method is still applicable to any local potential with or without hard cores, even if the potential has some energy dependence.

The off-shell \( T \) matrices have the symmetry property
\[
t_{i, l'}^{(l''+)}(p, k; z) = t_{i, l'}^{(l''+)}(k, p; z)
\]
for fixed energy parameters. The on-shell elements are represented only by the outer boundary term in Eq. (50). It is the result of the second derivative of the function \( \chi^{l, i}_{i, l'}(z, q, r) \) at the hard-core edge, i.e., Eq. (48), and corresponds to the fact that the usual on-shell \( T \) matrix is determined only by the phase shifts and coupling parameter which are determined under the conditions on the function \( u^{l, i}_{i, l'}(z, q, r) \). The signs of the real parts of the off-shell \( T \) matrices for small momenta \( p \) and \( k \) coincide with those of the potentials \( U_i(r) \), \( U_4(r) \) and \( U_5(r) \).

The following remarks seem to be worthy. In order to show that the present method can reproduce the usual on-shell \( T \) and \( K \) matrices, the bar phase shifts and coupling parameter introduced by Stapp et al. have been used when the linear-independent solutions of the homogeneous equation of Eq. (3) are determined. However we have no need to use any phase shifts and coupling parameters, for the functions \( \chi^{l, i}_{i, l'}(z, k, r) \) should be determined under the boundary conditions. Moreover, the conditions imposed on the special solution \( \phi^{l, i}_{i, l'}(z, k, r) \) may be removed. Any linear-independent solutions for \( \xi^{(3), i}_{2, l}(qr) \) and \( \eta^{(3), i}_2(qr) \) and any special solutions of Eqs. (3) and (5) for \( \phi^{l, i}_{i, l'}(z, k, r) \) may be used. Then the multiplicative complex constants \( A_{i, l'}^{1, (n)} \) and \( B_{i, l'}^{1, (n)} \) should be determined by the conditions imposed on the functions \( \chi^{l, i}_{i, l'}(z, k, r) \):
\[
A_{i, l'}^{1, (n)} \mathcal{J}_i(qr_e) + A_{i, l'}^{1, (n)} \mathcal{J}_i(qr_d) = -\delta_{i, l'} \mathcal{J}_i(kr_e),
\]
\[
A_{i, l'}^{1, (n)} \mathcal{J}_i(qr_e) + A_{i, l'}^{1, (n)} \mathcal{J}_i(qr_d) = -\delta_{i, l'} \mathcal{J}_i(kr_d)
\]
\[
A_{i, l'}^{1, (n)} \mathcal{J}_i(qr_e) + A_{i, l'}^{1, (n)} \mathcal{J}_i(qr_d) = -\delta_{i, l'} \mathcal{J}_i(kr_e),
\]
\[
A_{i, l'}^{1, (n)} \mathcal{J}_i(qr_e) + A_{i, l'}^{1, (n)} \mathcal{J}_i(qr_d) = -\delta_{i, l'} \mathcal{J}_i(kr_d).
\]
where primes on the functions denote the differentiation with respect to $r$ at $r = r_d$. These results must reproduce the same functions $\chi_{i,j}^{(+)}(z, k, r)$ as the one in § 2.

The present method can be applicable to the off-shell $T$ matrix with negative energy parameters $z < 0$. This kind of off-shell $T$ matrix will be obtained by analytic continuations from the previous and present off-shell $T$ matrices and will be useful for three-body bound state problems. These $T$ matrices will be treated elsewhere.

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**References**

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