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Discussion

Exact Stationary-Response Solution for Second-Order Nonlinear Systems Under Parametric and External White-Noise Excitations[1]

M. F. Dimentberg[2]. The authors should be congratulated on their paper, which illustrates a systematic approach to the search of exact solutions for stationary probability densities of the responses of dynamic systems to combined external and parametric white-noise excitations.

The authors refer to the paper [8] by this writer, in which one such solution had first been obtained. The writer, indeed, had not shown in [8], how that solution had been attained. However, this was by no means "accidental" and, therefore, some clarifying remarks seem appropriate.

The solution for stationary amplitude probability density in a SDOF system with nonlinear damping and combined external and parametric random excitation had been obtained by this writer as far back as the 1970's by the approximate stochastic averaging method (*Soviet Appl. Mech.*, Vol. 11, No. 4, 1975, pp. 401–404). It is evident that if the exact solution does exist, it should result in such expression for stationary probability density of the amplitude, that coincides with the approximate one. Therefore, to obtain the exact solution one may express the known amplitude probability density in terms of displacement and velocity and then substitute it into Fokker-Planck equation for the original system. The latter would be satisfied exactly, only provided that the system's parameters are related in a certain special way; thus, the problem is reduced to algebraic equations for this specific subset of parameters, for which this exact solution is valid indeed. This way of looking for exact solutions, based on the use of the approximate stochastic averaging results is, indeed, of the inverse type, but seems to be rather systematic, although it differs from that adopted by the authors.

By the way, the seemingly unexpected result in Example 4 – Gaussian property of the response of nonlinearly damped system with combined external and parametric white-noise excitation in a certain special case – has a very clear physical interpretation. Whilst random parametric excitation tends to prolong the tails of response probability density, the (positive) damping nonlinearity tends to make them shorter. In the above special case these two effects cancel each other, leading to the same shape of response probability density as in linear SDOF system with purely external random excitation.

A more detailed discussion of these topics can be found in the forthcoming book by this writer entitled "Statistical Dynamics of Nonlinear and Time-Varying Systems," which will be published by the Research Studies Press in 1988.

Authors' Closure

Professor Dimentberg's discussion on our paper has provided additional insights into the behavior of nonlinear systems under combined external and parametric white-noise excitations. Determining the exact joint probability density of displacement and velocity indirectly by way of the probability density of the amplitude obtained from the approximate stochastic averaging method is indeed a very useful approach, and it may be applicable to some cases, for example, [8]. We take this opportunity to point out, however, that since restrictive relations between the system parameters and spectral densities of the excitations must be imposed to obtain exact solutions if parametric and external white-noise excitations are both present, a more important issue is how to obtain approximate solutions without such restrictions. In a recent investigation [17], we consider a system

$$\ddot{X}+\alpha\dot{X}+X=XW_1(t)+\dot{X}W_2(t)+W_3(t) \tag{1}$$

where $W_j(t)$ are correlated Gaussian white noises. Although example (1) is linear, the stationary joint probability of $X(t)$ and $\dot{X}(t)$ is not obtainable at the present time. An approximate solution for the amplitude can be obtained using the stochastic averaging method, and when converted to the joint probability density of $X(t)$ and $\dot{X}(t)$, we obtain

$$p_1(x_1,\, x_2)=C_1[4K_{33}+(K_{11}+3K_{22})(x_1^2+x_2^2)]^{-\sigma} \tag{2}$$

where

$$\sigma=\frac{2(\alpha+\pi k_{22})}{\pi(K_{11}+3K_{22})} \tag{3}$$

where K_{jj} is the spectral density of $W_j(t)$, and where x_1 and x_2 are possible values of $X(t)$ and $\dot{X}(t)$, respectively. However, using a generalized version of the present approach [18] and an equal energy dissipation criterion [17], we obtain another approximation,

$$p_2\,(x_1,x_2)=C_2\,\{1+\nu[(1-\pi K_{12})(x_1-x_{1,0})^2+x_2]\}^{-\delta} \tag{4}$$

where K_{ij} is the cross-spectral density of $W_i(t)$ and $W_j(t)$ and

$$x_{1,0}=\frac{\pi K_{23}}{1-\pi K_{12}}$$

$$\nu=\frac{K_{11}+3K_{22}(1-\pi K_{12})}{4(1-\pi K_{12})(K_{11}x_{1,0}^2+2K_{13}x_{1,0}+K_{33})} \tag{5}$$

$$\delta=\frac{2(1-\pi K_{12})(\alpha+\pi K_{22})]}{\pi[K_{11}+3K_{22}\,(1-\pi K_{12})]} \tag{6}$$

It is of interest to note that p_2 contains the cross-spectral densities of the excitations whereas p_1 does not, and the stability

[1] By Y. Yong and Y. K. Lin, published in the June, 1987, issue of the ASME Journal of Applied Mechanics, Vol. 54, pp. 414–418.

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conditions implied in examples (2) and (4) are very much different.

Whether p_1 or p_2 is a better approximation is not immediately clear. However, for the linear system (1) it is possible to derive a set of *exact* equations for the moments $E[X(t)]$, $E[\dot{X}(t)]$, $E[X^2(t)]$, $E[X(t)\dot{X}(t)]$, and $E[\dot{X}^2(t)]$. It can be shown that these equations are satisfied by the moments computed from p_2 but not from p_1 [17]. Therefore, the method of stochastic averaging, even though very versatile, should be used with care.

Finally, the "unexpected" results shown in example 4 in our paper is no longer unexpected. We have since found that different stochastic systems can indeed share an identical probability distribution. They are called equivalent stochastic systems [19]. When the associated Fokker-Planck equations are identical, the equivalent stochastic systems must share both the stationary probability distribution and the transient nonstationary probability distribution under identical initial conditions. Such systems are said to be stochastically equivalent in the strict sense. A wider class, referred to as the class of equivalent stochastic systems in the wide sense, also includes those sharing only the stationary probability distribution but having different Fokker-Planck equations.

Additional References

17 Cai, G. Q., and Lin, Y. K., 1988, "A New Approximate Solution Technique for Randomly Excited Nonlinear Oscillators" to appear in *International Journal of Nonlinear Mechanics*.

18 Lin, Y. K., Yong, Y., and Cai, G. Q., 1987, "Exact Solution for Nonlinear Systems Under Parametric and External White-Noise Excitations," *Stochastic Structural Mechanics, Lecture Notes in Engineering 31*, Edited by Y. K. Lin and G. I. Schueller, Springer-Verlag, New York, pp. 268–280.

19 Lin, Y. K., and Cai, G. Q., 1988, "Equivalent Stochastic Systems," to appear in ASME Journal of Applied Mechanics.

Torsion of Cylinders With Shape Intrinsic Orthotropy[3]

J. G. Kennedy[4] and D. R. Carter.[4] It should be clarified that the central results stated by S. C. Cowin in (1987) had been established previously by Kennedy and Carter (1985a) and extensively cited by Kennedy et al., (1985b). A class of cylindrical orthotropy termed "circumferential" orthotropy is discussed in (1985a, 1985b) which is equivalent to the "shape intrinsic" orthotropy studied in (1987). It is shown explicitly in (1985a) that "an orthotropic shaft with circumferential planes of material symmetry tangent to the torsional stress lines which would occur if the shaft was (tranversely) isotropic (the defining property of circumferential orthotropy) will experience a torsional stress field identical to that which would occur if the shaft was (transversely) isotropic." Furthermore, ". . . the strain field is identical to that which would occur in an isotropic body having a shear modulus equal to G_{23} (the orthotropic shear modulus associated with the circumferential and longitudinal directions)." Consequently, in the presence of circumferential orthotropy, these results are identical to the central conclusion of Cowin that, "for a certain class of elastic cylinders with shape intrinsic orthotropy, the solution of the torsion problem is the same as the solution to the torsion problem for the isotropic cylinder of the same shape if the isotropic shear modulus G were replaced by the orthotropic shear modulus G_{iz} ($\equiv G_{23}$ above)." Furthermore, Cowin's discussion of the "experimental evaluation of the shear moduli for shape intrinsic orthotropy" uses one of the alternative experimental procedures discussed explicitly in (1985b).

It is unfortunate that Dr. Cowin overlooked these publications in the Biomechanics literature.

References

Cowin, S. C., 1987, "Torsion of Cylinders With Shape Intrinsic Orthotropy," ASME Journal of Applied Mechanics, Vol. 54, pp. 778–782.

Kennedy, J. G., and Carter, D. R., 1985a, "Long Bone Torsion: I. Effects of Heterogeneity, Anisotropy and Geometric Irregularity," ASME *Journal of Biomechanical Engineering*, Vol. 107, pp. 183–188.

Kennedy, J. G., Carter, D. R., Caler, W. E., 1985b, "Long Bone Torsion: II. A Combined Experimental and Computational Method for Determining an Effective Shear Modulus," ASME *Journal of Biomechanical Engineering*, Vol. 107, pp. 189–191.

Author's Closure

It is unfortunate that I overlooked these publications. There are many reasons why I should have seen them and read them. I apologize to the authors for my oversight.

A verbal statement of the central argument in my paper extending the St. Venant solution for torsion of certain classes of shafts with shape intrinsic orthotropy does appear in the appendix of Kennedy and Carter (1985a). In my opinion that is the only original element in common between the two papers. I disagree with the opening statement in the above discussion by Kennedy and Carter. My paper, Cowin (1987), contains the following results not contained in Kennedy and Carter (1985a, b):

1. An explicit mathematical condition defining the class of cylinders with shape intrinsic orthotropy for which the solution to the torsion problem is the same as the solution to the torsion problem for the isotropic cylinder of the same shape if the isotropic shear modulus G were replaced by the orthotropic shear modulus G_{tz};
2. Examples of cylinders satisfying this condition;
3. A formula for the torque in a laminated shape intrinsic cylinder if each laminate has a different G_{tz} shear modulus.

The key equations of Kennedy and Carter (1985a), their equations (9) and (10), cannot be used to establish the explicit formulas given in my paper for the three results detailed above because their equations (9) and (10) are expressed in terms of arbitrary orthogonal curvilinear coordinates.

If I had seen the papers of Kennedy and Carter before I wrote my paper, Cowin (1987), I would still have written it, but it would have been slightly modified to account for their earlier argument. I think my paper is quite different from those of Kennedy and Carter (1985a, b) and I believe that readers interested in that subject would benefit from reading all three papers.

Finally, it should be noted that the principal equations used by Kennedy and Carter (1985a) in their argument, their equations (9) and (10), contain an error. The right hand side of both equations should be multiplied by the factor

$$\frac{1}{2h_1h_2}\left(\frac{\partial}{\partial q_1}(q_1h_1)+\frac{\partial}{\partial q_2}(q_2h_1)\right)$$

where h_1, h_2, q_1 and q_2 are identified by Kennedy and Carter (1985a).

[3]By S. C. Cowin, published in the December, 1987 issue of the ASME Journal of Applied Mechanics, Vol. 54, No. 4, pp. 778–782.

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