The Comparison of the Earth's Gravitational Potential
derived from Satellite Observations with Gravity
Observations on the Surface

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Summary

Instead of regarding gravity as a function of height above sea-level and angular co-ordinates, it may be regarded as a function of geopotential and angular co-ordinates. Since the operations of spirit levelling give differences of geopotential directly, observed values of gravity on the surface of the Earth are obtained as functions of geopotential, while gravity as a function of geopotential may be calculated from the expression for the potential derived from satellite observations. Comparing these two functions, we obtain differences between observed and calculated gravity at the same values of geopotential, the most direct possible comparison between the two sets of observational data.

1. Introduction

One of the outstanding problems of the theory of the Earth's external gravity field that has been revived by the impact of the highly accurate estimates of potential derived from observations of artificial satellites is the theoretical basis on which these estimates may be compared with surface gravity measurements. I am not concerned here with the uncertainties due to the imperfect distribution of gravity observations over the surface; the question is how to deal with the fact that observations of gravity are made on the surface of the Earth at varying heights.

2. Principles

Many discussions of this problem have, in the past, been in terms of equipotential surfaces, either of the actual Earth or of various models, at points within the topography and therefore at places where Laplace's equation is not obeyed and where, accordingly, very intricate problems arise especially when considering isostatic anomalies or models. With some models, also, it has not been shown that the external field of the model agrees at all external points (and not just on the surface) with that of the actual Earth—in dealing with artificial satellites this is a fundamental consideration. The ideas I use in this paper derive to a large extent from those of Molodensky (1948), de Graaff-Hunter (1950), Levallois (1957) and Sir Harold Jeffreys (1952), in that I consider values of gravity at the physical
surface of the Earth and not on any internal surface, real or imaginary. On this basis, Jeffreys (1952) has given a treatment of the free-air reduction extrapolating the external field from gravity given on the actual surface so that the external field on and outside the actual surface is correctly reproduced. However for the comparison of satellite with surface data, an even more straightforward way of formulating the problem seems to be available which entirely avoids such questions as the form of equipotential surfaces, the definition of height and so on and concentrates attention entirely on observable quantities.

The observations which can be made at the surface of the Earth give values of the geopotential $U$ (actually differences from the sea-level value) and values of the normal gradient of $U$, that is values of gravity, $g$. Values of gravity are thus known as a function of geopotential $U$ and of positional co-ordinates such as longitude and geocentric colatitude. As has been pointed out by many authors, ordinary spirit-levelling measurements of height fundamentally yield differences of geopotential. Now from the observations of artificial satellites the external potential can be specified by, for instance, the values of the coefficients in a spherical harmonic expansion and hence again it is possible to express the value of gravity as a function of geopotential and other co-ordinates. The values so expressed may then be compared with the observed values on the surface.

3. Mathematical development

As one possible way of making the comparison between surface gravity and that derived from a "satellite" potential, I shall derive a Taylor series expansion for the latter.

Since spherical polar co-ordinates $(r, \theta, \phi)$ are the natural ones to use in dealing with the motion of artificial satellites (but see Vinti 1960), they will be used in the working below.

Let the geopotential be expressed in the form

$$U = \frac{fM}{a} \left( \frac{a}{r} \sum J_n \left( \frac{a}{r} \right) P_n(\cos \theta) \right) + \frac{1}{2} \omega^2 \sin^2 \theta,$$

where $a$ is the Earth's mean radius and the coefficients $J_n$ are determined from satellite observations. $J_n$ differs slightly from the coefficient associated with the equatorial radius. Terms that are functions of longitude are omitted since they cannot at present be found from satellite observations.

$g$, equal to $|\text{grad } U|$, may also be written as a function of $r$ and $\theta$. But equally $g$ may be regarded as a function of $U$ and $\theta$ so that near sea-level

$$g(U, \theta) = g(U_0, \theta) + \frac{\partial g}{\partial U} \delta U + ...$$

where $U_0$ is the value of $U$ at sea-level and $\partial g/\partial U$ is evaluated at sea-level. This is adequate for moderate heights above sea-level, but in general further terms of the Taylor series should be included.

Equation (2) therefore gives a value of gravity to be compared with observed values on the surface and, to evaluate it, $\partial g/\partial U$ has to be calculated for $U$ specified by equation (1).

Now the term $fM/a$ in equation (1) is one thousand times greater than the term $(fMa^2/r^3) \times J_2 P_2(\cos \theta)$ and that itself is nearly one thousand times greater than
any subsequent term in the gravitational potential, at least so far as present information goes. We therefore ignore all these subsequent terms in calculating \( \frac{\partial g}{\partial U} \) and take for the geopotential

\[
U = \frac{fM}{a} \left( \frac{a}{r} - \left( \frac{a}{r} \right)^3 J_2 P_2(\cos \theta) \right) + \frac{1}{2} r^2 \omega^2 \sin^2 \theta.
\]

Put \( m = a^2 \omega^2 / fM \). Then

\[
U = \frac{fM}{a} \left( \frac{a}{r} - \left( \frac{a}{r} \right)^3 J_2 P_2(\cos \theta) + \frac{1}{2} \left( \frac{r}{a} \right)^2 m \sin^2 \theta \right).
\]

If the development is restricted to first order in \( J_2 \),

\[
g = -\frac{\partial U}{\partial r} = \frac{fM}{a^2} \left[ \left( \frac{a}{r} \right)^3 - 3 \left( \frac{a}{r} \right)^4 J_2 P_2(\cos \theta) - \frac{r}{a} m \sin^2 \theta \right].
\]

Write \( g_m \) for the mean value of \( g \):

\[
g_m = \frac{fM}{a^2} (1 - \frac{2}{3} m).
\]

Then

\[
\frac{fM}{a} = a g_m (1 + \frac{2}{3} m).
\]

Now \( U \) can be written in the form

\[
\frac{fM}{a} \left( \frac{a}{r} - \alpha \left( \frac{a}{r} \right)^3 + \beta \left( \frac{a}{r} \right)^2 \right)
\]

where \( \alpha = J_2 P_2(\cos \theta) \) and \( \beta = \frac{1}{2} m \sin^2 \theta \) are both small quantities and \( a/r \) is of the order of unity.

Then to first order in \( m \) and \( J_2 \),

\[
a \frac{U}{r} = \frac{U a}{fM} + \alpha \left( \frac{U a}{fM} \right)^3 - \beta \left( \frac{U a}{fM} \right)^2
\]

or

\[
a \frac{U}{r} = \frac{U}{ag_m} (1 - \frac{2}{3} m) + \left( \frac{U}{ag_m} \right)^3 J_2 P_2(\cos \theta) - \frac{1}{2} \left( \frac{U}{ag_m} \right)^2 m \sin^2 \theta.
\]

We let \( U/ag_m = u \) and write

\[
\left( \frac{a}{r} \right)^n = u^n \left[ 1 - \frac{2}{3} nm + nu^2 J_2 P_2(\cos \theta) - \frac{1}{2} nu^{-3} m \sin^2 \theta \right].
\]

With the aid of this expansion, the formula for \( g \) becomes

\[
\frac{g}{g_m} = \left( 1 + \frac{2}{3} m \right) u^2 \left[ 1 - \frac{2}{3} m + u^2 J_2 P_2(\cos \theta) - 2 u^{-3} m \sin^2 \theta \right].
\]

Hence

\[
\frac{1}{g_m} \frac{\partial g}{\partial u} = 2 u (1 - \frac{2}{3} m) + 4 u^2 J_2 P_2(\cos \theta) + 2 u^{-2} m \sin^2 \theta.
\]
This gives the expression for $\partial g/\partial U$ that we require to insert in equation (2). We note that since $du/dh$ is approximately equal to $-a^{-1}$ and since $u$ is of the order of unity, this equation is equivalent to first order to the free-air relation

$$\frac{\partial g}{\partial h} = -\frac{2g_m}{a}.$$ 

Further,

$$\frac{1}{g_m} \frac{\partial g}{\partial u^2} = 2(1 - \frac{2}{3}m) + 12u^2 J_2 P_2(\cos \theta) - 4mu^{-3} \sin^2 \theta.$$ 

We may obtain a simpler expression for $\partial g/\partial u$, for now $u$ is equal to $(U_0 - \int g \, dh)/ag_m$ where the integral $\int g \, dh$ is evaluated from sea-level upwards and $U_0$ is the value of $U$ at sea-level, that is $ag_m(1 + \frac{1}{3}m)$. Thus $u$ is equal to

$$1 + \frac{1}{3}m - \frac{1}{ag_m} \int g \, dh$$

and to first order,

$$\frac{1}{g_m} \frac{\partial g}{\partial u} = 2(1 - \frac{1}{3}m) + 4J_2 P_2(\cos \theta) + 2m \sin^2 \theta$$

or

$$\frac{1}{g_m} \frac{\partial g}{\partial u} = 2 + e \frac{1}{2} P_2(\cos \theta)$$

where

$$e = \frac{2}{3} J_2 + \frac{1}{3} m.$$ 

It will be remembered that in all these formulae, $a$ is the Earth's mean radius and $J_2$ is $(C - A)/Ma^2$.

The difference between observed gravity and values derived from a "satellite" potential by equation (2) are gravity anomalies and numerically they will be almost identical with free-air anomalies. In principle, however, they are rather different, for with the anomalies suggested by Levallois (1957) the object is to compare observed gravity with that calculated from a formula at the same point in space; whereas the object here is to compare gravity at the same values of the geopotential, on the actual Earth and on the model as de Graaff-Hunter (1950) does. In general, these two comparisons are not the same.

I would also like to emphasize that the values of gravity calculated from the "satellite" potential are not conventional values like those derived from the International Gravity Formula but are just as much the results of observation as are surface gravity measurements. We are concerned with comparisons between two sets of observations, not between observation and convention.

It should be mentioned that this approach to the problem of comparing satellite results with surface gravity measurements is very similar to that of Marussi (1951) and Hotine (1957) in their discussion of geodetic problems. The essential point is that geopotential is taken as a co-ordinate instead of a length such as radius vector or height.
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References