same manner as used to convert functions of a single variable to functions of two. Obviously the amount of equipment required and the complexity of the problem increase when an additional input variable is introduced.

Discussion

E. S. Sherrard.[2] Other methods than those described in the paper may be used for generating a function of two variables with electronic components. Methods for handling quite arbitrary functions of two variables have been described by Jerrard and Jacobi,[3] and by Philbrick.[4] The isocline method of Meissinger[5] is useful for generating a particular class of functions of two variables. Since the equipment used by the author is simple, cheap, and much more frequently available than is the equipment used in Meissinger's method, the writer would appreciate his comment on the following:

1 What functions of two variables may be generated by both his method and Meissinger's method?

2 What functions of two variables, if any, may be easily generated by his method but are difficult or impossible to generate by Meissinger's method?

3 What functions of two variables, if any, are difficult or impossible to generate by his method, but are easily generated by Meissinger's method?

The author's method is capable of generating exactly or approximately the family of curves $z(x, y_i)$, according to whether Equations [33] and [34] or Equations [35] and [36] are exactly or approximately satisfied

$$z(x, y_i) = f_1(w) + f_3(\delta y_i) \text{ (see Fig. 5)} \quad [33]$$

$$w = x + f_2(\delta y_i), \quad \delta y_i = y_i - y_0 \quad [34]$$

$$z(x, y_i) = f_1(w) + x f_3(\delta y_i) \text{ (see Fig. 11)} \quad [35]$$

$$w = x + f_2(\delta y_i), \ \delta y_i = y_i - y_0. \quad [36]$$

where the "reference function" $f_1(w)$ is taken equal to $z(x, y_0)$, y_0 being the reference value of y, the parameter of the family. In the foregoing equations $f_2(\delta y_i)$ is the horizontal shift function and $f_3(\delta y_i)$ is the vertical shift function illustrated in Fig. 7 of the paper. The author's method of determining these shift functions is a cut-and-try procedure which requires the drawing of an overlay of $f_i(w)$. Such a method may be the best approach if Equations [33] and [34] or [35] and [36] are satisfied approximately rather than exactly. If these equations are not satisfied approximately, the cut-and-try procedure may be both tedious and ineffective.

A simple test for the satisfaction of Equations [33] and [34] may be made by using the first differences with respect to x of $z(x, y_i)$ and the first differences of $f_1(w)$ with respect to w. If Equation [33] is satisfied, then differentiation of Equation [33] yields

$$\frac{\partial z(x, y_i)}{\partial x} = \frac{df_1(w)}{dw} \quad [37]$$

Equation [37] will be satisfied for corresponding values of w and x, (w_1, x_1), (w_2, x_2), (w_3, x_3), etc. The values of x_1, x_2, x_3, etc., corresponding to chosen values of w_1, w_2, w_3, etc., may be found by determining from the difference tables of $z(x, y_i)$ the values of x at which the first differences of $z(x, y_i)$ are equal to the first difference of $f_1(w)$. Then, if Equation [34] is satisfied $f_2(\delta y_i)$ is given by

$$f_2(\delta y_i) = w_1 - x_1 = w_2 - x_2 = w_3 - x_3, \text{ etc.}$$

Also, for Equation [33]

$$f_3(\delta y_i) = z(x_1, y_i) - f_1(w_1) = z(x_2, y_i) - f_1(w_2), \text{ etc.}$$

The constancy of $f_2(\delta y_i)$ and $f_3(\delta y_i)$ as w and x assume the values of (w_1, x_1), (w_2, x_2), (w_3, x_3), etc., determines whether or not Equations [33] and [34] are good or poor approximations for the given function $z(x, y_i)$.

If $z(x, y_i)$ is satisfied by Equations [35] and [36], the second differences of $z(x, y_i)$ and $f_1(w)$ may be used to test Equations [35] and [36]. Differentiation of Equation [35] yields

$$\frac{\partial^2 z(x, y_i)}{\partial x^2} = \frac{d^2 f_1(w)}{dw^2} \quad [38]$$

From the tables of second-order differences, corresponding values (w_1, x_1), (w_2, x_2), (w_3, x_3), etc., that satisfy Equation [38] may be determined. Then, $f_2(\delta y_i)$ is again equal to

$$w_1 - x_1 = w_2 - x_2 = w_3 - x_3, \text{ etc.}$$

and $f_3(\delta y_i)$ is determined from Equation [35] with (w, x) set equal to (w_1, x_1), (w_2, x_2), (w_3, x_3), etc. As before, the constancy of $f_2(\delta y_i)$ and $f_3(\delta y_i)$ will determine whether Equations [35] and [36] are a good or a poor approximation of $z(x, y_i)$.

An accurate graphical solution of Equation [37] may be made by using the well-known "mirror method" of graphical differentiation. To do so, one may determine the slope of $f_1(w)$ at a chosen value of w, w_1. x_1 is the value of x at which the slope of $f(x, y_i)$ is equal to the slope of $f_1(w)$ at $w = w_1$. Other pairs of corresponding values (w_2, x_2), (w_3, x_3), etc., may be found by the same method. Then, as before,

$$f_2(\delta y_i) = w_1 - x_1, \ w_2 - x_2, \ w_3 - x_3, \text{ etc.}$$

and $f_3(\delta y_i)$ is found from Equation [33] for (w, x) equal to (w_1, x_1), (w_2, x_2), (w_3, x_3), etc. There does not seem to be any simple and accurate graphical method for determining corresponding values of w and x that satisfy Equation [38].

Author's Closure

The comments of Mr. Sherrard are greatly appreciated. The methods which he describes add a considerable improvement in mathematical treatment to the graphical principles which have been described.

In answer to his specific questions concerning the differences between the Meissinger method and that described here, the following comparisons may be made:

1 Both methods can generate functions which have constant spacing between isoclines.

2 Functions which contain intersecting lines of y = const may be generated easily by the method shown in the paper. This is a more difficult case for the Meissinger method than that of nonintersecting curves having a laminated pattern.

3 The method described here cannot generate functions in which the curves of y = const are greatly dissimilar in basic

[2] Analog Systems Section, Data Processing Systems Division, U. S. Department of Commerce, National Bureau of Standards, Washington, D. C.

[3] "Generation of a Function of Two Variables," by R. P. Jerrard and G. T. Jacobi, presented at the Annual Conference of the Association for Computing Machinery, Cambridge, Mass., September, 1953.

[4] "Continuous Electric Representation of Non-Linear Functions of n-Variables," by G. A. Philbrick, A Palimpsest on the Electronic Analog Art, G. A. Philbrick Researches, Inc., 1955.

[5] "An Electronic Circuit for the Generation of Functions of Several Variables," by H. F. Meissinger, Institute of Radio Engineers Spring Meeting, 1955.

shape. The Meissinger method can generate some cases of this nature using a three-channel diode network.

The basic Meissinger cases of convergent radiating isoclines develop families of curves which expand outward from the convergent point. These cannot be generated directly by the methods which have been described here. However, by substitution of multiplications for the summations shown in Fig. 5, functions of this nature can be represented.

The other methods for generating functions of two variables which Mr. Sherrard has mentioned are certainly more general and will provide more uniform accuracy than that described here. As he points out, "map readers" of this nature are also much more expensive, and involve considerably more complexity of equipment.

We have also used a map-reading technique of this nature for simulating difficult functions. The method generates a family of arbitrary curves, but requires a function generator for each curve, a multiplier to interpolate between each pair of lines, and three amplifiers per curve. Although this is effective for an occasional three-dimensional function, such a technique becomes prohibitive if many functions must be simulated in a given problem. The method described in the paper has been presented as a useful procedure where simplicity and economy of equipment are desired.