Analysis on the Two Meson Theory

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Abstract

Possible models for the two mesons, \( \pi \) and \( \mu \) are selected by testing them with the present experimental knowledges, concerning the \( \beta \)-decay, the \( \mu \)-meson capture, and the \( \pi-\mu \) decay.

1. Introduction.

After the discoveries of \( \pi \) and \( \mu \) mesons, various models for these two mesons are discussed extensively by Yukawa,\(^1\) Taketani,\(^2\) Wheeler,\(^3\) etc. Of these the choice of the models I, II, and III will be analysed in this paper.

2. \( \beta \)-Decay. (model I)

The model I proposed by Inoue and Ogawa\(^4\) following Sakata-Tanikawa's model of the two mesons, is based upon the original Yukawa's idea; that is the beta-decay of the nucleus goes through virtual \( \pi \) mesons. It is generally believed that this model has a defect that a free \( \pi \) meson decays faster into the light particles than into \( \mu \) mesons. So far as the scalar, vector and pseudovector \( \pi \) mesons are concerned, there is no room for doubt. For pseudoscalar \( \pi \) meson, as was once pointed out by Ozaki\(^5\) and Shono\(^6\) partly, this defect can be removed, if the interactions between \( (\pi-\epsilon, \nu) \) and \( (\pi-\mu, \nu) \) could be limited to the pseudovector type. But in this case, it is claimed that usual perturbation calculation cancels out the matrix elements of the allowed transitions of \( \beta \)-decay through \( \pi \) mesons, and it starts from the matrix elements of \( j \sigma \).

If it is true, the troubles accompanying this result are that the Sargent law of the allowed \( \beta \)-decay and the forbidden beta spectra can not consistently be reproduced. Meanwhile, the choice of the direct interaction in the original Hamiltonian may be adjusted to bring about the allowed type transition, as it was proposed by Miyazima\(^7\) and Sakata,\(^8\) in which case the model of Inoue and Ogawa can escape the difficulty of the Sargent law. Of course, in this case the \( \beta \)-decay of the nucleus is to be explained by mixing of meson theory and Fermi
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theory. In this point, Lagrangian method recently proposed by Kobat\(^9\) will settle the ambiguities concerning direct interactions involved in the Hamiltonian. But if the experiment now going on the forbidden \(\beta\)-ray spectra requires the interaction other than that of pseudovector type\(^10\), this will not hold good. Anyhow, meson theory of \(\beta\)-decay will not conclusively be excluded for the time being.

3. \(\pi\mu\) Decay of Spin \(\frac{1}{2}\) \(\mu\) meson. (Model II)

The main problem on the model (II) is which process, \((\pi \rightarrow \mu, \nu)\) or \((\pi \rightarrow e, \nu)\), is more predominant if both takes place through virtual nucleon pair. So far various authors\(^{11,12,13}\), have treated the process (II). According to Steinberger,\(^13\)

(A) Only if \(\pi\) is pseudoscalar, we have the term \((m_1 + m_2)^3\) (where \(m_1\) and \(m_2\) are the masses of two particles in the final state) in the matrix element as well as in the volume element of the momentum space for the final state. This term is \(4 \times 10^4\) times smaller for \((\pi \rightarrow e, \nu)\) than for \((\pi \rightarrow \mu, \nu)\), which favors the agreement of the theory with the experiment.

(B) If one disposes of the divergent integral by regulator, the absolute value of life time for \((\pi \rightarrow \mu, \nu)\) becomes extraordinarily greater than the experimental value. Therefore, one cannot take this procedure.

On the other hand, according to Finkelstein and Ruderman\(^12\) the absolute life time of \((\pi \rightarrow \mu, \nu)\) can be adjusted to the experimental value only in the case of pseudoscalar \(\pi\) meson, by cutting off the logarithmic divergent term instead of using the regulator.

Thus, in order to finish the analysis of this problem on the present stage, we shall check one by one the life times \(\tau_{\pi\mu}\) of all kinds of \(\pi\) meson for all types of the interactions between nucleons and \((\mu, \nu)\) or \((e, \nu)\).

As for the method of calculation, we followed the theory of Tomonaga and Schwinger.

1) Interaction.

The Hamiltonian for the direct coupling of light particles and nucleons is assumed to be the same

\[
H = \frac{-\hbar e}{c} \left( \psi_1^* \vec{r} \cdot \vec{A}_1 - \psi_2^* \vec{r} \cdot \vec{A}_2 \right)
\]

where \(\vec{A}_1\) and \(\vec{A}_2\) are the vector potentials for the fields of \(\pi\) and \(\mu\), respectively.

Table I. Selection rules for \(\pi\mu\) decay. (Spin \(\frac{1}{2}\) \(\mu\) meson)

<table>
<thead>
<tr>
<th>(N\rightarrow e, \nu) or (N\rightarrow \mu, \nu)</th>
<th>(S)</th>
<th>(V)</th>
<th>(t)</th>
<th>(PV)</th>
<th>(PS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi\rightarrow N)</td>
<td>(\pi)</td>
<td>(S)</td>
<td>*</td>
<td>*</td>
<td>(L)</td>
</tr>
<tr>
<td>(\pi\rightarrow N)</td>
<td>(\pi)</td>
<td>(V)</td>
<td>*</td>
<td>(D)</td>
<td>(D)</td>
</tr>
<tr>
<td>(\pi\rightarrow N)</td>
<td>(\pi)</td>
<td>(V)</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>(\pi\rightarrow N)</td>
<td>(\pi)</td>
<td>(T)</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>(\pi\rightarrow N)</td>
<td>(\pi)</td>
<td>(PV)</td>
<td>(L)</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>(\pi\rightarrow N)</td>
<td>(\pi)</td>
<td>(T)</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>(\pi\rightarrow N)</td>
<td>(\pi)</td>
<td>(PS)</td>
<td>(L)</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>(\pi\rightarrow N)</td>
<td>(\pi)</td>
<td>(PV)</td>
<td>(L)</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

where * : Furry's Theorem
\(L\) : Lorentz Invariance
\(\varepsilon\) : Parity
\(D\) : Divergence Theorem

prohibit the process.
form as that of $\mu$ meson and nucleons. As a result of the calculation following Tomonaga-Schwinger's method, we have obtained the following forms (2). The quadratically divergent term is caused by the same origin as in the case of the self-energy of the photon, which contradicts with the divergent theorem, the equivalent theorem and gauge invariance. For these reasons, it can be reasonaly dropped off.

2) Selection rules.

Taking into consideration various types of Fermi coupling between the nucleon, $N$, and $(e, \nu)$ or $N$ and $(\mu, \nu)$ and various types of mesons, we have the following selection rules. (See Table I)

This selection rule is exactly the same with that of Finkelstein and Ruderman but slightly differs from that of Steinberger.

3) The life times, $\tau_{\mu\nu}$, of $\pi-\mu$ decay are given as follows:

$$\frac{1}{\tau_{\mu\nu}} = \left(\frac{\mu}{2\pi}\right)^{\frac{1}{8}} \sqrt{(1-\theta^2)(1-\theta'^2)}L,$$

where $L$ takes the following forms depending on the type of $\pi$ meson.

$$L_\delta = 18(1-\theta^2)(\lambda^2 A-B)\frac{F^2_\delta F^2_\delta}{\mu^2},$$

$$L_\tau = \frac{2}{3}(1-\theta'^2)\left\{ (2+\theta^2)(2BF_\tau+\lambda^{-1}AG_\tau)\gamma^2_{\tau} 
+ 12\theta(2BF_\tau+\lambda^{-1}AG_\tau)(\lambda^{-1}AF_\tau+\lambda^{-2}A+B)G_\tau f_\tau f_\tau 
+ 4(1+2\theta^2)(\lambda^{-1}AF_\tau+\lambda^{-2}A+B)G_\tau f_\tau f_\tau \right\},$$

$$L_{pF} = \frac{8}{3}(1-\theta^2)\left\{ (2+\theta^3)F^2_{pF}f^2_{pF} + (1+2\theta^3)G^2_f f^2_f \right\}(\lambda^{-2}A-B)^2,$$

$$L_{ps} = 2(1-\theta^3)\left\{ f_{ps}[F_{ps}(-\lambda^{-2}A+3B)+G_{ps}A^{-1}A] 
+ f_{pv}\theta A\lambda^{-1}(-F_{ps}+2\lambda^{-2}G_{ps}) \right\},$$

where $\theta = \frac{m+m'}{\mu}$, $\theta' = \frac{m-m'}{\mu}$, $\lambda = \frac{\mu}{x}$,

$m, m'$: (masses of $\mu$ meson and neutrino) x $c$,

$x$: nucleon mass x $c$,

$\mu$: $\pi$ meson mass x $c$,

$$A = 2 \log \infty - \int_1^{\infty} \frac{dv}{2} \log \left(1-\frac{1-v^2}{4}x^2\right),$$

$$B = \frac{1}{3} \log \infty - \int_1^{\infty} \frac{dv}{2} \frac{1-v^2}{4} \log \left(1-\frac{1-v^2}{4}x^2\right),$$

$$= \frac{1}{3} \log \infty + \frac{1}{30}x^2 + \frac{1}{280}x^4 + \ldots,$$
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\[ \log \infty = \lim_{k \to \infty} \left( \log \left( \frac{k + \sqrt{k^2 + k^2}}{x} \right) - \frac{k}{\sqrt{x^2 + k^2}} \right). \]  

(5)

The value of \( \log \infty \) cut off at suitably chosen value of \( k \) is denoted by \( \tilde{A} \).

Then the values of \( \tilde{A} \) and cut-off momentum are shown in the following Table II.

It is obvious from this table that the value of \( \tilde{A} \) is sensitively dependent upon the cut-off momentum.

4) The life of \( \pi \to (e, \nu), \tau_{\pi e} \).

If \( m \) and \( m' \) in the above \( \tau_{\pi e} \) formula is substituted by the masses of the electron and the neutrino and Fermi coupling term of \( N \) and \((\mu, \nu)\) by that of \( N \) and \((e, \nu)\), the resulting formula gives the values for \( \tau_{\pi e} \).

Table II. \( \tau_{\mu} \) and \( \tau_{\pi e} \)

<table>
<thead>
<tr>
<th>( \text{Cut off of } k )</th>
<th>( x )</th>
<th>( 10x )</th>
<th>( 100x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{A} )</td>
<td>0.18</td>
<td>1.77</td>
<td>4.1</td>
</tr>
</tbody>
</table>

Table III. \( \tau_{\pi e} \) and \( \tau_{\pi e}/\tau_{\mu} \)

<table>
<thead>
<tr>
<th>( \pi )</th>
<th>( N \leftrightarrow \pi )</th>
<th>( \text{Fermi Coupling} )</th>
<th>( \tau_{\pi e} (\sec) )</th>
<th>( \log \infty = 0 )</th>
<th>( \log \infty = \tilde{A} )</th>
<th>( \tau_{\pi e}/\tau_{\mu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>( S )</td>
<td>( f_3 )</td>
<td>6.27</td>
<td>3.26 \times 10^{-5}</td>
<td>1.13 \times 10^{-10} \tilde{A}^{-3}</td>
<td>5.7</td>
</tr>
<tr>
<td>( V )</td>
<td>( V )</td>
<td>( f_V )</td>
<td>4.26</td>
<td>4.26</td>
<td>2.13 \times 10^{-5} \tilde{A}^{-3}</td>
<td>4.4</td>
</tr>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( f_T )</td>
<td>1.49 \times 10^{-4}</td>
<td>1.49 \times 10^{-4}</td>
<td>5.03 \times 10^{-4} \tilde{A}^{-3}</td>
<td>4.4</td>
</tr>
<tr>
<td>( PV )</td>
<td>( PV )</td>
<td>( f_{PV} )</td>
<td>1.06 \times 10^{-4}</td>
<td>1.06 \times 10^{-4}</td>
<td>1.54 \times 10^{-4} \tilde{A}^{-3}</td>
<td>2.9</td>
</tr>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( f_T )</td>
<td>2.38</td>
<td>1.01 \times 10^{-4}</td>
<td>3.33 \times 10^{-10} \tilde{A}^{-3}</td>
<td>2.7</td>
</tr>
<tr>
<td>( PS )</td>
<td>( PS )</td>
<td>( f_{PS} )</td>
<td>22.57</td>
<td>3.26 \times 10^{-4}</td>
<td>9.78 \times 10^{-10} \tilde{A}^{-3}</td>
<td>5.7</td>
</tr>
<tr>
<td>( PV )</td>
<td>( PV )</td>
<td>( f_{PV} )</td>
<td>1.80 \times 10^{-8}</td>
<td>1.80 \times 10^{-8}</td>
<td>4.51 \times 10^{-8} \tilde{A}^{-3}</td>
<td>5.7</td>
</tr>
<tr>
<td>( PS )</td>
<td>( PS )</td>
<td>( f_{PS} )</td>
<td>2.22 \times 10^{-4}</td>
<td>2.22 \times 10^{-4}</td>
<td>1.89 \times 10^{-4} \tilde{A}^{-3}</td>
<td>1.3 \times 10^{-4}</td>
</tr>
<tr>
<td>( PV )</td>
<td>( PV )</td>
<td>( f_{PV} )</td>
<td>5.7 \times 10^{-1}</td>
<td>5.7 \times 10^{-1}</td>
<td>4.06 \times 10^{-1} \tilde{A}^{-2}</td>
<td>1.3 \times 10^{-4}</td>
</tr>
</tbody>
</table>

Spin 0 \( \mu \) meson

| \( S \) | \( S \) | \( f_3 \) | 1.3 \times 10^{-2} | 1.3 \times 10^{-2} | 3 \times 10^{-6} \tilde{A}^{-3} | 1.2 \times 10^{-3} |

5) Results.

Substituting the numerical values into the above results of the calculation, we obtain the following Table III of \( \tau_{\pi e} \) and \( \tau_{\pi e}/\tau_{\mu} \). These should be compared with the experimental values

\[ \tau_{\pi e} \sim 10^{-8} \text{ sec}, \]

\[ \tau_{\pi e}/\tau_{\mu} \leq 10^{-8}. \]

where the constants are given as follows:
\[ \tau_\beta = F_2^2 \left( \frac{\mu}{2\pi} \right)^2 \approx 5.63 \times 10^{-7}, \]
\[ \frac{F_2^2}{4\pi} = \frac{G_2^2}{4\pi} = 0.1, \]

\[ 2\pi^2 \frac{\hat{S}}{g_\mu} \frac{g_\pi}{m_e^2 c^2} = 4.4 \times 10^3 \text{ sec} \] which corresponds to the characteristic time of \( \beta \)-decay. This is adjusted in agreement with the value obtained by J. A. Wheeler and J. Tiomno\(^{10} \) for the Fermi coupling constant of \( \beta \)-decay;

\[ f = 2 \times 10^{-59} \text{ g} \cdot \text{cm}^3. \]

The characteristic time of \( \beta \)-decay is usually taken to be \( 3 \times 10^3 \) sec. In this case the factor \( 30/44 \) must be multiplied to the lifetime of \( \pi^-\mu \) decay shown in the table.

From this calculation, we have the following conclusions.

A) Competition of \( \pi^- \rightarrow (e, \nu) \) and \( \pi^- \rightarrow (\mu, \nu) \).

Except the case of pseudoscalar \( \pi \) meson and pseudovector Fermi coupling, all the results are incompatible with the experiment. (c.f. \( \tau_\pi/\tau_\mu \geq 10^{-2} \) in Table III)

B) The lifetime of \( \pi^-\mu \) decay.

Different results are deduced according to the treatment of the divergent integrals.

i) Regulator \( \log \infty = A \) give results which are at variances with the experiments.

ii) \( \log \infty = A \) cut off

With the suitable choice the cut-off momentum \( k \), we can in general deduce the results which agree with the experiments. (c.f. \( \log \infty = A \) in Table III). Let us examine the case where the agreement with the experiment is exclusively attained for the competition of \( \pi^- \rightarrow (\mu, \nu) \) and \( \pi^- \rightarrow (e, \nu) \).

\( \pi \); Pseudoscalar

Fermi couplings between \( (\pi^- \rightarrow e, \nu) \) or \( (\pi^- \rightarrow \mu, \nu) \) involves pseudovector.

If the coupling of \( \pi^- \rightarrow N \) is pseudoscalar, \( k = 100 \) and \( \tau_\pi = 10^{-8} \) sec. and if that of \( \pi^- \rightarrow N \) is pseudovector, \( k = x \) and \( \tau_\pi = 1.25 \times 10^{-8} \) sec.

Therefore, it is found out that, in the conclusion of Finkelstein et al., the case (7) was overlooked.

C) Selection Rule for \( \beta \)-decay.

In the recent analysis\(^{10} \) of the forbidden spectrum of \( \beta \)-decay, it was concluded that the Fermi coupling must involve pseudoscalar and pseudovector as well as tensor interactions. According to the conclusion of the above section, however,
the Fermi couplings can not involve pseudoscalar type in order that with the model II we may expect consistency with the empirical result \( \tau_{\pi\mu}/\tau_{\pi\tau} \leq 10^{-4} \). Therefore, taking into consideration on both the two facts, we can conclude:

In the model II, \( \pi \) meson must be pseudoscalar and Fermi couplings must be pseudovector (for the latters, scalar, tensor, vector are available, but pseudoscalar must be excluded).

4. \( \pi\mu \) Decay of Spin 0 \( \mu \) meson (model II)

\( \mu \) meson of spin 0 was involved as one of the models in the two meson theory put forward by Tanikawa\(^{3b}\) in 1942. According to Tiomno,\(^{18}\), the electronic spectrum due to the \( \beta \)-decay of \( \mu \) meson can be reproduced by assuming suitable interactions between \( \mu \) and the light particles. R. Latter and R. F. Christ\(^{17}\) takes the same model (III) for the calculation of \( \pi\mu \) decay and \( \mu \)-capture.

We shall now discuss the validity of assumption of spin 0 \( \mu \) mesons in the model II. The method of analysis will follow the above procedures taken for the \( \mu \) meson of spin \( \frac{1}{2} \).

1) Probability of \( \mu \)-capture (Spin 0 \( \mu \) meson).

Capture life time for capturing \( \mu \) meson in the \( K \)-shell is calculated.

\[
\frac{1}{\tau_{\mu\text{ cap}}} = \frac{|M|^2}{\tau_0} \frac{1}{4\pi} \frac{[4(1-\epsilon^2)]^{1+\frac{1}{2}}}{\Gamma(2\tau+3)} \left( \frac{m_\mu (cr_\mu)}{\hbar} \right)^{2\tau} \left( \frac{\Delta \tau + \epsilon}{\epsilon} \right).
\]

where \( \tau_0 = \frac{4\pi \hbar^2}{m_\mu^2 \frac{3^3}{2} c^2} \). \( m_\mu \) is the mass of \( \mu \) meson.

On the other hand, experimental evidences for \( \mu \) capture is given, for \( Z = 10 \), as follows:

\[
\tau_{\mu\text{ cap}} = 1.49 \times 10^{-8} \text{ sec.}^{18}
\]

If we put \( \Delta \tau = 20/276 \), we have from (8)

\[
\tau_\epsilon = 1.1 \times 10^{-7} \text{ sec.}
\]

The direct coupling, \( f \), of (\( N\rightarrow\pi\mu, \nu \)) derived from the experiment of \( \mu \)-capture will be given

\[
\frac{f^2 m_\mu^3 c^2}{\hbar} = 3.6 \pi \times 10^7 \text{ sec}^{-1}.
\]

2) \( \pi\mu \) decay (Spin 0 \( \mu \) meson).

\( \pi\mu \) decay of \( \pi \)-mesons in the case of spin 0 \( \mu \) mesons will be dealt with just the same way as in the case of spin \( \frac{1}{2} \) \( \mu \) mesons. Selection rules for the processes will be given in Table IV.
According to the table, pseudoscalar $\pi$-meson must be eliminated since $\pi-\mu$ decay cannot take place at all. For the scalar $\pi$, the life $\tau_{\pi\mu}$ is given as follows:

$$1/\tau_{\pi\mu} = \frac{1}{8\pi} \left(1-\left(\frac{m_\pi}{m_\omega}\right)^3\right) \left(\frac{m_\omega}{2\pi}\right)^3 \times \frac{\epsilon^2 f^2}{\vec{p}} \frac{F^2}{4\pi}[\delta]^3,$$

(11)

where $m_\pi/m_\omega = 215/276$, $F^2/4\pi = 0.1$.

Inserting (10) into (11), we get

$$\tau_{\pi\mu} = 3 \times 10^{-5} \left[\log \infty\right]^{-3} \text{ sec.}$$

(12)

$\log \infty$ is just the same meaning as before (see Table II).

As is obvious from this table, the regulator method and $\log \infty = 0$ procedure are at variances with experimental value of $\tau_{\pi\mu}$. If we cut off this $\log \infty$ at $k = 10\pi$, $\tau_{\pi\mu} = 10^{-8}$ sec. But the competition of $\pi-\mu$ decay and $\pi-\nu$ decay cannot agree with the experiment. The analysis for the vector or pseudovector $\pi$-meson can be performed in just the same manner as before. In this case, too, the competition $\tau_{\pi\mu}/\tau_{\pi\nu} \gg 1$, being contradictory with the experiment.

3) Results.

For the spin 0 $\mu$, $\nu$ meson there remains the following possibilities for the type of $\pi$ mesons consistent with experiment, provided the validity of cut off procedure, is assumed.

A) $\pi$: scalar and the Fermi couplings of ($N\rightarrow\pi, \nu$) do not involve scalar interaction.

B) $\pi$: vector and the Fermi couplings of ($N\rightarrow\tau, \nu$) do not involve vector and tensor interaction.

For the pseudoscalar $\pi$ mesons, only if either of $\mu$ and $\nu$ mesons is scalar and pseudoscalar respectively, the selection rule can be satisfied. In this case the following condition is also required to explain the experiments concerned.

D) $\pi$ is pseudoscalar and the Fermi couplings of ($N\rightarrow\tau, \nu$) do not involve pseudoscalar interactions. Of course, the cut off procedure is assumed to be accepted.

5. Discussion of $\mu$-capture (Model I, III)

For the model I, III some alteration is to be mentioned to our previous discussion. We have mentioned before that the nuclear matrix element for $\mu$-meson capture $|\langle \vec{F} \cdot \vec{D} \rangle |^2 \leq 1$. Since every proton in the initial nucleus has
chance to capture the $\mu$-meson, it should be $|\{\Phi\}| \leq Z$, where $Z$ is the atomic number of the nucleus. The important conclusion we can draw from this revision is that only the pseudoscalar $\pi$ meson may be not at variances with the requirement of explaining $\mu$-capture and $\pi$-$\mu$ decay by the model III and I. Another types for $\pi$ meson, i.e. vector, pseudovector and scalar contradicts these experiments in explaining by the model III and I.

There is no more question in so far as direct interaction between elementary particles is assumed. Indirect interactions through virtual process, such as $\beta$-decay and $\mu$-capture for model I, $\pi$-$\mu$ decay for model II, and $\mu$-capture for model III, are to be studied in the light of experimental evidences.

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