$$F^0L^0T^0 = \left(\frac{1}{T}\right)^1 \cdot L^{\alpha_3} \cdot \left(\frac{F}{L^2}\right)^{\beta_3} \cdot \left(\frac{1}{T}\right)^{\gamma_3} \tag{35}$$

Equating the exponents of like dimensions in equation (35) results in $\beta_3 = 0$, $\alpha_3 = 0$, $\gamma_3 = -1$. Hence

$$\pi_3 = \left(\frac{d\epsilon}{dT}\right) \Big/ \frac{dN}{dT} \tag{36}$$

Equation (27) can now be written as

$$\pi_1 = f(\pi_2, \pi_3)$$

or

$$\frac{da}{dN} \cdot \frac{1}{a} = f\left[\frac{\hat{\sigma}_G}{c}, \frac{\frac{d\epsilon}{dT}}{\frac{dN}{dT}}\right] \tag{37}$$

With Specimen of Finite Width. It is assumed that

$$\frac{da}{dN} = f\left[\hat{\sigma}_G, a, W, c, \frac{d\epsilon}{dT}, \frac{dN}{dT}\right] \tag{38}$$

Equation (38) is the same as equation (27) except that finite width W is involved. To perform the dimensional analysis, one has the values given in Table 5.

In this case there will be four π-equations which are expressed as follows:

$$\pi_1 = \left(\frac{da}{dN}\right)^1 \cdot W^{\alpha_1} \cdot c^{\beta_1} \cdot \left(\frac{dN}{dT}\right)^{\gamma_1} \tag{39}$$

$$\pi_2 = (\hat{\sigma}_G)^1 \cdot W^{\alpha_2} \cdot c^{\beta_2} \cdot \left(\frac{dN}{dT}\right)^{\gamma_2} \tag{40}$$

$$\pi_3 = (a)^1 \cdot W^{\alpha_3} \cdot c^{\beta_3} \cdot \left(\frac{dN}{dT}\right)^{\gamma_3} \tag{41}$$

$$\pi_4 = \left(\frac{d\epsilon}{dT}\right)^1 \cdot W^{\alpha_4} \cdot c^{\beta_4} \cdot \left(\frac{dN}{dT}\right)^{\gamma_4} \tag{42}$$

It has been chosen that in equations (39) to (42) four linear independent sets of exponents are

$$\left(\frac{da}{dN}\right)^1 \cdot \hat{\sigma}_G{}^0 \cdot a^0 \cdot \left(\frac{d\epsilon}{dT}\right)^0,$$

$$\left(\frac{da}{dN}\right)^0 \cdot \hat{\sigma}_G{}^1 \cdot a^0 \cdot \left(\frac{d\epsilon}{dT}\right)^0,$$

$$\left(\frac{da}{dN}\right)^0 \cdot \hat{\sigma}_G{}^0 \cdot a^1 \cdot \left(\frac{d\epsilon}{dT}\right)^0,$$

and

$$\left(\frac{da}{dN}\right)^0 \cdot \hat{\sigma}_G{}^0 \cdot a^0 \cdot \left(\frac{d\epsilon}{dT}\right)^1.$$

One now has four sets of algebraic equations for four sets of unknown exponents:

$$\alpha_1, \beta_1, \gamma_1, \quad \alpha_2, \beta_2, \gamma_2, \quad \alpha_3, \beta_3, \gamma_3, \quad \text{and} \quad \alpha_4, \beta_4, \gamma_4.$$

The dimensional equation for π_1 is

$$F^0L^0T^0 = L^1 \cdot L^{\alpha_1} \cdot \left(\frac{F}{L^2}\right)^{\beta_1} \cdot \left(\frac{1}{T}\right)^{\gamma_1} \tag{43}$$

Equating like dimensions in equation (43) yields $\beta_1 = 0$, $\alpha_1 = -1$, $\gamma_1 = 0$. Hence

$$\pi_1 = \left(\frac{da}{dN}\right)\frac{1}{W} \tag{44}$$

The dimensional equation for π_2 is

Table 5

Symbol	Units	Relationship
$\frac{da}{dN}$	L	π_1
$\hat{\sigma}_G$	F/L^2	π_2
a	L	π_3
W	L	Dimensionally independent
c	F/L^2	Dimensionally independent
$\frac{d\epsilon}{dT}$	$1/T$	π_4
$\frac{dN}{dT}$	$1/T$	Dimensionless independent

$$F^0L^0T^0 = \left(\frac{F}{L^2}\right)^1 (L)^{\alpha_2} \cdot \left(\frac{F}{L^2}\right)^{\beta_2} \left(\frac{1}{T}\right)^{\gamma_2} \tag{45}$$

Equating like dimensions in equation (45) results in $\beta_2 = -1$, $\alpha_2 = 0$, $\gamma_2 = 0$. Hence

$$\pi_2 = \hat{\sigma}_G \cdot \frac{1}{c} \tag{46}$$

The dimensional equation of π_3 is

$$F^0L^0T^0 = (L)^1 \cdot L^{\alpha_3} \cdot \left(\frac{F}{L^2}\right)^{\beta_3} \cdot \left(\frac{1}{T}\right)^{\gamma_3} \tag{47}$$

Equating like dimensions in equation (47) gives $\beta_3 = 0$, $\alpha_3 = -1$, $\gamma_3 = 0$. Hence

$$\pi_3 = \frac{a}{W} \tag{48}$$

The dimensional equation for π_4 is

$$F^0L^0T^0 = \left(\frac{1}{T}\right)^1 \cdot (L)^{\alpha_4} \cdot \left(\frac{F}{L^2}\right)^{\beta_4} \cdot \left(\frac{1}{T}\right)^{\gamma_4} \tag{49}$$

Equating like dimensions in equation (49) yields $\beta_4 = 0$, $\alpha_4 = 0$, $\gamma_4 = -1$. Hence

$$\pi_4 = \frac{d\epsilon}{dT} \Big/ \frac{dN}{dT} \tag{50}$$

Therefore

$$\pi_1 = f(\pi_2, \pi_3, \pi_4)$$

or one has

$$\frac{da}{dN} \cdot \frac{1}{W} = f\left[\frac{\hat{\sigma}_G}{c}, \frac{a}{W}, \frac{d\epsilon/dT}{dN/dT}\right] \tag{51}$$

DISCUSSION

James R. Rice[2]

The author's paper constitutes a useful addition to the available experimental data on crack propagation and provides further support of Paris' [16, 17][3] suggestion that fatigue crack growth rates may be correlated in terms of the variation of the elastic stress-intensity factor for the crack tip. There are, however, some questions as to the validity of the author's theoretical work on the dimensional analysis of crack growth.

In the first dimensional analysis given in the Appendix for a

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[3] Numbers in brackets designate Additional References at end of this discussion.

crack in an infinite sheet, Yang introduces as the relevant variables:

Variable	Dimension	Meaning
$\frac{da}{dN}$	L	Crack growth per load cycle
a	L	Current crack length
σ_G	F/L^2	Maximum applied stress (in this case identical to stress range)
C	F/L^2	Material constant
$\frac{d\epsilon}{dT}$	$\frac{1}{T}$	Strain rate at crack tip
$\frac{dN}{dT}$	$\frac{1}{T}$	Frequency of loading

An immediate objection is that the strain rate $\frac{d\epsilon}{dT}$ is not a primary variable, but rather a variable that is derived from a proper solution of the problem. If one wishes to deal with time-dependent material effects, a material constant such as some characteristic relaxation time should be introduced; or if one wishes to treat dynamic effects, the mass density of the material is appropriate. Yang's treatment amounts essentially to ignoring rate effects anyway, so let us drop the last two variables and procede to a proper dimensional analysis. The only dimensionless combinations are $\frac{1}{a}\frac{da}{dN}$ and $\frac{\sigma_G}{C}$, implying a propagation law

$$\frac{da}{dN} = af\left(\frac{\sigma_G}{C}\right) \tag{52}$$

as Yang obtains. This is quite in disagreement with both Yang's and Paris' experimental results.

Yang's reasoning in the introduction of the elastic stress-intensity factor is not particularly clear; however, the factor may be introduced into the dimensional analysis in a rational way. In accord with Paris' data and with one's intuition, it is reasonable to assume that so long as significant plastic deformation occurs only in a small region near the crack tip, the influence of external loadings and specimen geometry is sensed solely through the elastic stress-intensity factor K. In this case, the variables which Yang introduces as significant are modified, for rate insensitive materials, to

Variable	Dimension	Meaning
$\frac{da}{dN}$	L	Crack growth per load cycle
K	$F/L^{3/2}$	Maximum stress-intensity factor of applied loads (in this case identical to stress-intensity factor range)
C	F/L^2	Material constant

There is now only one dimensionless combination, $\frac{C^2}{K^2}\frac{da}{dN}$, and the crack growth rate law is

$$\frac{da}{dN} = (\text{const})\frac{K^2}{C^2} \tag{53}$$

This is the law introduced by Liu [18] from a similar dimensional analysis and again is in disagreement with available experimental data.

The reason for the discrepancy between these theoretical laws and available data is rather simple. As is well known, a dimensional analysis leads to proper results only if all the relevant variables are taken into account. On the other hand, the dimensional analyses of Yang and of Liu [18] assume that the only material constants entering into the problem have dimensions of stress (for example, C in Yang's paper, which may stand for the elastic modulus or the yield stress). Suppose we add to the foregoing lists of relevant variables a variable S:

Variable	Dimension	Meaning
S	F/L	Material constant

There are good physical reasons to expect the appearance of a material constant with such dimensions. Since fatigue crack growth involves the separation of surfaces, one might expect the surface energy (which has dimensions F/L) to be important. Also, a material constant entering into the Griffith-Irwin theory of fracture is the critical stress-intensity factor, and this reasonably may be expected relevant to fatigue crack growth. When squared and divided by a material constant with dimensions of stress, the resulting combination has dimensions F/L.

Let us now add S to the foregoing lists of relevant variables. Equation (52) is modified since now the only independent dimensionless combinations are $\frac{C}{S}\frac{da}{dN}, \frac{\sigma_G}{C}, \frac{\sigma_G a}{S}$, resulting in

$$\frac{da}{dN} = \frac{S}{C} f\left(\frac{\sigma_G}{C}, \frac{\sigma_G a}{s}\right) \tag{54}$$

Similarly, in the case where it is appropriate to assume the influence of loads and geometry are sensed solely through the stress-intensity factor, the independent dimensionless combinations are $\frac{C}{S}\frac{da}{dN}, \frac{K^2}{CS}$, leading to

$$\frac{da}{dN} = \frac{S}{C} f\left(\frac{K^2}{CS}\right) \tag{55}$$

Equation (55) is sufficiently general to account for Paris' observation that fatigue crack growth rate is a function of the variation in stress intensity factor, or to account for particular power laws such as the fourth power law which Paris found to be most suitable over the entire range of available data or higher power laws which Yang finds suitable for the limited range of data he obtains near the critical stress-intensity factor level of K.

Additional References

16 P. C. Paris and F. Erdogan, "A Critical Analysis of Crack Propagation Laws," JOURNAL OF BASIC ENGINEERING, TRANS. ASME, Series D, vol. 85, 1963, p. 528.

17 P. C. Paris, "The Fracture Mechanics Approach to Fatigue," in *Fatigue–An Interdisciplinary Approach*, eds. Burke, Reed, and Weiss, Syracuse University Press, 1964, pp. 107–127.

18 H. W. Liu, "Fatigue Crack Propagation and Applied Stress Range—An Energy Approach," JOURNAL OF BASIC ENGINEERING, TRANS. ASME, Series D, vol. 85, 1963, p. 116.

Author's Closure

As a general rule, in dimensional analysis [2] it is very important to list all variables that may influence the quantity of interest. If there is doubt about including a particular quantity, it is better to list it than to omit it. It is far less serious to include one too many variables than to omit one. An entirely erroneous result will be obtained from a dimensional analysis that does not include all important variables, whereas an extra variable would probably be dropped out in the analysis process. The worst thing that could happen due to the inclusion of one too many variables would be that an expression or equation would result which would be correct, though more complicated than necessary. Therefore, the author preferred to include $d\epsilon/dt$ in the variables of interest at the beginning, even though it might not be considered a prime variable.

Equation (53) by Mr. Rice already has been included by the author in equation (11), which is the same as equation (52). This was mentioned in the text on p. 488. In equation (52), if one lets $f(\sigma_G/c) = (\text{constant})\,(\sigma_G/c)^2$, equation (52) becomes equation (53) directly.

In the author's analysis, equation (20) is the last step of dimensional analysis. To get a definitive function of f_1 in equation (20), the author intended to plot da/dN versus $\hat{\sigma}_G$ (when a/w is kept constant) and to plot da/dN versus a/w (when $\hat{\sigma}_G$ is kept constant). The former would yield the power index of (σ_G/c) and the latter, the index of (a/w). This would seemingly lead to a power law equation. However, specific test data were not available. Therefore, Professor Paris' stress-intensity factor approach was finally followed.

It is a nice thought of Mr. Rice that he derived a power law equation, equation (52), directly from dimensional analysis by introducing a stress-intensity factor, K. But then, if one presumes that K is the key factor in the problem, there really won't be much of a problem. This is not the author's original intention in working on this subject. Besides, the author has certain reservations about Mr. Rice's introduction of the surface energy and the ratio of critical stress-intensity factor, K_c, to a certain stress, as material constants. Whereas the former is not commonly used quantitatively in mechanics, the latter, K_c, is not a material constant at all, according to the test results from the author's laboratory and recent private communications with knowledgable people in the field.

Therefore, although the $f(K^2/cs)$ term in equation (55) could account for Professor Paris and the author's experimental power laws generally, the validity of the equation is questionable in practice.

The author would like to thank Mr. Rice for his interest in the problem.