A Realistic Model of Convergent Relativistic Quantum Mechanics of Interacting Particles

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A model of convergent relativistic quantum mechanics of interacting particles in which the particle number is not conserved is obtained. The model somewhat resembles the so-called $\phi^4$ theory, but differs from the latter at the following points. i) It is given in terms of creation and annihilation operators in momentum space representation, and the Hamiltonian does not contain those terms which are products of only creation operators or only annihilation operators like $A^-A^-A^-A^-$ or $AAAA$. Hence the model is free from the divergence difficulties arising from $\langle 0 | AAAAAA^-A^-A^- | 0 \rangle$. ii) The model incorporates the invariant form factors from the start, and hence is free from the ultraviolet divergence difficulties. iii) The model is obtained as a solution of the fundamental commutator equations for ten generators of the Poincaré group. On solving the equations, the primary interaction Hamiltonian which is the sum of the terms $A^-A^-A^-A^-$, $A^-A^-A^-$ and $A^-A^-A^-$ multiplied by the form factors is used as an input. Thus the model substantially forms a unitary reducible representation of the Poincaré group.

The S-matrix elements for two-particle elastic scattering and two-particle production processes are calculated up to second order with respect to the coupling constant. The dressing effect and the contributions to real processes of the particle-number-changing interaction terms are examined. The implication of these results on the future application of this formalism to quantum electrodynamics is mentioned.

§ 1. Introduction and summary

In a previous paper a model of convergent relativistic quantum mechanics of interacting particles has been investigated. There, the convergence of the theory is confirmed by the form factor (the vertex function) which is incorporated into the theory from the start. The relativistic invariance of the theory is guaranteed by the fact that it constitutes a unitary reducible representation $U(A_{\mu\nu}, \tau_{\rho})$ of the Poincaré group. The model is characterized by the set of infinitesimal generators of the Poincaré group which satisfies the fundamental commutator equations:

\[
\begin{align*}
[P_\mu, P_\nu] &= 0, \\
[M_\mu, P_\nu] &= i\delta_{\mu\nu}P_\sigma - i\delta_{\nu\sigma}P_\mu, \\
[M_\mu, M_\nu] &= i\delta_{\mu\sigma}M_{\nu\sigma} - i\delta_{\nu\sigma}M_{\mu\sigma} + i\delta_{\nu\sigma}M_{\mu\rho} - i\delta_{\mu\sigma}M_{\nu\rho}.
\end{align*}
\] (1.1)

The reducibility corresponds to the reducible structure of the Hilbert space spanned by the eigenstates of the $P_\mu$. The state space is decomposed into the sum
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of various invariant subspaces, each being the representation space of a unitary irreducible representation of the Poincaré group. The one-particle states constitutes one of such invariant subspaces. Since the particle number is a constant of motion in that model, any scattering state is characterized by the particle number. The set of $n$-particle scattering states can be decomposed into the sum of various invariant subspaces characterized by the eigenvalues of the $P^s$ and the Pauli-Lubanski operator. As has been emphasized in Ref. 1), the perfect parallelism between the $P_i$ and $M_i$ in their forms is essential for the validity of the simple transformation law of the continuum states $|q^{(1)}_A, \ldots, q^{(n)}_A\rangle$ with incident waves plus outgoing (or ingoing) scattered waves boundary conditions:

$$U(A_{2^n}, \tau_A)|q^{(1)}_A, \ldots, q^{(n)}_A\rangle = \exp(i\tau_A(q^{(1)}_p + \ldots + q^{(n)}_p))|A q^{(1)}_A, \ldots, A q^{(n)}_A\rangle.$$  \hfill (1·2)

The model which has been obtained in Ref. 1), however, is so simple that the particle number does not change and consequently no dressing effect of particles occurs. It is the purpose of this paper to obtain a more realistic model in which the particle number is not a constant of motion and thereby to search for a reasonable way of describing the dynamics of elementary particles. For such a generalization, however, a very important restriction will be imposed, in § 2, on the form of the interaction Hamiltonian. This restriction lies in forbidding, in the interaction Hamiltonian, any term which has only creation operators or only annihilation operators. By virtue of this restriction and the incorporation of form factors in the theory from the start, there occurs no divergence difficulties. If such terms as mentioned above were included in the interaction Hamiltonian, then they would give rise to divergence difficulties of a particular type which could not be overcome in spite of the incorporation of the form factors. In the usual theory of quantized fields, the appearance of the terms having only creation operators or only annihilation operators in the interaction Hamiltonian is inevitable, because the Lagrangian taken there has interaction parts which essentially are products of field operators and each of these field operators is a certain sum of creation and annihilation operators. In contrast with the usual theory of quantized fields, the approach adopted in this paper and the preceding papers $^{1·6}$ has great advantages in that it is never necessary to start with field operators, and consequently it is admitted without destroying the relativistic invariance of the theory to exclude terms which contain only creation operators or only annihilation operators. Thus, in § 2, the simplest model of convergent relativistic quantum mechanics of interacting particles in which the particle number is not a constant of motion is obtained by solving the fundamental commutator equations (1·1) successively with respect to the coupling constant under the Ansatz:

$$P_i = P_i^{(0)}, \quad M_{im} = M_{im}^{(0)},$$

$$P_i = P_i^{(0)} + \sum_{n=1}^{\infty} P_i^{(n)}, \quad M_{it} = M_{it}^{(0)} + \sum_{n=1}^{\infty} M_{it}^{(n)}.$$  \hfill (1·3)
where the $P_{\mu}$ and $M_{\mu}$ are the Poincaré generators for the non-interacting spinless particles of mass $m$, and the first order interaction Hamiltonian $H = -i\hat{P}_i$ is, as an input, given in terms of creation and annihilation operators by

$$
\hat{P}_i = \frac{i}{3!} \int \frac{dp dp' dp'' dq}{p_0 p'_0 p''_0 q_0} \delta(p + p' + p'' - q) \times \hat{F}(p, p', p''|q) A^\dagger(p) A^\dagger(p') A^\dagger(p'') A(q)
$$

$$
+ \frac{i}{2!} \int \frac{dp dp' dp dq'}{p_0 p'_0 q_0 q'_0} \delta(p + p' - q - q') \times \hat{F}(p, p'|q, q') A^\dagger(p) A^\dagger(p') A(q) A(q')
$$

$$
+ \frac{i}{3!} \int \frac{dp dq dq''}{p_0 q_0 q''_0} \delta(p - q - q' - q'') \times \hat{F}(p|q, q', q'') A^\dagger(p) A(q) A(q') A(q'').
$$

The primary vertex functions (the form factors) $\hat{F}$ are real invariant functions which have a certain symmetry properties with respect to their arguments and which damp sufficiently fast at large values of the arguments. For any higher $N$ one can get the $P_{(N)}$ and $M_{(N)}$ by solving Eq. (1·1) successively in such a way that the perfect parallelism between the $P_{(N)}$ and $M_{(N)}$ in their forms holds. In § 2, explicit expressions for the $P_{(N)}$ and $M_{(N)}$ up to $N=2$ are given. In § 3, the physical one-particle states are obtained and the dressing effect by the first and third terms on the right-hand side of Eq. (1·4) is calculated up to second order with respect to the coupling constant. In § 4, the scattering states $|q^{(1)} q^{(2)}\rangle$ and $|q^{(1)} q^{(2)}, q^{(3)}\rangle$ are obtained up to second order with respect to the coupling constant and then the $S$-matrix elements $\langle p^{(1)} p^{(2)}|S|q^{(1)} q^{(2)}\rangle$ and $\langle p^{(1)} p^{(2)} p^{(3)} p^{(4)}|S|q^{(1)} q^{(2)}\rangle$ are calculated. The resulting $S$-matrix elements up to second order are as follows:

$$
\langle p^{(1)} p^{(2)}|S|q^{(1)} q^{(2)}\rangle
$$

$$
= \sqrt{p_0^{(1)} p_0^{(2)}} \delta(p^{(1)} p^{(2)}|q^{(1)} q^{(2)}) \sqrt{q_0^{(1)} q_0^{(2)}}
$$

$$
- \pi i \delta(p^{(1)} + p^{(2)} - q^{(1)} - q^{(2)}) \{(q^{(1)} q^{(2)}|F(p^{(1)} p^{(2)})|q^{(1)} q^{(2)})
$$

$$
- \pi i \frac{1}{2!} \int \frac{dk dk'}{k_0 k'_0} F(p^{(1)} p^{(2)}|k, k') \delta(k_0 + k'_0 - q^{(1)} - q^{(2)})
$$

$$
	imes \hat{F}(k, k'|q^{(1)} q^{(2)}\rangle
$$

$$
+ O(g^3).
$$

(1·5)
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\[ \langle p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)} | S | q^{(1)}, q^{(2)} \rangle \]
\[ = - \frac{2\pi^2}{\sqrt{4!}} \delta(p^{(1)}_\mu + p^{(2)}_\mu + p^{(3)}_\mu + p^{(4)}_\mu - q^{(1)}_\mu - q^{(2)}_\mu) \]
\[ \times \sum_p \int \frac{dk^{(1)}}{k_5} \tilde{F}(p^{(1)}, p^{(2)}, p^{(3)} | k) \delta(p^{(1)}_\mu + k_\mu - q^{(1)}_\mu - q^{(2)}_\mu) \]
\[ \times \tilde{F}(p^{(1)}, k | q^{(3)}, q^{(4)}) \]
\[ + O(g^{(4)}) \]  

where \( \sum_p \) denotes the sum of the succeeding expression over four combinations of dividing the \( p^{(1)}, \ldots, p^{(4)} \) into \( \langle p^{(1)}, p^{(2)}, p^{(3)} \rangle \). Thus the relativistic invariance and the convergence of the theory manifest themselves also in the explicit expressions of the \( S \) matrix. It is important, however, to notice that products such as \( \tilde{F}(p^{(1)}, p^{(2)} | q^{(3)}, k^{(1)}, k^{(2)}) \tilde{F}(p^{(1)}, k^{(1)}, k^{(2)} | q^{(1)}) \) do not contribute to the \( \langle p^{(1)}, p^{(2)} | S | q^{(1)}, q^{(2)} \rangle \) in second order with respect to \( g \). Based on such observation of the result, in § 5, the effects of those primary interaction terms, in which the particle number changes, on the various real processes are examined from the standpoint of the structural mechanism of the formalism, and its implication in the future application of this formalism to quantum electrodynamics is mentioned.

§ 2. Model in which the particle number changes

In order to get a model which is more realistic than the previous one in that the particle number is not conserved and also in that composite particle states can be contained, let us put the Ansatz (1.3) for the Poincaré generators, set the primary interaction Hamiltonian \( \tilde{H} = -i \tilde{P}_4 \) as Eq. (1.3), and then solve Eq. (1.1) successively with respect to the coupling constant \( g \) with an additional care to hold the perfect parallelism between the \( \tilde{P}_i \) and the \( \tilde{M}_\mu \) in their forms.

The \( \tilde{P}_4 \) given by Eq. (1.4) is composed of three terms. Among three, the second term is just the \( \tilde{P}_4 \) of the previous model. The conditions which should be imposed on the vertex function \( \tilde{F}(p, p'| q, q') \) have been stated in Ref. 1). The first term and the third one in Eq. (1.4) are the added terms which violate the particle number conservation. We impose on the new vertex functions \( \tilde{F}(p, p', p'' | q) \) and \( \tilde{F}(p | q, q', q'') \) the following conditions similar to those on \( \tilde{F}(p, p' | q, q') \):

(i) real,

(ii) \( \tilde{F}(p, p', p'' | q) = \tilde{F}(q, p, p', p'') \) and symmetric with respect to \( p, p' \) and \( p'' \).
(iii) invariant function of the argument, namely, 
\[ \mathcal{M}_\mu \mathcal{F}(p, p', p'' | q) = 0 = \mathcal{M}_\mu \mathcal{F}(q, q', q'') , \]
(iv) damp sufficiently fast at the large values of the arguments.

Now one may wonder why terms such as 
\[ P^{(\text{bad})} = \frac{i}{4!} \int \frac{dp dp' dp'' dp'''}{p p' p'' p'''} \delta(p + p' + p'' + p''') \]
\[ \times \mathcal{F}(p, p', p'', p''') A^\dagger(p) A^\dagger(p') A^\dagger(p'') A^\dagger(p'''',) \]
\[ + \frac{i}{4!} \int \frac{dq dq' dq'' dq'''}{q q' q'' q'''} \delta(q + q' + q'' + q''') \]
\[ \times \mathcal{F}(q, q', q'', q''') A(q) A(q') A(q'') A(q'''',) \]
with real \( \mathcal{F}(p, p', p'', p''') \) and \( \mathcal{F}(q, q', q'', q''') \) should be excluded from \( P^4 \). The reason is as follows. If the \( P^{(\text{bad})} \) were included in the \( P^4 \), then the resulting \( (P^4)^2 \) would contain a term such as 
\[ -\frac{1}{(4!)^2} \int \frac{dp dp' dp'' dp'''}{p p' p'' p'''} \frac{dq dq' dq'' dq'''}{q q' q'' q'''} \delta(p + p' + p'' + p''') \]
\[ \times \mathcal{F}(p, p', p'', p''') \mathcal{F}(q, q', q'', q''') A(q) A(q') A(q'') A(q''',) \]
\[ \times A^\dagger(p) A^\dagger(p') A^\dagger(p'') A^\dagger(p'''',) , \]
which, when operated onto the free vacuum state \( |0\rangle \), would surely give rise to an infinity 
\[ -\delta(0) \frac{1}{4!} \int \frac{dp dp' dp'' dp'''}{p p' p'' p'''} \delta(p + p' + p'' + p'''') \]
\[ \mathcal{F}(p, p', p'', p''') [\mathcal{F}(q, q', q'', q''')] . \]

From this expression it is evident that such \( \delta(0) \)-type infinity cannot be overcome by whatever form factor \( \mathcal{F}(p, p', p'', p''') \), in so far as the \( P^4 \) contains the \( P^{(\text{bad})} \). This is the reason why the author excludes, in the definition (1.4) of \( P^4 \), such terms as \( P^{(\text{bad})} \). By forbidding the terms which consist of only creation operators or only annihilation operators to get in the generators of the Poincaré group, one can always remain in the same Hilbert space which is spanned by the state \( |0\rangle \) and the states obtained by operation of creation operators onto the vacuum state.

The infinity which originates from terms of the above-mentioned type in a Hamiltonian was analyzed by Dirac\(^6\) in a simple model Hamiltonian. He showed that, for systems with infinite degrees of freedom and with terms of the above-mentioned type in its Hamiltonian, such infinity is inherent in the Schrödinger picture, while it is eliminated in the Heisenberg picture. For this reason he
developed his quantum field theory in the Heisenberg picture, and proposed its physical interpretation in terms of intensity instead of probability. As to other types of infinities which cannot be eliminated even in the Heisenberg representation, he used cutoffs with somewhat reasonable explanation, by which, however, the relativistic invariance of his theory was destroyed.

In the usual theory of quantized fields, the appearance of the terms having only creation operators or only annihilation operators is inevitable, because the Lagrangian taken there has interaction parts which essentially are products of field operators and each of these field operators is a certain sum of creation and annihilation operators; if the Lagrangian contains, for example, a term like $A^tA^tA^t$, then it necessarily contains also terms like $A^tA^tA^tA^tA^tA^tA^t$ and $A^tA^tA^tA^tA^tA^tA^tA^tA^t$. In addition to this situation, the local character of interactions between fields is essential to the ultra-violet divergence difficulties of the usual theory of quantized fields.

In these respects the author emphasizes here that the approach adopted in this paper and the preceding papers has great advantages in contrast with the usual theory of quantized fields. Firstly it is never necessary to start from field operators, and consequently it is admitted to exclude terms which contain only creation operators or only annihilation operators. Secondly it is possible to work, from the very start, with form factors in a relativistically invariant way, and hence to get a finite theory of interacting particles without destroying relativistic invariance. The theory obtained by this approach is essentially of a nonlocal nature.

Now let us go back to the problem of solving Eqs. (1·1) successively with respect to the coupling constant $g$. In the first order of $g$, Eqs. (1·1) reduce to

\[
\begin{align*}
&[M_{4i}, \hat{P}_m] = -i\delta_{im}\hat{P}_i, \\
&[M_{4i}, \hat{M}_m] = i\delta_{im}M_{4i} - i\delta_{ij}M_{4m}, \\
&[\hat{M}_{4i}, \hat{P}_j] + [\hat{M}_{4i}, \hat{P}_j] = 0, \\
&[\hat{M}_{4i}, \hat{M}_m] + [\hat{M}_{4i}, \hat{M}_m] = 0.
\end{align*}
\]

By putting the expression (1·4) of the $\hat{P}_i$ into Eqs. (2·1), and using the formulae (6) ~ (15) of Ref. 2) and the imposed properties of the vertex functions $F$ we get the following solution for the $\hat{M}_{4i}$:

\[
\begin{align*}
\hat{M}_{4i} = \frac{1}{3!} \int \frac{d\rho d\rho' d\rho'' dq}{\rho \rho' \rho'' q} \frac{\partial \delta(p + p' + p'' - q)}{\partial \rho_i} \\
	imes F(p, p', p''|q) A^t(p) A^t(p') A^t(p'') A(q)
\end{align*}
\]
The expression thus obtained for the $\mathcal{M}$ has a perfect parallelism to the one for the $\mathcal{P}$. Next, in the second order of $\mathcal{g}$ Eqs. (1·1) reduce to

\[
\begin{align*}
\{\mathcal{P}_4, \mathcal{P}_4\} &= 0, \\
\{\mathcal{P}_4, \mathcal{M}_4m\} &= 0, \\
\{\mathcal{M}_44, \mathcal{P}_4\} &= -i\delta_{4m}\mathcal{P}_4, \\
\{\mathcal{M}_44, \mathcal{M}_4m\} &= i\delta_{4m}\mathcal{M}_4 - i\delta_{44}\mathcal{M}_4m, \\
\{\mathcal{M}_44, \{\mathcal{P}_4, \mathcal{P}_4\}\} &= -\{\mathcal{M}_44, \mathcal{P}_4\}, \\
\{\mathcal{M}_44, \{\mathcal{M}_44, \mathcal{M}_4m\}\} &= -\{\mathcal{M}_44, \mathcal{M}_4m\}. 
\end{align*}
\]

By the use of Eqs. (1·4) and (2·2), the commutators $\{\mathcal{M}_44, \mathcal{P}_4\}$ and $\{\mathcal{M}_44, \mathcal{M}_4m\}$ can be calculated. The calculation shows that both the commutators are composed of nine terms of different type. For example, the $A^4A$-type terms of the commutators are as follows:

\[
\begin{align*}
\{\mathcal{M}_44, \{\mathcal{P}_4, \mathcal{P}_4\}\} &= i \int \frac{dpdq}{p_0q_0} \delta(p - q) G_1(p|q) A^4(p) A(q), \\
\{\mathcal{M}_44, \{\mathcal{M}_44, \mathcal{M}_4m\}\} &= i \int \frac{dpdq}{p_0q_0} \left\{ \frac{\delta(p - q)}{\delta p_n} G_1(p|q) - \frac{\delta(p - q)}{\delta p_i} G_m(p|q) \right\} A^4(p) A(q), 
\end{align*}
\]

where

\[
G_1(p|q) = -\frac{1}{2 \cdot 3!} \int \frac{dkdk'k''}{k_0k'_0k''_0} \left\{ \frac{\delta(p - k - k' - k'')}{\delta k_1} + \frac{\delta(k + k' + k'' - q)}{\delta k_1} \right\} \times F(p|k, k', k'') F(k, k', k''|q).
\]
We can solve Eqs. (2·3) for the $^{(2)}P_i$ and $^{(2)}M_i$ in the following way. Let us assume both the $^{(2)}P_i$ and $^{(2)}M_i$ to be the sums of nine terms of the same types as those in the $^{(1)}P_i$, $^{(1)}M_i$. For example we write the $A^A$ type terms of the $^{(2)}P_i$ and $^{(2)}M_i$ as

$$
^{(2)}(P_i)_{A^A} = i \int \frac{dp dq}{p_0 q_0} \delta(p - q)^{^{(2)}} F(p|q) A^1(p) A(q),
$$

$$
^{(2)}(M_i)_{A^A} = \int \frac{dp dq}{p_0 q_0} \delta(p - q)^{^{(2)}} F(p|q) A^1(p) A(q),
$$

with the function $^{^{(2)}}F(p|q)$ to be determined. Then, by virtue of the formulae (6) ~ (15) of Ref. 2), the $A^A$-type parts of Eqs. (2·3) give the following equations for the $^{(2)}F(p|q)$:

$$
\begin{align*}
\delta(p - q) M_{im}^{^{(2)}} F(p|q) &= 0, \\
\frac{\partial \delta(p - q)}{\partial p_i} M_{im}^{^{(2)}} F(p|q) &= 0, \\
\delta(p - q) \{ M_{si}^{^{(2)}} F(p|q) + G_i^{^{(2)}}(p|q) \} &= 0, \\
\frac{\partial \delta(p - q)}{\partial p_m} \{ M_{im}^{^{(2)}} F(p|q) + G_i^{^{(2)}}(p|q) \} \bigg|_{p_i} &= 0.
\end{align*}
$$

As the simplest solution of Eqs. (2·9) for the $^{^{(2)}}F(p|q)$ we get

$$
^{^{(2)}}F(p|q) = -\frac{1}{2 \cdot 3!} \int \frac{dk dk' dk''}{k_0 k_0' k_0''} \left\{ \delta(p - k - k' - k'') \left( \frac{\delta(p - k - k' - k'')}{p_0 - k_0 - k_0' - k_0''} \right) - \frac{\delta(k + k' + k'' - q)}{k_0 + k_0' + k_0'' - q_0} \right\} \times \{^{(1)}F(p|k, k', k'')^{(1)}F(k, k', k'')|q\}. 
$$

In quite the same way we can get the other eight terms of both the $^{(2)}P_i$ and $^{(2)}M_i$. Thus we obtain the $^{(2)}P_i$ and $^{(2)}M_i$ whose forms bear a perfect parallelism with each other.

$$
^{(2)}P_i = \sum_{n=1}^{2n-1} \frac{i}{(2n-m)!} \int \frac{dp_1 \cdots dp_{2n-m}}{p_0^{(i)} \cdots p_0^{(2n-m)}} \frac{dq_1 \cdots dq_m}{q_0^{(i)} \cdots q_0^{(m)}} \times \delta(p^{(i)} + \cdots + p^{(2n-m)} - q^{(i)} - \cdots - q^{(m)}) \{^{(2)}F(p^{(i)}, \cdots, p^{(2n-m)})|q^{(i)}, \cdots, q^{(m)}\}
$$
\[ M_{4t} = \sum_{n=1}^{2n-2} \sum_{m=1}^{2n-1} \frac{1}{(2n-m)! m!} \int \cdots \int \frac{dp^{(1)} \cdots dp^{(2n-m)}}{p^{(1)} \cdots p^{(2n-m)}} \frac{dq^{(1)} \cdots dq^{(m)}}{q^{(1)} \cdots q^{(m)}} \]
\[ \times \left\{ \delta \left( p^{(1)} + \cdots + p^{(2n-m)} - q^{(1)} - \cdots - q^{(m)} \right) \right\} \]
\[ \times F(p^{(1)}, \ldots, p^{(2n-m)}|q^{(1)}, \ldots, q^{(m)}) \]
\[ \times A'(p^{(1)}) \cdots A'(p^{(2n-m)}) A(q^{(1)}) \cdots A(q^{(m)}), \quad (2.11) \]

where the second order vertex functions \( F(p^{(1)}, \ldots, p^{(2n-m)}|q^{(1)}, \ldots, q^{(m)}) \) are given in terms of the first order vertex functions as follows:

\[ F(p, p', p''|q) = F(q|p, p', p'') \]
\[ \frac{1}{2 \cdot 2!} \sum_{r} \int \cdots \int \frac{dk dk'}{k k'} \left\{ \delta \left( p + p' - k - k' \right) + \delta \left( k + k' + p'' - q \right) \right\} \]
\[ \times F(p, p'|k, k') F(k, k'|q), \quad (2.13) \]

\[ F(p, p'|q, q') = \frac{1}{2 \cdot 2!} \int \cdots \int \frac{dk dk'}{k k'} \left\{ \delta \left( p + p' - k - k' \right) + \delta \left( k + k' + q - q' \right) \right\} \]
\[ \times F(p, p'|k, k') F(k, k'|q, q'), \quad (2.14) \]

\[ F(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}, p^{(5)}|q) = F(q|p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}, p^{(5)}) \]
\[ \frac{1}{2} \sum_{r} \int \frac{dk}{k} \left\{ \delta \left( p^{(5)} + p^{(4)} + p^{(3)} - k \right) + \delta \left( k + p^{(1)} + p^{(2)} - q \right) \right\} \]
\[ \times F(p^{(5)}, p^{(4)}, p^{(3)}|k) F(k, p^{(1)}, p^{(2)}|q), \quad (2.15) \]

\[ F(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}|q^{(1)}, q^{(2)}) = F(q^{(1)}, q^{(2)}|p^{(3)}, p^{(4)}|p^{(5)} p^{(6)} \]
\[ \frac{1}{2} \sum_{r} \int \frac{dk}{k} \left\{ \delta \left( p^{(5)} + p^{(4)} + p^{(3)} - k \right) + \delta \left( k + p^{(1)} + q^{(1)} - q^{(2)} \right) \right\} \]
\[ \times F(p^{(5)}, p^{(4)}, p^{(3)}|k) F(k, p^{(1)}|q^{(1)}, q^{(2)}). \quad (2.16) \]
\[ -\frac{1}{2} \sum \sum \int \frac{dk}{k_0} \left\{ \frac{\delta(p^{(1)} + p^{(2)} - q^{(1)} - k)}{p_0^{(1)} + p_0^{(2)} - q_0^{(1)} - k_0} - \frac{\delta(k + p^{(3)} - q^{(3)})}{k_0 + p_0^{(3)} - q_0^{(3)}} \right\} \]
\[ \times \mathcal{F}(p^{(1)}, p^{(2)}|q^{(1)}, k^{(1)}), \mathcal{F}(k, p^{(3)}|q^{(3)}), \quad (2.16) \]

\[ \mathcal{F}(p^{(1)}, p^{(2)}, p^{(3)}|q^{(1)}, q^{(2)}, q^{(3)}) \]
\[ = -\frac{1}{2} \sum \sum \int \frac{dk}{k_0} \left\{ \frac{\delta(p^{(1)} + p^{(2)} - k)}{p_0^{(1)} + p_0^{(2)} - k_0} - \frac{\delta(q^{(1)} - q^{(2)} - q^{(3)})}{k_0 - q_0^{(1)} - q_0^{(2)} - q_0^{(3)}} \right\} \]
\[ \times \mathcal{F}(p^{(1)}, p^{(2)}, p^{(3)}|q^{(1)}, q^{(2)}, q^{(3)}) \]
\[ -\frac{1}{2} \sum \sum \int \frac{dk}{k_0} \left\{ \frac{\delta(p^{(3)} - q^{(3)} - k)}{p_0^{(3)} - q_0^{(3)} - k_0} - \frac{\delta(k + p^{(3)} - q^{(3)} - q^{(1)})}{k_0 + k_0^{(3)} - q_0^{(3)} - q_0^{(1)}} \right\} \]
\[ \times \mathcal{F}(p^{(3)}, p^{(3)}|q^{(1)}, q^{(3)}) \]
\[ = -\frac{1}{2} \sum \sum \int \frac{dk}{k_0} \left\{ \frac{\delta(p^{(3)} - q^{(3)} - k)}{p_0^{(3)} - q_0^{(3)} - k_0} - \frac{\delta(k + p^{(3)} - q^{(3)} - q^{(1)})}{k_0 + k_0^{(3)} - q_0^{(3)} - q_0^{(1)}} \right\} \]
\[ \times \mathcal{F}(p^{(3)}, p^{(3)}|q^{(1)}, q^{(3)}) \]. \quad (2.17) \]

In the above integrals, principal values should be taken whenever the energy denominators vanish.

By the same procedure as the above, one can get the \( P_i \) and \( M_{ii} \) for any higher \( N \). As can be easily seen, the \( P_i \) and \( M_{ii} \) do not contain any pathological term like \( A_i A_i \cdots A_i \) or \( AA \cdots A \). Consequently, in this model, the free vacuum state \( |0\rangle \) is the simultaneous eigenstate of the \( P_\mu \) and \( M_{\mu\nu} \) with eigenvalues zero. The physical vacuum is identical with the free vacuum.

\section*{§ 3. Physical one-particle states}

The physical one-particle state or dressed one-particle state \( |q\rangle_{\text{dressed}} \) with the four-momentum \( q_\mu \) can be obtained from the equation
\[ P_\mu |q\rangle_{\text{dressed}} = q_\mu |q\rangle_{\text{dressed}} \quad (3.1) \]

by putting
\[ |q\rangle_{\text{dressed}} = Z^{-1/2}(q) \left\{ A^\dagger(q) \right\} \]
\[ + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \left( \int \frac{dp^{(1)} \cdots dp^{(2n+1)}}{p^{(1)}_0 \cdots p^{(2n+1)}_0} \delta(p^{(1)} + \cdots + p^{(2n+1)} - q) \right) \]
\[ \times \phi(p^{(1)}, \cdots, p^{(2n+1)}|q) A^\dagger(p^{(1)}) \cdots A^\dagger(p^{(2n+1)}) \} |0\rangle. \quad (3.2) \]
We normalize the states by
\[ \langle p | q \rangle_{\text{dressed}} = \sqrt{p_0} \delta(p - q) \sqrt{q_0}, \]  
so that
\[ Z(q) = 1 + \frac{1}{q_0} \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \int \cdots \int \prod_{k_1, \ldots, k_{n+1}} \frac{dk_1 \cdots dk_{n+1}}{k_1 \cdots k_{n+1}} \delta(k_1 + \cdots + k_{n+1} - q) \times |\varphi(k_1, \ldots, k_{n+1})|^2. \]  
By perturbative calculations the amplitudes \( \varphi(p^{(i)}, \ldots, p^{(n+1)}|q) \) can be obtained in the power series expansion with respect to the coupling constant \( g; \)
\[ \varphi(p^{(i)}, p^{(2)}, p^{(2)}, p^{(3)}, p^{(4)}, p^{(5)}|q) = \frac{1}{p^{(1)} + p^{(2)} + p^{(3)} + p^{(4)} + p^{(5)} - q_0} \left[ -F(p^{(1)}, p^{(2)}, p^{(3)}|q) \right. 
+ \left. \frac{1}{2} \sum_p \int \left. \frac{dk_d k_d'}{k_d k_d'} \left( \delta(p^{(1)} + p^{(2)} - k - k') + \delta(k + p^{(3)} - q) \right) \right. 
\times \left. F(p^{(1)}, p^{(2)}, k, p^{(3)}|q) \right] + O(g^3), \]
and so forth. Thus we get the following power series expansion of the normalization factor:
\[ Z^{-1/2}(q) = 1 \]
\[ - \frac{1}{2q_0} \frac{1}{3!} \int \cdots \int \frac{dk_d k_d' k_d''}{k_d k_d' k_d''} \left( \delta(k + k' + k'' - q) \right) \frac{1}{(k_d + k_d' + k_d'' - q_0)} \times \varphi(k, k', k'') \varphi(k, k', k'') \]  
\[ + O(g^3). \]  
The dressing effect of the first and third terms in the primary interaction Hamiltonian \( -i\hat{P} \) is now evident in the above results. As will be shown in the
next section, the normalization factor $Z^{-1/2}(q)$ of the physical one-particle state plays its proper role in the calculations of scattering states and $S$-matrix elements.

§ 4. Scattering states and $S$-matrix elements

First let us consider the two-particle scattering state with incident-plane-wave plus outgoing-(or ingoing)-scattered-wave-boundary condition. For this purpose we deal with the equation

$$ (H - q_0^{(1)} - q_0^{(3)} \mp i\epsilon)|q^{(1)}, q^{(3)}\rangle_{\pm i\epsilon} = 0, \quad (4.1) $$

where $H$ is the total Hamiltonian $-iP_4$ of the system and $\epsilon$ is a small positive parameter. We write the state vector as

$$ |q^{(1)}, q^{(3)}\rangle_{\pm i\epsilon} $$

$$ = \frac{1}{\sqrt{2}} Z^{-1/2}(q^{(1)}, q^{(3)}) \left\{ A^\dagger(q^{(1)}) A^\dagger(q^{(3)}) $$

$$ + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \cdots \int \frac{dp^{(1)} \cdots dp^{(2n)}}{p_{q_0}^{(1)} \cdots p_{q_0}^{(2n)}} \delta(p^{(1)} + \cdots + p^{(2n)} - q^{(1)} - q^{(3)}) $$

$$ \times \phi^{(\pm i\epsilon)}(p^{(1)}, \cdots, p^{(2n)}|q^{(1)}, q^{(3)}) A^\dagger(p^{(1)}) \cdots A^\dagger(p^{(2n)} \} |0\rangle, \quad (4.2) $$

where the $Z^{-1/2}(q^{(1)}, q^{(3)})$ is a normalization factor to be determined later. The first term in the curly brackets represents the incident part and the other terms stand for the scattered parts. The scattered waves $\phi^{(\pm i\epsilon)}$ can be determined from Eq. (4.1) in the perturbative expansion of the form

$$ \phi^{(\pm i\epsilon)}(p^{(1)}, p^{(3)}|q^{(1)}, q^{(3)}) = \sum_{n=1}^{(N)} \phi^{(\pm i\epsilon)}(p^{(1)}, \cdots, p^{(n)}|q^{(1)}, q^{(3)}), \quad (4.3) $$

$$ \phi^{(\pm i\epsilon)}(p^{(1)}, \cdots, p^{(4)}|q^{(1)}, q^{(3)}) = \sum_{n=1}^{(N)} \phi^{(\pm i\epsilon)}(p^{(1)}, \cdots, p^{(4)}|q^{(1)}, q^{(3)}), \quad (4.4) $$

$$ \phi^{(\pm i\epsilon)}(p^{(1)}, \cdots, p^{(6)}|q^{(1)}, q^{(3)}) = \sum_{n=1}^{(N)} \phi^{(\pm i\epsilon)}(p^{(1)}, \cdots, p^{(6)}|q^{(1)}, q^{(3)}), \quad (4.5) $$

It follows that

$$ \phi^{(\pm i\epsilon)}(p^{(1)}, p^{(3)}|q^{(1)}, q^{(3)}) = - \frac{F(p^{(1)}, p^{(3)}|q^{(1)}, q^{(3)})}{p_0^{(1)} + p_0^{(3)} - q_0^{(1)} - q_0^{(3)} \mp i\epsilon}, \quad (4.6) $$

$$ \delta(p^{(1)} + \cdots + p^{(4)} - q^{(1)} - q^{(3)}) \phi^{(\pm i\epsilon)}(p^{(1)}, \cdots, p^{(4)}|q^{(1)}, q^{(3)}) $$

$$ = - \sum_p \sum_q \frac{p_0^{(1)}}{p_0^{(1)} + p_0^{(3)} - q_0^{(1)} - q_0^{(3)} \mp i\epsilon} \frac{F(p^{(1)}, p^{(3)}, p^{(4)}|q^{(1)}, q^{(3)})}{p_0^{(1)} + p_0^{(3)} + p_0^{(4)} - q_0^{(1)} - q_0^{(3)} \mp i\epsilon}, \quad (4.7) $$
\[
\phi^{(\pm \epsilon)}(p^{(1)}, p^{(2)}|q^{(1)}, q^{(2)}) = \frac{1}{p_0^{(1)} + p_0^{(2)} - q_0^{(1)} - q_0^{(2)} + \epsilon} \left\{ -F(p^{(1)}, p^{(2)}|q^{(1)}, q^{(2)}) + \frac{1}{2} \int \frac{dkdk'}{k_0k'_0} \frac{F(p^{(1)}, p^{(2)}|k, k')}{k_0 + k'_0 - q_0^{(1)} - q_0^{(2)} + \epsilon} \delta(k + k' - q^{(1)} - q^{(2)}) + \frac{1}{2} \sum_{p} \sum_{q} \int \frac{dkdk'}{k_0k'_0} \frac{F(p^{(3)}|k, k', q^{(2)})}{k_0 + k'_0 + p_0^{(1)} - q_0^{(3)} + \epsilon} \times F(k, k', p^{(1)}|q^{(1)}) \right\}, \tag{4.8}
\]

\[
\delta(p^{(1)} + \cdots + p^{(4)} - q^{(1)} - q^{(2)}) \phi^{(\pm \epsilon)}(p^{(1)}, \cdots, p^{(4)}|q^{(1)}, q^{(2)}) = -\sum_{p} \sum_{q} p_0^{(3)} \delta(p^{(3)} - q^{(3)}) \frac{F(p^{(3)} + p^{(3)} + p^{(4)} - q^{(2)})}{p_0^{(3)} + p_0^{(3)} + p_0^{(4)} - q_0^{(5)} + \epsilon} + \frac{1}{2} \int \frac{dkdk'}{k_0k'_0} \frac{F(p^{(3)}|p^{(1)}, p^{(2)}|k, k')}{k_0 + k'_0 + p_0^{(1)} - q_0^{(3)} + \epsilon} \times F(p^{(3)} + k + k' - q^{(2)}) \times F(p^{(2)}|k, k', q^{(2)}) + \frac{1}{2} \sum_{p} \sum_{q} \int \frac{dk}{k_0} \frac{F(p^{(3)}|p^{(3)}, p^{(4)}|k)}{p_0^{(3)} + k_0 - q_0^{(3)} - q_0^{(5)} + \epsilon} F(p^{(1)}, k|q^{(1)}, q^{(2)}) + \frac{1}{2} \sum_{p} \sum_{q} \int \frac{dk}{k_0} \frac{F(p^{(3)}|p^{(3)}, p^{(4)}|q^{(1)}, k)}{p_0^{(3)} + p_0^{(3)} + k_0 - q_0^{(3)} + \epsilon} F(p^{(1)}, p^{(3)}, k|q^{(2)}) + \frac{1}{2} \sum_{p} \sum_{q} \int \frac{dk}{k_0} \frac{F(p^{(3)}|p^{(3)}, p^{(4)}|q^{(1)}, q^{(2)})}{p_0^{(3)} + p_0^{(3)} + k_0 - q_0^{(3)} + \epsilon} F(p^{(1)}, p^{(3)}, k|q^{(2)}) \right\}, \tag{4.9}
\]

and so forth. By calculations of inner products of these state vectors we get

\[
\lim_{\epsilon \to 0} \frac{\langle p^{(1)}, p^{(2)}|q^{(1)}, q^{(2)} \rangle^{(\pm \epsilon)}}{\epsilon} = \sqrt{p_0^{(1)} p_0^{(2)}} \delta(p^{(1)}, p^{(2)}; q^{(1)}, q^{(2)}) \sqrt{q_0^{(1)} q_0^{(2)}} Z^{-1}(q^{(1)}, q^{(2)}) \times \left\{ 1 + \sum_{q} \frac{1}{3} \int \frac{dkdk'dk''}{k_0 k'_0 k''_0} \delta(k + k' + k'' - q^{(3)}) \times F(q^{(3)}|k, k', k'') F(k, k', k''|q^{(1)}) + O(p') \right\}. \tag{4.10}
\]

Hence, by choosing the normalization factor as
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\[ Z^{-1/3}(q^{(3)}, q^{(0)}) \]

\[ = 1 - \frac{1}{Z} \sum_{q} \frac{1}{q_0^{(1)}} \frac{1}{3!} \int \cdots \int \frac{dkdk'dk''}{k_0k'_0k''_0} \frac{\delta(k+k'+k''-q^{(0)})}{(k_0+k'_0+k''_0-q_0^{(0)})^3} \]

\[ \times F(q^{(1)}|k, k', k'') F(k, k', k''|q^{(0)}) \]

\[ + O(g^3) \]

\[ = Z^{-1/3}(q^{(3)}) Z^{-1/3}(q^{(0)}) , \quad (4.11) \]

we can normalize the state vectors as follows:

\[ \lim_{\epsilon \to 0} \langle p^{(1)}, p^{(2)}|q^{(1)}, q^{(2)}\rangle^{(\pm \epsilon)} = \sqrt{p_0^{(1)} p_0^{(2)}} \delta(p^{(1)}, p^{(2)}; q^{(1)}, q^{(2)}) \sqrt{q_0^{(1)} q_0^{(2)}} . \quad (4.12) \]

The four-particle scattering states \( |p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}\rangle^{(\pm \epsilon)} \) can also be calculated in the same fashion. We get

\[ \frac{1}{\sqrt{4!}} Z^{-1/3}(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}) \]

\[ \times \left\{ A^\dagger(p^{(1)}) A^\dagger(p^{(2)}) A^\dagger(p^{(3)}) A^\dagger(p^{(4)}) \right. \]

\[ + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \cdots \int \frac{dq^{(1)}\cdots dq^{(2n)}}{q_0^{(1)}\cdots q_0^{(2n)}} \delta(q^{(1)} + \cdots + q^{(2n)} - p^{(1)} - p^{(2)} - p^{(3)} - p^{(4)}) \]

\[ \times \phi^{(\pm \epsilon)}(q^{(1)}, \ldots, q^{(2n)}|p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)} A^\dagger(q^{(1)}) \cdots A^\dagger(q^{(2n)}) \right\} 0^\rangle , \quad (4.13) \]

where the normalization factor is given by

\[ Z^{-1/3}(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}) = Z^{-1/3}(p^{(1)}) Z^{-1/3}(p^{(2)}) Z^{-1/3}(p^{(3)}) Z^{-1/3}(p^{(4)}) , \quad (4.14) \]

and the scattered amplitudes can be expressed in power series with respect to the coupling constant:

\[ \phi^{(\pm \epsilon)}(q^{(1)}, q^{(2)}|p^{(1)}, \ldots, p^{(4)}) = \sum_{N=1}^{\infty} \frac{(N)}{N!} \phi^{(\pm \epsilon)}(q^{(1)}, q^{(2)}|p^{(1)}, \ldots, p^{(4)}) , \quad (4.15) \]

\[ \phi^{(\pm \epsilon)}(q^{(1)}, \ldots, q^{(6)}|p^{(1)}, \ldots, p^{(4)}) = \sum_{N=1}^{\infty} \frac{(N)}{N!} \phi^{(\pm \epsilon)}(q^{(1)}, \ldots, q^{(6)}|p^{(1)}, \ldots, p^{(4)}) , \quad (4.16) \]

\[ \phi^{(\pm \epsilon)}(q^{(1)}, \ldots, q^{(6)}|p^{(1)}, \ldots, p^{(4)}) = \sum_{N=1}^{\infty} \frac{(N)}{N!} \phi^{(\pm \epsilon)}(q^{(1)}, \ldots, q^{(6)}|p^{(1)}, \ldots, p^{(4)}) , \quad (4.17) \]

\[ \phi^{(\pm \epsilon)}(q^{(1)}, \ldots, q^{(6)}|p^{(1)}, \ldots, p^{(4)}) = \sum_{N=1}^{\infty} \frac{(N)}{N!} \phi^{(\pm \epsilon)}(q^{(1)}, \ldots, q^{(6)}|p^{(1)}, \ldots, p^{(4)}) , \quad (4.18) \]

\[ \ldots \]
with
\[
\delta(q^{(1)} + q^{(2)} - p^{(1)} - p^{(2)} - p^{(3)} - p^{(4)}) \varphi^{(z \pm i \epsilon)}(q^{(3)}, q^{(4)}|p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)})
\]
\[
= - \sum_q \sum_p q_0^{(3)} \delta(q^{(1)} - p^{(1)}) \delta(q^{(2)} - p^{(2)} - p^{(3)} - p^{(4)}) \frac{(1) \hat{F}(q^{(3)}, q^{(4)}|p^{(0)}, p^{(0)}, p^{(0)})}{q_0^{(3)} - p_0^{(3)} - p_0^{(3)} - p_0^{(3)} + i \epsilon},
\]

(4.19)

\[
\delta(q^{(3)} + \cdots + q^{(6)} - p^{(1)} - \cdots - p^{(4)}) \varphi^{(z \pm i \epsilon)}(q^{(3)}, q^{(4)}|p^{(1)}, \cdots, p^{(4)})
\]
\[
= - 2! \sum_q \sum_p q_0^{(3)} q_0^{(4)} \delta(q^{(3)}, q^{(4)}; p^{(3)}, p^{(4)})
\]
\[
\times \delta(q^{(1)} + q^{(2)} - p^{(1)} - p^{(2)} - p^{(3)} - p^{(4)}) \frac{(1) \hat{F}(q^{(1)}, q^{(2)}|p^{(0)}, p^{(0)}, p^{(0)})}{q_0^{(2)} + q_0^{(3)} + q_0^{(4)} - p_0^{(4)} + i \epsilon},
\]

(4.20)

\[
\delta(q^{(1)} + q^{(2)} - p^{(1)} - \cdots - p^{(6)}) \varphi^{(z \pm i \epsilon)}(q^{(3)}, q^{(4)}|p^{(1)}, \cdots, p^{(4)})
\]
\[
= - 3! \sum_q \sum_p q_0^{(3)} q_0^{(4)} q_0^{(5)} \delta(q^{(3)}, q^{(4)}, q^{(5)}; p^{(1)}, p^{(3)}, p^{(5)})
\]
\[
\times \delta(q^{(1)} + q^{(2)} + q^{(3)} - p^{(1)} - p^{(3)} - p^{(4)}) \frac{(1) \hat{F}(q^{(3)}, q^{(4)}, q^{(5)}|p^{(0)}, p^{(0)}, p^{(0)})}{q_0^{(3)} + q_0^{(4)} + q_0^{(5)} - p_0^{(4)} + i \epsilon},
\]

(4.21)

\[
\delta(q^{(1)} + q^{(2)} - p^{(1)} - \cdots - p^{(4)}) \varphi^{(z \pm i \epsilon)}(q^{(3)}, q^{(4)}|p^{(1)}, \cdots, p^{(4)})
\]
\[
= - \sum_q \sum_p q_0^{(3)} \delta(q^{(1)} - p^{(1)}) \delta(q^{(2)} - p^{(2)} - p^{(3)} - p^{(4)}) \frac{(1) \hat{F}(q^{(3)}, q^{(4)}|p^{(0)}, p^{(0)}, p^{(0)})}{q_0^{(2)} + q_0^{(3)} + q_0^{(4)} - p_0^{(4)} + i \epsilon}
\]
\[
+ \sum_q \sum_p q_0^{(3)} \delta(q^{(1)} - p^{(1)}) \frac{\delta(q^{(2)} - p^{(2)} - p^{(3)} - p^{(4)})}{q_0^{(2)} - p_0^{(2)} - p_0^{(4)} + i \epsilon}
\]
\[
\times \frac{1}{2} \left\{ \int \frac{dk dk' |F(q^{(3)}|k, k', p^{(3)})}{k_0 k'_0} \delta(k + k' - p^{(3)} - p^{(4)}) \frac{1}{k_0 + k'_0 - p_0^{(3)} - p_0^{(4)} + i \epsilon} \right\}
\]
\[
\times \frac{(1) \hat{F}(k, k'|p^{(0)}, p^{(0)})}{k_0 + k'_0 - p_0^{(3)} - p_0^{(4)} + i \epsilon}
\]
\[
+ \frac{\delta(q^{(1)} + q^{(2)} - p^{(1)} - \cdots - p^{(4)})}{q_0^{(1)} + q_0^{(2)} + p_0^{(1)} - \cdots - p_0^{(4)} + i \epsilon} \left\{ - (2) \hat{F}(q^{(3)}, q^{(4)}|p^{(0)}, \cdots, p^{(0)}) \right\}
\]
\[
+ \sum_p \int \frac{dk |(1) \hat{F}(q^{(3)}, q^{(4)}|k, p^{(1)})}{k_0 - p_0^{(2)} - p_0^{(3)} - p_0^{(4)} + i \epsilon} \delta(k - p^{(2)} - p^{(3)} - p^{(4)})
\]
\[
\times \frac{(1) \hat{F}(k, p^{(3)}, p^{(4)})}{k_0 + q_0^{(3)} + q_0^{(4)} - p_0^{(4)} + i \epsilon}
\]
\[
+ \sum_q \sum_p \int \frac{dk |(1) \hat{F}(q^{(3)}|k, p^{(1)}, p^{(3)})}{k_0 + q_0^{(3)} + q_0^{(4)} - p_0^{(4)} + i \epsilon} \delta(k + q^{(3)} - p^{(3)} - p^{(4)})
\]
\[
\times \frac{(1) \hat{F}(k, q^{(3)}|p^{(0)}, p^{(0)})}{k_0 + q_0^{(3)} + q_0^{(4)} - p_0^{(4)} + i \epsilon}
\]
\[
\right\},
\]

(4.22)

\[
\delta(q^{(1)} + \cdots + q^{(5)} - p^{(1)} - \cdots - p^{(4)}) \varphi^{(z \pm i \epsilon)}(q^{(3)}, q^{(4)}|p^{(1)}, \cdots, p^{(4)})
\]
\[
= - 2! \sum_q \sum_p q_0^{(3)} q_0^{(4)} \delta(q^{(3)}, q^{(4)}; p^{(3)}, p^{(4)}) \delta(q^{(1)} + q^{(2)} - p^{(1)} - p^{(2)})
\]
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\[ \times \frac{\delta(q^{(q)} + p^{(q)})}{q^{(q)} + p^{(q)} - p^{(q)} - p^{(q)} + \iota \epsilon} \]

\[ - \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(q^{(q)} + q^{(q)} + q^{(p)} - p^{(p)} - p^{(p)})}{q^{(q)} + q^{(q)} + q^{(p)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(q^{(q)} + q^{(q)} - p^{(p)} - p^{(p)})}{q^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]

\[ + \sum_{q} \sum_{p} q^{(q)} \delta(q^{(q)} - p^{(q)}) \delta(q^{(q)} + q^{(p)} + p^{(p)} - p^{(q)} - p^{(q)} - p^{(q)}) \]

\[ \times \frac{\delta(k + q^{(q)} - p^{(p)} - p^{(p)})}{k^{(q)} + q^{(q)} - p^{(p)} - p^{(p)} + \iota \epsilon} \]
\[ + \sum_{q} \sum_{p} q^{(i)} \delta (q^{(i)} - p^{(i)}) \frac{\delta (q^{(3)} + q^{(2)} - p^{(3)} - p^{(2)})}{\frac{q^{(3)} + q^{(2)} - p^{(3)} - p^{(2)}}{q^{(3)} + q^{(2)} - p^{(3)} - p^{(2)}} + i\epsilon} \times \int \frac{dk^{(1)}}{k^{2}} F^{(1)}(q^{(3)}|k, p^{(1)}, p^{(2)}) \frac{\delta (k + q^{(1)} + q^{(2)} - p^{(2)})}{k^{2} + q^{(1)} + q^{(2)} - p^{(2)} + i\epsilon} F^{(1)}(k, q^{(1)}, q^{(2)}|p^{(3)}), \]

and so forth.

It is not difficult to verify that the state \(|p^{(1)}, p^{(2)}, p^{(3)}, p^{(6)}\rangle\) is orthogonal to the state \(|q^{(1)}, q^{(2)}\rangle\).

Now the S-matrix elements can be obtained by taking the limits \(\epsilon \rightarrow 0^{+}\) and \(\epsilon' \rightarrow 0^{+}\) for the inner products \(<-i\omega l<|S|q^{(1)}, q^{(2)}>\) of these scattering states. In the approximation up to second order with respect to the coupling constant \(g\), we obtain for the elastic two-particle scattering and the two-particle production processes the following results:

\[ \langle p^{(1)}, p^{(2)}|S|q^{(1)}, q^{(2)}\rangle = \lim_{\epsilon \rightarrow 0^{+}, \epsilon' \rightarrow 0^{+}} <p^{(1)}, p^{(2)}|q^{(1)}, q^{(2)}\rangle \]

\[ = \sqrt{\rho^{(1)} \rho^{(2)}} \delta (p^{(1)} + p^{(2)} - q^{(1)} - q^{(2)}) \sqrt{q^{(1)} q^{(2)}} \]

\[ - \frac{\pi i}{4} \left\{ \frac{1}{2} \int \int dk dk' \frac{1}{k k'} \frac{1}{F^{(1)}(p^{(1)}, p^{(2)}|k, k') \delta (k^{2} + k^{2}' - q^{(1)} - q^{(2)})}{F^{(1)}(p^{(1)}, p^{(2)}|q^{(1)}, q^{(2)})} \right\} + O(g^{2}). \]

\[ \langle p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}|S|q^{(1)}, q^{(2)}\rangle = \lim_{\epsilon \rightarrow 0^{+}, \epsilon' \rightarrow 0^{+}} <p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}|q^{(1)}, q^{(2)}\rangle \]

\[ = - \frac{2\pi^{2}}{\sqrt{4!} \sqrt{2!} \delta (p^{(1)} + p^{(2)} + p^{(3)} + p^{(4)} - q^{(1)} - q^{(2)})} \times \sum_{p} \int \frac{dk^{(1)}}{k^{2}} F^{(1)}(p^{(3)}, p^{(4)}|k) \delta (p^{(1)} + k^{2} - q^{(1)} - q^{(2)}) F^{(1)}(p^{(3)}, k|q^{(1)}, q^{(2)}) \]

\[ + O(g^{2}). \]

§ 5. Concluding remarks

At a glance of the result (4·24), one may wonder why terms such as \(F(p^{(2)}|k, k', q^{(1)}) F(p^{(1)}, k, k'|q^{(2)})\), namely, the products of the particle-number-changing primary interaction vertex functions do not contribute to the \(\langle p^{(1)}, p^{(2)}|S|q^{(1)}, q^{(2)}\rangle\)
in second order with respect to the coupling constant. In order to understand this situation, it is better to examine in detail the cancellation mechanism of the terms in the course of calculations. From Eqs. (4·2) ~ (4·5) we first get

\[ \langle p^{(1)}, p^{(2)}|q^{(1)}, q^{(2)}\rangle^{(+\epsilon, -\epsilon)} \]

\[ = Z^{-1/2}(p^{(1)}, p^{(2)})Z^{-1/2}(q^{(1)}, q^{(2)}) \]

\[ \times \left[ 2! \sqrt{\rho^{(1)}_0 \rho^{(2)}_0} \delta(p^{(1)}, p^{(2)}; q^{(1)}, q^{(2)}) \sqrt{q^{(1)}_0 q^{(2)}_0} \right. \]

\[ + \frac{1}{2} \left( \phi^{(1)} + \phi^{(2)} \right) (q^{(1)}, q^{(2)}|p^{(1)}, p^{(2)}) \]

\[ + \frac{1}{2} \left( \phi^{(1)} + \phi^{(2)} \right) (q^{(1)}, q^{(2)}|p^{(1)}, p^{(2)}) \]

\[ + \frac{1}{2} \left( \phi^{(1)} + \phi^{(2)} \right) (q^{(1)}, q^{(2)}|p^{(1)}, p^{(2)}) \]

\[ + \frac{1}{2} \left( \phi^{(1)} + \phi^{(2)} \right) (q^{(1)}, q^{(2)}|p^{(1)}, p^{(2)}) \]

\[ + \left. \frac{1}{2} \right] \int \frac{dk dk' dk''}{k_0 k'_0 k''_0} \delta(k + k' - p^{(1)} - p^{(2)}) \phi^{(1)}(k, k'|p^{(1)}, p^{(2)}) \]

\[ \times \delta(k + k' - q^{(1)} - q^{(2)}) \phi^{(1)}(k, k'|q^{(1)}, q^{(2)}) \]

\[ + \frac{1}{2} \left( \phi^{(1)} + \phi^{(2)} \right) (k^{(1)}, \cdots, k^{(6)}|p^{(1)}, p^{(2)}) \]

\[ \times \delta(k + \cdots - q^{(1)} - q^{(2)}) \phi^{(1)}(k^{(1)}, \cdots, k^{(6)}|q^{(1)}, q^{(2)}) \]

\[ + O(g^6) \right]. \]

(5·1)

After insertion of Eqs. (4·6) ~ (4·8) into Eq. (5·1) and the limiting procedures for \( \epsilon \) and \( \epsilon' \), we obtain first, as the second order contributions to the \( \langle p^{(1)}, p^{(2)}|S|q^{(1)}, q^{(2)}\rangle \), the following five terms:

\[ \langle p^{(1)}, p^{(2)}|S|q^{(1)}, q^{(2)}\rangle \]

\[ = (I) + (II) + (III) + (IV) + (V), \]

where

\[ (I) = -p^{(1)}_0 p^{(2)}_0 \delta(p^{(1)}, p^{(2)}; q^{(1)}, q^{(2)}) \]

\[ \times \frac{1}{q^{(1)}_0 q^{(2)}_0} \int \frac{dk dk' dk''}{k_0 k'_0 k''_0} \tilde{F}(p^{(1)}|k, k', k'') \delta(k + k' + k'' - q^{(1)}_0) \]

\[ \times \tilde{F}(k, k', k''|p^{(1)}), \]
The term (I) comes from the normalization factor $Z^{-1/2} \left( p^{(1)}_0, p^{(2)}_0 \right)$ in Eq. (5·1). All of the terms (II), (III) and (IV) come from the $\varphi^*$- and $\varphi^-$ terms, the $\varphi^* \left( k, k', p^{(1)}, p^{(2)} \right) \varphi \left( k, k' \right) q^{(1)}, q^{(2)}$ term and a part of the $\varphi^* \left( k^{(1)}, \ldots, k^{(4)} \right) p^{(1)}, p^{(2)} q^{(1)}, q^{(2)}$ term. The last term (V) comes from the remaining part of the $\varphi^* \left( k^{(1)}, \ldots, k^{(4)} \right) p^{(1)}, p^{(2)} q^{(1)}, q^{(2)}$ term in Eq. (5·1).

Now the terms (I) and (V) cancel out with each other. Furthermoe, by virtue of Eq. (2·14), the term (II) cancels out with the principal value integrals in the terms (III) and (IV). Also the $\delta \left( p^{(2)}_0 + k_0 + k_\mu' - q^{(3)}_0 \right)$ in the (IV) vanishes because the vector $p^{(2)}_0 + k_0 + k_\mu' - q^{(3)}_0$ cannot become zero for our $k_0 = \sqrt{m^2 + k^2}$ and $k_\mu' = \sqrt{m^2 + (k')^2}$. Thus the only contribution to $\langle p^{(1)}, p^{(2)} \left| S \right| q^{(1)}, q^{(2)} \rangle$ is the one from the $\delta \left( k_0 + k^\mu_\mu' - q^{(3)}_0 - q^{(3)}_0 \right)$ part in the term (III). That is, we get

$$
\langle p^{(1)}, p^{(2)} \left| S \right| q^{(1)}, q^{(2)} \rangle = (\pi i)^2 \delta \left( p^{(1)}_0 + p^{(2)}_0 - q^{(1)}_0 - q^{(3)}_0 \right)
\times \frac{1}{2!} \int \frac{dk dk' \delta \left( p^{(1)}_0 + p^{(2)}_0 - q^{(1)}_0 - q^{(3)}_0 \right)}{k_0 k_0' k_0''} \int \frac{d^3k}{\left( k_0 + k_\mu' - q^{(3)}_0 \right)^3} \delta \left( k_0 + k_\mu' - q^{(3)}_0 \right) F \left( k, k', q^{(1)}, q^{(2)} \right).
$$

For the $S$-matrix elements $\langle p^{(1)}, p^{(2)} \left| S \right| q^{(1)}, q^{(2)} \rangle$ also we can analyse the cancellation mechanism in the same way.
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Originally, as can be seen from the way of construction of the $P_s$ and $M_s$ in § 2, the higher order interaction Hamiltonians $-i\Pi^{(N)}_s(N \geq 2)$ have been introduced into the model just for the reason of relativistic invariance; they are the least necessities in order that the $P_s$ which contains the primary interaction Hamiltonian $-i\Pi^{(1)}_s$ satisfies the fundamental commutator equations (1.1). Therefore the higher order vertex functions $F^{(N)}(N \geq 2)$ consist of only principal value integrals whose integrands are essentially products of the primary vertex functions $F^{(1)}$ divided by energy denominators. From the relativistic invariance of the theory, however, it follows automatically that the $S$-matrix elements in the model can be expressed only in terms of the invariants such as the primary vertex functions $F^{(1)}$ and those $\delta$-functions expressing energy-momentum conservation which come from the scattered wave boundary condition, and the invariant integrations $\int dk/k_0$. This is the essence of the cancellation mechanism due to the structure of the formalism.

We have thus obtained in this paper a model of convergent relativistic quantum mechanics of interacting particles in which the particle number can change. It is of interest to apply the formalism adopted here to the system of interacting photons and electrons and thereby to see whether the convergent relativistic quantum electrodynamics which agrees with the empirical facts can be obtained. In order to describe real emission and absorption processes of a photon by a charged particle in interaction with other charged particles and also to describe real creation or annihilation processes of an electron-positron pair, it will be necessary to assume, as the primary interaction Hamiltonian, terms such as $\gamma\epsilon\epsilon\gamma, \epsilon\epsilon\gamma, \gamma\epsilon\epsilon, \epsilon\epsilon\gamma, \epsilon\epsilon\gamma$ and $\gamma\epsilon\epsilon$. Here by the symbols $\gamma, \epsilon$ and $\gamma$ we mean the annihilation operators of a photon, electron and positron respectively. The terms such as $\gamma\epsilon\epsilon\gamma$ and $\epsilon\epsilon\gamma$, however, should not be contained in the primary interaction Hamiltonian for the reason described in § 2; they should be excluded from the beginning. Furthermore, based on considerations of the cancellation mechanism illustrated above, it will be necessary to assume additional terms like $\epsilon\epsilon\epsilon\epsilon, \epsilon\epsilon\epsilon\epsilon$ and $\epsilon\epsilon\epsilon\epsilon$ in the second order Hamiltonian.

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References

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