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A Bivariate Extreme Value Distribution Applied to Flood Frequency Analysis

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This article presents a procedure for use of the Gumbel logistic model to represent the joint distribution of two correlated extreme events. Parameters of the distribution are estimated using the method of moments. On the basis of marginal distributions, the joint distribution, the conditional distributions, and the associated return periods can be deduced. The applicability of the model is demonstrated by using multiple episodic flood events of the Harricana River basin in the province of Quebec, Canada. It is concluded that the model is useful for describing joint probabilistic behavior of multivariate flood events.

Introduction

Floods are intrinsically multivariate random events that are characterized by their peak, volume, and duration, which might be mutually correlated. The severity of a flood is determined not only by its peak flow, but also by other aspects such as its volume and duration. However, flood frequency analysis has mainly focused on a single variable such as flood peak or/and volume. Comprehensive reviews of single-variable flood frequency analysis have been undertaken by Cunnane (1987) and Bobée and Rasmussen (1994). Flood peak or volume frequency analysis provides a limited assessment of flood events, while the solution of many hydrological problems requires the knowledge of complete information concerning the flood event (flood peak, flood volume, and flood duration). Meaningful attempts to address this topic include the works by Correia (1987), Sackl and Bergmann (1987), Krstanovic and Singh (1987), Loganathan *et al.* (1987), Choulakian *et al.* (1990), Kelly and Krzysztofowicz (1997), Goel *et al.* (1998), Yue (1999, 2000), Yue *et al.* (1999). Most of these studies used the bivariate normal distribution to describe the joint distribution of two correlated random variables. A prerequisite of use of a bivariate normal distribution is that two correlated random variables are normally distributed or can be normalized by transformation techniques such as the Box-Cox transformation (Box and Cox 1964). Jain and Singh (1986) have demonstrated that transformation approaches can not always transform sample data to follow the normal distribution.

In practice, extreme events such as flood peak and flood volume may be represented by the Gumbel distribution (Gumbel 1958; Todorovic 1978; Castillo 1988; Watt *et al.* 1989). Thus, it may be advantageous to directly use a bivariate extreme value distribution to analyze the joint behavior of two correlated Gumbel distributed random variables. Several bivariate extreme value distributions have been studied by Gumbel (1960a, 1960b, 1961), Gumbel and Mustafi (1967), Oliveria (1975, 1982), Pickands (1981), Buishand (1984), Raynal-Villasenor and Salas (1987), Tawn (1988), Joe *et al.* (1992), Coles and Tawn (1991, 1994), and others. However, these models have mainly remained their theoretical developments and seldom succeeded in resolving practical problems in the field of hydrological frequency analysis. The recent work by Yue *et al.* (1999) explored the usefulness of a bivariate extremal distribution, the Gumbel mixed model (Gumbel 1960a) for describing multivariate flood events. But the correlation coefficient between two random variables must be in the range: $0 \leq \rho \geq 2/3$. In practice, a large number of hydrological extreme events may be closely correlated and the correlation between them may be greater than 2/3. In such cases, the Gumbel mixed model is no longer valid.

This study makes an attempt to use another bivariate extreme value distribution with Gumbel marginals, the Gumbel logistic model with the correlation coefficient $0 \leq \rho < 1$ (Gumbel 1960b, 1961), to represent multiple episodic flood events. It presents a procedure for the use of this model to describe the joint probability distribution of flood peak and volume and the joint probability distribution of flood volume and duration. Based on the marginal distributions of these random variables, one can readily derive the joint distributions, the conditional distributions, and the associated return periods. The usefulness of the bivariate extreme value distribution is illustrated by modeling joint statistical properties of annual maximum flood events of the Harricana basin in the province of Quebec, Canada. For most basins larger than 500 km^2 in the province of Quebec, snowmelt usually motivates the annual maximum flood both in flood peak and volume (Watt *et al.* 1989). Therefore, we analyze the joint probability distributions of flood peak and volume as well as flood volume and duration using the annual maximum series. In general, there exist closely correlated relationships between flood peak and volume and between flood volume and duration, while there is no significant correlation between flood peak and duration. It is especially true in the study region. Thus, the bivariate extreme distribution can be utilized to analyze the different two-way combinations of the flood event: the joint distribution of flood peak and volume and the joint distribution of flood volume and duration.

The Gumbel Logistic Model

The Gumbel logistic model with standard Gumbel marginal distributions was originally proposed by Gumbel (1960b, 1961) as follows

$$F(x,y) = \exp\left\{-\left[(-\ln F_X(x))^m + (-\ln F_Y(y))^m\right]^{1/m}\right\} \quad (m \geq 1) \tag{1}$$

where $F_X(x)$ and $F_Y(y)$ are the marginal distributions of random variables X and Y, respectively, and are expressed as

$$F_X(x) = \exp\left[-\exp(-x)\right] \tag{2a}$$

$$F_Y(y) = \exp\left[-\exp(-y)\right] \tag{2b}$$

and where m ($m \geq 1$) is the parameter describing the association between the two random variables X and Y. The estimator of m is given by (Gumbel and Mustafi 1967; Johnson and Kotz 1972)

$$m = \frac{1}{\sqrt{(1-\rho)}} \quad (0 \leq \rho < 1) \tag{3}$$

where ρ is the product-moment correlation coefficient given by

$$\rho = \frac{E\left[(X-M_X)(Y-M_Y)\right]}{S_X S_Y} \tag{4}$$

in which M_X and M_Y, and S_X and S_Y are the sample mean and sample standard deviation of X and Y, respectively. When $m = 1$, the product-moment correlation coefficient ρ is equal to zero. This represents the independent case and the bivariate distribution splits into the product of the two marginal distributions

$$F(x,y) = F_X(x)F_Y(y) \tag{5}$$

By setting the probability density function (pdf) and cumulative distribution function (cdf) of the marginal distributions of X and Y to take the following forms

$$f_x(x) = \frac{1}{\alpha_x} \exp\left[-\frac{x-u_x}{\alpha_x} - \exp\left(-\frac{x-u_x}{\alpha_x}\right)\right] \tag{6a}$$

$$F_X(x) = \exp\left[-\exp\left(-\frac{x-u_x}{\alpha_x}\right)\right] \tag{6b}$$

$$f_y(y) = \frac{1}{\alpha_y}\exp\left[-\frac{y-u_y}{\alpha_y} - \exp\left(-\frac{y-u_y}{\alpha_y}\right)\right] \tag{6c}$$

$$F_Y(y) = \exp\left[-\exp\left(-\frac{y-u_y}{\alpha_y}\right)\right] \tag{6d}$$

the joint pdf can be derived using Eq. (1) and is expressed as follows

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{F(x,y)}{\alpha_x \alpha_y}\left(e^{-\frac{m(x-u_x)}{\alpha_x}} + e^{-\frac{m(y-u_y)}{\alpha_y}}\right)^{\frac{1-2m}{m}} \times$$
$$\left[\left(e^{-\frac{m(x-u_x)}{\alpha_x}} + e^{-\frac{m(y-u_y)}{\alpha_y}}\right)^{\frac{1}{m}} + m-1\right] e^{-m\left(\frac{x-u_x}{\alpha_x}+\frac{y-u_y}{\alpha_y}\right)} \tag{7}$$

The joint cdfs of X and Y takes the same form as Eq. (1). Thus, the conditional pdf of X given $Y = y$ can be derived as follows

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{F(x,y)}{\alpha_x}\left(e^{-\frac{m(x-u_x)}{\alpha_x}} + e^{-\frac{m(y-u_y)}{\alpha_y}}\right)^{\frac{1-2m}{m}} \times$$
$$\left[\left(e^{-\frac{m(x-u_x)}{\alpha_x}} + e^{-\frac{m(y-u_y)}{\alpha_y}}\right)^{\frac{1}{m}} + m-1\right] e^{-m\left(\frac{x-u_x}{\alpha_x}+\frac{y-u_y}{\alpha_y}\right)} e^{\left(\frac{y-u_y}{\alpha_y}+\exp\left(-\frac{y-u_y}{\alpha_y}\right)\right)} \tag{8a}$$

The conditional cdf of X given $Y = y$ can be given by

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(u|y)\,du = \frac{\int_{-\infty}^{x} f(u,y)\,du}{f_Y(y)} = \frac{\partial F(x,y)/\partial y}{f_Y(y)}$$

$$= F(x,y)\left(e^{-\frac{m(x-u_x)}{\alpha_x}} + e^{-\frac{m(y-u_y)}{\alpha_y}}\right)^{\frac{1-m}{m}} e^{\frac{(1-m)(y-u_y)}{\alpha_y}+\exp\left(-\frac{y-u_y}{\alpha_y}\right)} \tag{8b}$$

Similarly, the conditional pdf $f_{Y|X}$ and $(y \mid x)$ the cdf $F_{Y|X}(y \mid x)$ of Y given $X = x$ can be expressed by equivalent formulae.

In practice, the conditional distribution of X given $Y \leq y$ may be of interest to hydrological engineers. Thus we also provide this non-standard conditional distribution. The conditional cdf $F'_{X|Y}(x \mid y) = \Pr[X \leq x \mid Y \leq]$ of X given $Y \leq y$ as follows

$$F'_{X|Y}(x|y) = \frac{F(x,y)}{F_Y(y)} = \exp\left\{ e^{-\frac{y-u_y}{\alpha_y}} - \left[e^{-\frac{m(x-u_x)}{\alpha_x}} + e^{-\frac{m(y-u_y)}{\alpha_y}} \right]^{\frac{1}{m}} \right\} \tag{9}$$

Similarly, the conditional cdf $F'_{Y|X}(y \mid x)$ of Y given $X \leq x$ can be expressed by an equivalent formula.

The return period associated with single event $X > x$ is represented as follows

$$T_X = \frac{1}{1-F_X(x)} \quad (F_X(x) = \Pr\{X \leq x\}) \tag{10a}$$

and the return period associated with single event $Y > y$ is given by an equivalent equation. One the basis of the same principle, the joint return period $T(x,y)$ of X and Y associated with the event that either x or y or both is exceeded ($X > x$, $Y > y$, or $X > x$ and $Y > y$) can be represented by

$$T(x,y) = \frac{1}{1-F(x,y)} \quad (F(x,y) = \Pr\{X \leq x, Y \leq y\}) \tag{10b}$$

Similarly, the joint return period $T'(x,y)$ of X and Y associated with the event that both x and y is exceeded ($X > x$ and $Y > y$) is

$$T'(x,y) = \frac{1}{1-F_X(x)-F_Y(Y)+F(x,y)} \tag{10c}$$

The conditional return period of X given $Y = y$ associated with the event $X > x \mid Y = y$ is

$$T(x|y) = \frac{1}{1-F_{X|Y}(x|y)} \quad (F_{X|Y}(x|y) = \Pr\{X \leq x | Y=y\}) \tag{10d}$$

and the conditional return period of Y given $X = x$ associated with the event $Y > y \mid X = x$ can be presented by an equivalent equation. The conditional return period of X given $Y \leq y$ associated with the event $X > x \mid Y \leq y$ is

$$T'(x|y) = \frac{1}{1-F'_{X|Y}(x|y)} \quad (F'_{X|Y}(x|y) = \Pr\{X \leq x | Y \leq y\}) \tag{10e}$$

Similarly, the conditional return period Y given $X \leq x$ associated with the event $Y > y \mid X \leq x$ is given by an equivalent formula.

Application

Description of the Basin of Study

The Harricana basin has an area of 3,680 km^2, located in the province of Quebec, Canada. Winter lasts about four months and precipitation is in the form of snowfall that is mainly stored in the basin. Spring represents the high flow season due to the contribution of spring snowmelt to river runoff. Generally, the combination of snowmelt and rainstorms generates the greatest annual floods both in flood peak and volume. Daily streamflow data from 1933 to 1995 are available at the gauging station 04NA001 at latitude 48:36:02 N and longitude 78:06:34 W, near the outlet of the basin.

Characteristics of Flood Events

The most significant characteristics of a flood event are the flood peak (Q), flood volume (V), and flood duration (D), as illustrated in Fig. 1. The determination of flood duration involves the identification of the times of start and end of flood runoff. Generally, time boundaries of a flood are marked by a rise in stage and discharge from base flow (start of flood runoff) and a return to base flow (end of flood runoff). For the basin of study, start of surface runoff is usually marked by the abrupt rise of the hydrograph. The end of flood runoff can be identified by the flattening of the recession limb of the hydrograph. As the characteristics of surface runoff recession are very different from those of base flow, there exists a significant change in the slope of the hydrograph as the transition occurs from surface runoff to base flow. Based on these criteria, the start date (SD_i)and end date (ED_i) of flood runoff for the i-th year can be determined, and then the flood duration series $\{D_i = ED_i - SD_i, i=1,2, \dots 63\}$ can be constructed. The flood volume series can be constructed using the following formula

$$V_i = \sum_{j=SD_i}^{ED_i} q_{ij} - \frac{1}{2}(q_{is}+q_{ie}) \quad (i=1,2,\dots,63) \tag{11}$$

where q_{ij} is the j-th day observed daily streamflow value for the i-th year; q_{is} and q_{ie} are observed daily streamflow values on the start date and end date of flood runoff for the i-th year, respectively.

The flood peak series are constructed by

$$Q_i = \max\{q_{ij}, j=SD_i, SD_i+1, \dots, ED_i\} \quad (i=1,2 \dots 63) \tag{12}$$

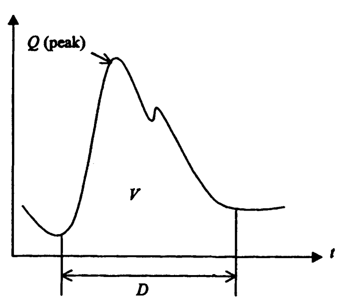

Fig. 1. Characteristic values of a flood event.

Marginal Distributions of Flood Peak, Volume and Duration

The non-exceedance probability is estimated using the Gringorten position-plotting formula (Gringorten 1963; Cunnane 1978; and Guo 1990)

$$P_k = \frac{k-0.44}{N-0.12} \tag{13}$$

where P_k is the cumulative frequency, the probability that a given value is less than the k-th smallest observation in the data set of N observations.

To test the goodness of fit of the Gumbel distribution, the Kolmogorov-Smirnov test (Kanji 1993) is executed. The Kolmogorov-Smirnov test statistics $D_{\max}$ are 0.1146 for the flood peak, 0.0607 for the flood volume, and 0.1194 for the flood duration. The critical Kolmogorov-Smirnov value $D_N(\alpha)$ with the sample size $N = 63$ and the significance level $\alpha = 0.05$ is $D_{63}(0.05) = 0.1713$. Therefore, it can be concluded that all these three characteristics of flood events can be represented by the Gumbel distribution. Fig. 2(a)-(c) illustrates the fit of the Gumbel distribution to the flood peak, volume, and duration, on the Gumbel paper. The parameters of the Gumbel distribution are estimated by the method of moments (MM) and are given by

$$\alpha = \frac{\sqrt{6}}{\pi} S \tag{14a}$$

$$u = M - 0.577\alpha \tag{14b}$$

where M and S are the mean and standard deviation of the sample. The estimated mean and standard deviation of the flood peak (Q), volume (V), and duration (D), from the sample data are listed in Table 1. The estimated parameters of the Gumbel distribution are also presented in Table 1.

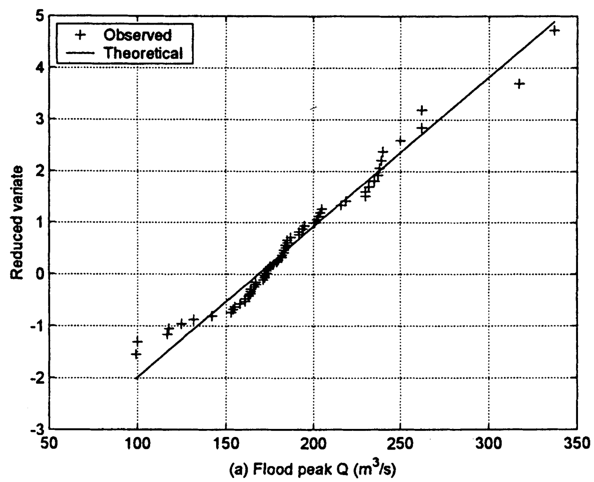

Fig. 2(a). Distribution of flood peaks.

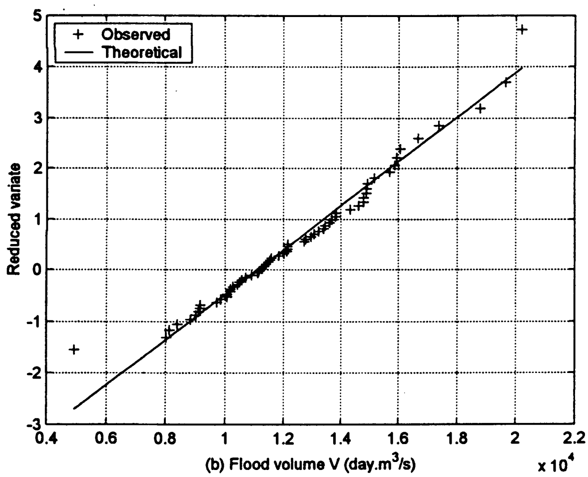

Fig. 2(b). Distribution of flood volumes.

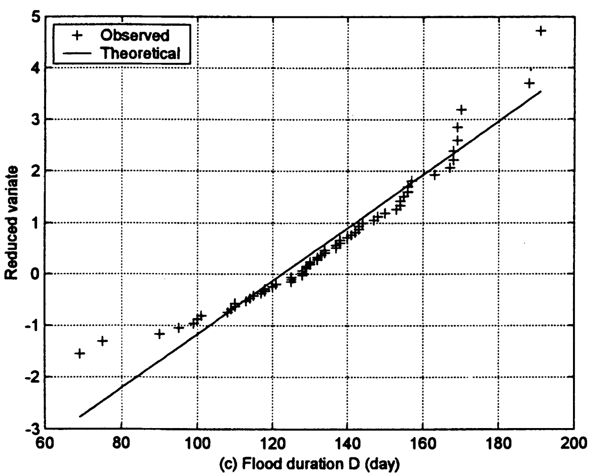

Fig. 2(c). Distribution of flood durations.

Table 1 – Statistics of the flood peak (Q), volume (V) and duration (D), and parameters of the Gumbel distribution

	Statistics		Parameters of Gumbel model	
	M	S	u	α
Q (m^3/s)	188.30	44.16	168.43	34.43
V (day.m^3/s)	12426	2933	11106	2287
D (days)	133.81	24.83	122.63	19.36

Associations between Flood Peak, Volume, and Duration

The product-moment correlation coefficients (ρ) between the flood peak and volume, between the flood duration and volume, and between the flood peak and duration are estimated using Eq. (4), and are 0.742, 0.615, and 0.095, respectively. There is no significant correlation between the flood peak and duration. Thus we can employ the bivariate extreme distribution to model the joint behavior of the flood event, *i.e.*, to analyze the different two-way combinations of the flood event: the joint distribution of the flood peak and volume and the joint distribution of the flood volume and duration. The association parameters m between the flood peak and volume and between the flood volume and duration are estimated using Eq. (3), and are 1.969 and 1.612, respectively.

Statistics of the Joint Distribution of the Flood Peak (Q) and Volume (V)

Validity of the Proposed Model – Empirical joint probabilities are computed using the approach proposed by Yue *et al.* (1999). A two-dimensional table is first constructed in which the variables Q and V are arranged in ascending order. The element in row i and column j of the table is defined as the joint frequency function of the two random variables and is estimated by

$$f(q_i, v_j) = \Pr(Q=q_i, V=v_j) = \frac{n_{ij}}{N} \tag{15}$$

where N is the total number of observations (N=63), and n_{ij} is the number of occurrences of the combinations of q_i and v_j.

The joint cumulative frequency (non-exceedance joint probability) takes an

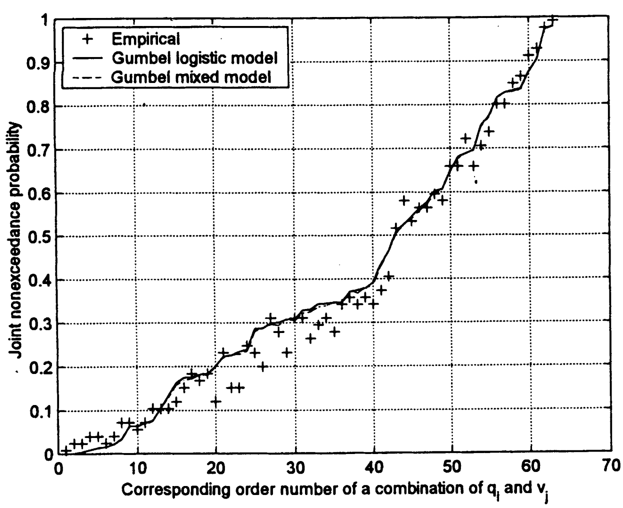

Fig. 3. Comparison of empirical and theoretical probabilities of flood peaks and volumes.

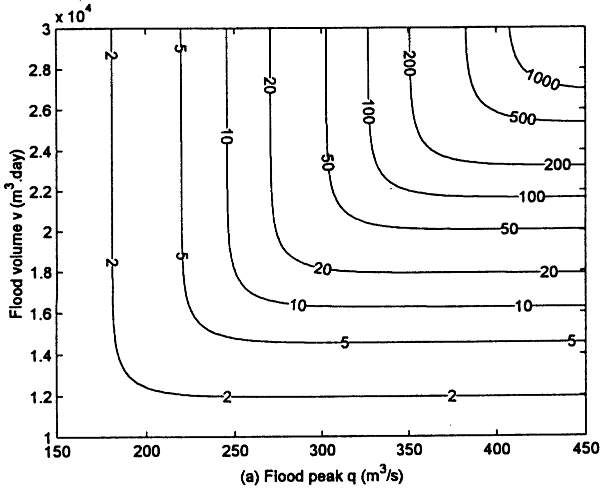

Fig. 4(a). Contours of the joint return period $T(q,v)$ associated with the event that either x or y or both is exceeded ($X>x$, $Y>y$, or $X>x$ and $Y>y$).

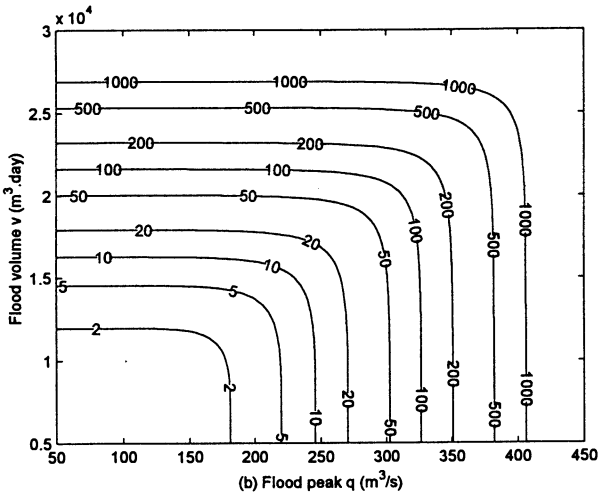

Fig. 4(b). Contours of the joint return period $T'(q,v)$ associated with the event that both x and y is exceeded ($X>x$ and $Y>y$).

equivalent form of Eq. (13) in order to keep consistent with the marginal case, and is given as

$$F(q,v) = \Pr(Q \le q_i, V \le v_j) = \frac{\sum_{m=1}^{i} \sum_{l=1}^{j} n_{ml} - 0.44}{N+0.12} \tag{16}$$

Theoretical joint probabilities of the real occurrence combinations of q_i and v_j are estimated using Eq. (1). The empirical and theoretical joint probabilities are plotted in Fig. 3, in which the solid-line represents the theoretical joint probabilities of flood peaks and volumes, which are arranged in ascending order, and the corresponding empirical joint probabilities are expressed by the plus sign. The x-axis is the corresponding order number of a combination of q_i and v_j. It can been seen that the theoretical probabilities fit the empirical ones well. It is therefore concluded that the model is suitable for representing the joint distribution of flood peak and volume.

Joint Return Periods of Q and V – The joint return periods $T(q,v)$ and $T'(q,v)$ are often of interest to hydrological engineers. Only these two joint return periods are presented. The values of $T(q,v)$ and $T'(q,v)$ corresponding to given $Q = 50$ (5) 450 and $V = 5000$ (200) 30000 are computed using Eqs.(10b) and (10c), respectively. The corresponding contour lines of the flood peaks and volumes are automatically obtained by MATLAB program and are plotted in Figs. 4(a) and 4(b), respectively. Based on the contours of the joint return periods, given a return period, one can obtain various combinations of flood peaks and volumes, and *vice versa*. These various scenarios can be useful for analysis and assessment of the risk associated with several hydrological problems, such as spillway design and flood control, in which both flood peak and volume are required. This information can not be derived by single-variable frequency analysis.

Conditional Return Periods – The conditional return period $T(q|v)$ of flood peak Q given flood volume V is estimated based on Eqs. (8b) and (10d), and is plotted in Fig. 5(a). The conditional return period $T'(q|v)$ is calculated using Eqs. (9) and (10e), and is shown in Fig. 5(b). From these diagrams, we can obtain the return periods of flood peaks under the condition that flood volumes are given, which can not be obtained from single-variable frequency analysis.

For comparison, the return period of the marginal distribution of the flood peak is displayed by the dashed line in Figs. 5(a) and 5(b). It is evident that for two positively correlated random variables, the two conditional distributions of a variable are dramatically different from its marginal distribution. However, in the case when two random variables are independent, the two conditional distributions of a variable are the same as its marginal distribution. Similarly, the conditional return periods $T(v|q)$ and $T'(v|q)$ of flood volume given flood peak can be derived, which are omitted here owing to space limitations.

Statistics of the Joint Distribution of the Flood Volume (*V*) and Duration (*D*)

Similar to the procedure as in the flood peak and volume, the empirical and theoretical joint probabilities for the flood durations and volumes are illustrated in Fig. 6. Again, the result indicates that no significant differences can be detected between the empirical and theoretical probabilities. Therefore, the model is concluded to be

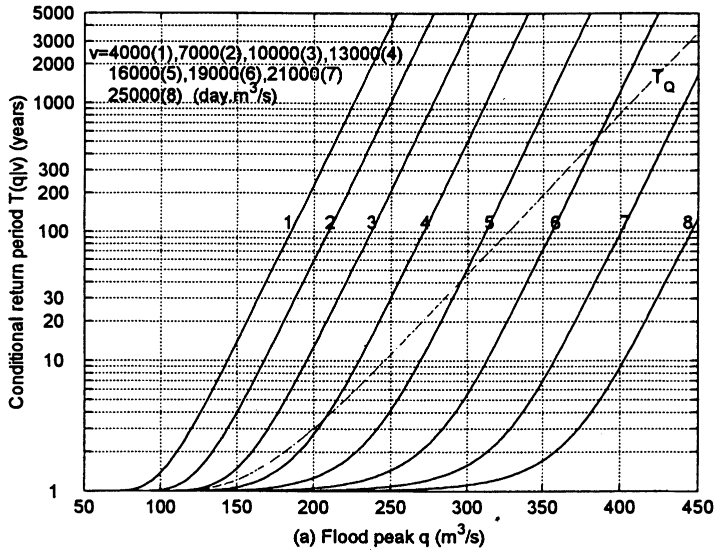

Fig. 5(a). Conditional return period $T(q|v)$ associated with the event $(Q > q|V = v)$.

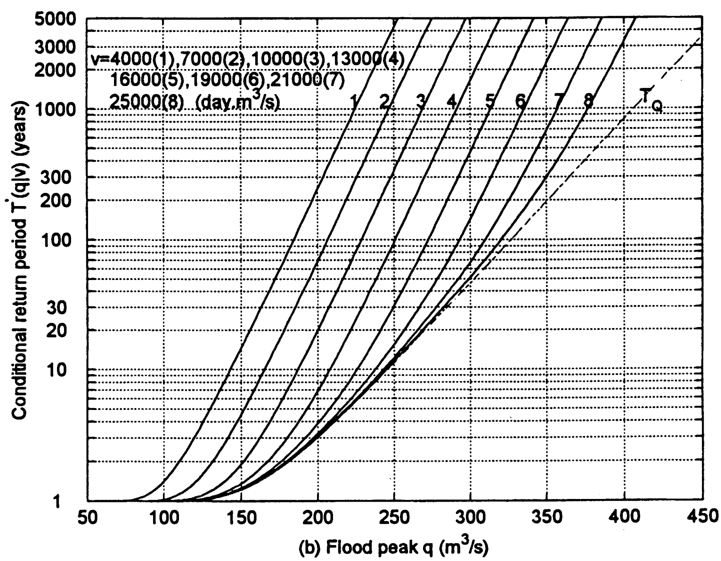

Fig. 5(b). Conditional return period $T'(q|v)$ associated with the event $(Q > q|V \leq v)$.

useful for representing the joint distribution of the flood duration and volume. The joint return periods and the conditional return periods can be derived using the same manner as in the previous section, omitted here for the sake of saving paper pages. Indeed, asides from flood peak, the duration and the volume of a given flood repre-

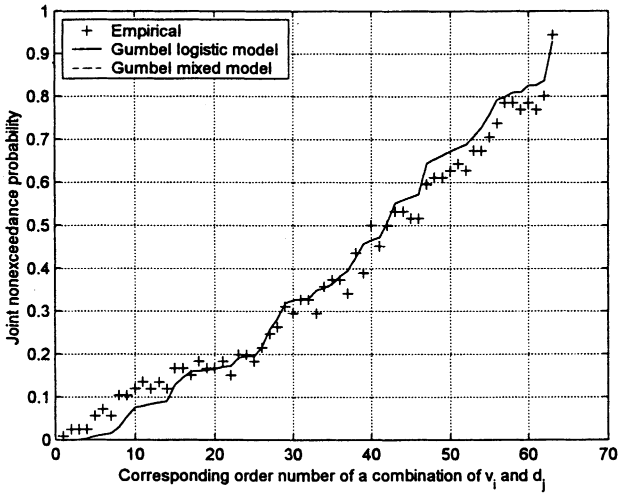

Fig. 6. Comparison of empirical and theoretical probabilities of flood durations and volumes.

sent two main flood characteristics upon which flood damages depend. Results produced by the proposed model may be useful for calibrating flood damage functions that are used by insurance companies and local agencies as estimation tools in pre-flood or post-flood studies.

A Comparison of the Gumbel Logistic Model and the Gumbel Mixed Model

The previous work of Yue *et al.* (1999) has used the Gumbel mixed model for analyzing the joint probabilistic behavior of the flood events of the Ashuapmushuan river basin, where both correlation coefficients between flood peak and volume, and between flood volume and duration are smaller than 2/3. For comparison, the theoretical joint probabilities of the real occurrence combinations of the flood peaks and volumes as well as the flood volumes and durations of the Harricana basin are also computed using the Gumbel mixed model without concerning the limitation on the correlation parameter. These theoretical probabilities are displayed by the dashed lines in Figs. 3 and 6, respectively. It is obvious that the joint probabilities of the flood volumes and durations computed by the Gumbel mixed model are the same as those by the Gumbel logistic model (see Fig. 6), in which the correlation coefficient is 0.615 and the association parameter is 0.930 for the Gumbel mixed model.

Fig 3. indicates that the joint probabilities of the flood peaks and volumes by the Gumbel mixed model are smaller than those by the Gumbel logistic model, in which the correlation coefficient is 0.742 and the association parameter (θ) of the Gumbel mixed model is 1.102. This is because the correlation coefficient (0.742) or the association parameter (1.102) exceeded the upper limitation of the Gumbel mixed model. As the association parameter (θ =1.102) is not greatly different from its upper limitation $\theta = 1$, the differences between the joint probabilities by the Gumbel mixed model and by the Gumbel logistic model are not large. A detailed comparison of these two models is beyond the scope of this paper.

Conclusions

This study provides a procedure for the use of the Gumbel logistic model to analyze the joint distributions of two positively correlated Gumbel distributed random variables. The parameters of the model are proposed to be estimated using the method of moments from the marginal distributions of random variables. The model was employed to represent the joint statistical properties of the flood events of two different basins: the Harricana river basin and the Ashuapmushuan river basin. The Ashuapmushuan basin, also located in the province of Quebec, Canada, has an area of 15,300km^2 and 33-year daily streamflow data from 1963 to 1995 are available at the gauging station 061901 at latitude 48.686389°N and longitude 72.488056°W. Although the agreement between the empirical and theoretical joint probabilities of the Ashuapmushuan basin is not as good as that of the Harricana basin, there are also no significant differences between empirical and theoretical joint probabilities of flood peaks and volumes, as well as flood volumes and durations. This study only presents the computation results of the Harricana river basin due to space limitations.

The analyzing results from these two basins demonstrate that the model is useful for representing the joint distributions of flood peak and volume, as well as flood volume and duration if their marginals can be represented by the Gumbel distributions. The proposed method provides additional information such as joint return periods, conditional return periods of two positively correlated flood characteristics, which cannot be obtained by single-variable flood frequency analysis. These results can contribute meaningfully in the analysis and assessment of the risk associated with several hydrological problems, such as spillway design and flood control.

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