A FORMULA FOR THE FUNDAMENTAL SOLUTION OF ANISOTROPIC ELASTICITY

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[Received 27 October 1995. Revise 24 June 1996]

SUMMARY

A formula for the fundamental solution of anisotropic elasticity is given in terms of the eigenvectors and the generalized eigenvectors of the associated six-dimensional eigenvalue problem called Stroh's eigenvalue problem. From this formula an explicit closed form of the fundamental solution for transversely isotropic media can be obtained.

1. Introduction

Let \( C = (C_{ijkl})_{1 \leq i,j,k,l \leq 3} \) be a three-dimensional homogeneous linear anisotropic elastic tensor which satisfies the following symmetry and strong convexity conditions:

(hyperelasticity)
\[
C_{ijkl} = C_{klji} \quad (1 \leq i, j, k, l \leq 3),
\]  

(strong convexity)
\[
\exists \delta > 0; \quad \sum_{i,j,k,l=1}^{3} C_{ijkl} e_{ij} e_{kl} \geq \delta \sum_{i,j=1}^{3} e_{ij}^2
\]  

for any real matrix \( E = (e_{ij}) \).

Let \( x \in \mathbb{R}^3 \) and let \( G_{km} = G_{km}(x) \) be a solution to
\[
\sum_{i,j,k,l=1}^{3} C_{ijkl} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} G_{km} + \delta_{lm} \delta(x) = 0 \quad \text{in} \ \mathbb{R}^3 \quad (1 \leq i, m \leq 3),
\]  

where \( (x_1, x_2, x_3) \), \( \delta_{lm} \) and \( \delta(x) \) are the Cartesian coordinates of \( x \), Kronecker delta and Dirac delta function, respectively. Physically, the solution \( G_{km} \) describes the displacement at the point \( x \) in the \( x_k \)-direction due to a point force at the origin in the \( x_m \) direction. We call
\[
G = G(x) = (G_{km} : k \downarrow 1, 2, 3; m \rightarrow 1, 2, 3)
\]
the fundamental solution to
\[
\sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} u_k(x) + f_i(x) = 0 \quad \text{in } \mathbb{R}^3 \quad (1 \leq i \leq 3);
\]
it generates the solution \( u(x) = (u_1(x), u_2(x), u_3(x)) \) in the following way:
\[
u(x) = \int_{\mathbb{R}^3} G(x - y)f(y) \, dy,
\]
where \( f(x) = (f_1(x), f_2(x), f_3(x)) \).

From (1.1), \( G(x) \) is given as the inverse Fourier transform of a homogeneous function. Using this expression, Lifschitz and Rozenzweig (1), Synge (2) and Barnett (3) gave a formula for \( G(x) \) in terms of an integral on the interval \([0, 2\pi]\) whose integrand is a smooth periodic function with period \(2\pi\). This formula is useful to evaluate \( G(x) \) numerically by using the Euler–Maclaurin expansion when specific values are given for \( C_{ijkl} \) (1 ≤ i, j, k, l ≤ 3).

Besides this integral formula for \( G(x) \), Malén (4) gave another formula. That is, he showed that \( G(x) \) can be expressed in terms of the eigenvectors of Stroh's eigenvalue problem provided that all the eigenvalues are distinct. Malén's formula was useful for the perturbation argument for the fundamental solution (4 to 7) and the estimation of the field around a straight dislocation (4, 8). Moreover, Malén's formula was rewritten in a more explicit form (see, for example, (9, 10)). However, the cases where the eigenvalue problem has multiple eigenvalues were not seriously considered. The assumption of distinctness of the eigenvalues is too strict. In fact, for most crystals which are typical examples of three-dimensional homogeneous linear anisotropic elastic media, the condition that all the eigenvalues are distinct does not hold because of the additional symmetries which these crystals have.

In this paper we give a formula for \( G(x) \) in terms of the eigenvectors and the generalized eigenvectors of Stroh's eigenvalue problem without assuming the distinctness of the eigenvalues. We prove this formula by mathematically rigorous and systematic arguments. As a byproduct we give an explicit closed form of \( G(x) \) for transversely isotropic media written in terms of the elastic tensor and the coordinates of \( x \), because the explicit forms of the eigenvectors and the generalized eigenvectors for the associated eigenvalue problem are available in the case of transversely isotropic media. The eigenvectors used in the formulae in (4, 9, 10) are normalized, while the eigenvectors and the generalized eigenvectors used in our formula are not necessarily normalized. This enables us to compute \( G(x) \) more straightforwardly for transversely isotropic media. Like Malén's formula, our formula will be useful for the perturbation argument for the fundamental solution,
since it only requires a perturbation argument for the eigenvalue problems in Stroh's sextic formalism.

To compute the displacement fields and the stress fields produced by dislocations, it is necessary to obtain the expressions of the derivatives of $G(x)$ (see Willis (11), Wang (12, 13)). For general anisotropic elasticity Wang (12, 13, 14) obtained formulae for $G(x)$ and its derivatives in closed forms, using Radon's transform, which contain the roots of a certain sixth-order equation, and he also studied the case when these roots are not distinct. However, it will be difficult to obtain a unified form of $G(x)$ by expressing these roots in terms of the elastic tensor when we apply Wang's formulae to transversely isotropic media.

Our closed form of $G(x)$ for transversely isotropic media is written in a unified expression. Hence we can obtain the derivatives of $G(x)$ by differentiating $G(x)$ directly. Moreover, for the boundary-element method in numerical analysis, the explicit closed form of the fundamental solution will be useful for computing the displacement and the stress fields in the elastic medium.

For transversely isotropic media, using various approaches many authors have given expressions of $G(x)$ (1, 15 to 19). We shall refer to these in Section 4.

Although some of the lemmas necessary to obtain our formula have already been proven or partly proven, we give their complete proofs to make this paper as self-contained as possible. Finally in the Appendix we give a mathematically rigorous justification of the formula in (1, 2, 3), because we have no recollection of any mathematical proof of it in spite of its importance.

**Remark** In the theory of classical linear elasticity the following symmetry conditions (which include hyperelasticity condition) are assumed:

\[
C_{ijkl} = C_{ijlk} = C_{klij} \quad (1 \leq i, j, k, l \leq 3).
\] (A.1')

However, in this paper, we develop our theory without the condition

\[
C_{ijkl} = C_{ijkl} \quad (1 \leq i, j, k, l \leq 3).
\]

2. Result

Let $x \neq 0$. Take any two orthogonal unit vectors $v$ and $w \in \mathbb{R}^3$ such that

\[
x/|x| = v \wedge w,
\] (2.1)

where $v \wedge w$ denotes the vector product. Define the matrices $Q$, $R$ and $T$ by

\[
Q = \langle v, v \rangle, \quad R = \langle v, w \rangle, \quad T = \langle w, w \rangle,
\]

where

\[
\langle v, w \rangle = \langle (v, w)_i, 1 \leq 1, 2, 3; k \rightarrow 1, 2, 3 \rangle, \quad (v, w)_k = \sum_{j, l=1}^{3} C_{ijkl} w_j w_l
\]
for \( v = (u_1, u_2, u_3), \ w = (w_1, w_2, w_3) \). Moreover define the matrix \( N \) by
\[
N = \begin{bmatrix}
-T^{-1}R^T & T^{-1} \\
-Q + RT^{-1}R^T & -RT^{-1}
\end{bmatrix}
\]
and consider the eigenvalue problem
\[
N \xi = p \xi.
\]
According to (A.2) its eigenvalues \( p_\alpha (1 \leq \alpha \leq 6) \) are not real. Together with this and the fact that \( N \) is a real matrix, we can renumerate \( p_\alpha (1 \leq \alpha \leq 6) \) in the following way:
\[
\tilde{p}_\alpha = p_{\alpha + 3}, \quad \text{Im} p_\alpha > 0 \quad (1 \leq \alpha \leq 3).
\]
Let \( \xi_\alpha = \begin{bmatrix} a_\alpha \\ l_\alpha \end{bmatrix} \) be the eigenvector or the generalized eigenvector corresponding to the eigenvalue \( p_\alpha \) where \( a_\alpha, l_\alpha \in \mathbb{C}^3 \).

**Theorem 2.1**
\[
G(x) = \frac{1}{4\pi |x|} (\text{Im} \{ L A^{-1} \})^{-1},
\]
where \( A = [a_1, a_2, a_3], \ L = [l_1, l_2, l_3] \) and \( \text{Im} \{ \cdot \} \) denotes the imaginary part of a matrix \( \{ \cdot \} \).

**Remark** The matrix \( (\text{Im} \{ L A^{-1} \})^{-1} \) is invariant under the choice of the two orthogonal unit vectors \( v \) and \( w \) satisfying (2.1).

### 3. Proof of Theorem 2.1
To start with we quote the formula in (1, 2, 3).

**Theorem 3.1** For \( x \neq 0 \), \( G(x) \) is given by
\[
G(x) = \frac{1}{8\pi^2 |x|} \int_0^{2\pi} \langle \eta^0(\psi), \eta^0(\psi) \rangle^{-1} d\psi,
\]
where \( \eta^0(\psi) = \cos \psi v + \sin \psi w \).

The proof of Theorem 3.1 is given in the Appendix.

Now following Chadwick and Smith (9), we define the Stroh tensors \( S_1, S_2, S_3 \) as follows. Let \( m^0 = v, n^0 = w \) and define \( m, n \) by
\[
m = \cos \psi m^0 + \sin \psi n^0, \quad n = -\sin \psi m^0 + \cos \psi n^0.
\]
Moreover let
\[
Q(\psi) = \langle m, m \rangle, \quad R(\psi) = \langle m, n \rangle, \quad T(\psi) = \langle n, n \rangle
\]
and define \( N_1(\psi), N_2(\psi), N_3(\psi) \) by
\[
\begin{align*}
N_1(\psi) &= -T^{-1}(\psi)R^T(\psi), \\
N_2(\psi) &= T^{-1}(\psi), \\
N_3(\psi) &= R(\psi)T^{-1}(\psi)R^T(\psi) - Q(\psi).
\end{align*}
\]
Then $S_j \ (j = 1, 2, 3)$ are defined by
\[ S_j = \frac{1}{2\pi} \int_0^{2\pi} N_j(\psi) \, d\psi. \]

Observe that
\[ \frac{1}{2\pi} \int_0^{2\pi} \langle \eta^0(\psi), \eta^0(\psi) \rangle^{-1} \, d\psi = S_2 \]
and $S_2$ is invertible.

Theorem 2.1 follows from the following key lemma.

**Lemma 3.1**
\[ S\xi_\alpha = i\xi_\alpha \quad (1 \leq \alpha \leq 3), \]

where
\[
S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_2^T \end{bmatrix}.
\]

**Proof of Theorem 2.1.** From (3.6) it follows that
\[ S_1 a_\alpha + S_2 I_\alpha = i a_\alpha \quad (1 \leq \alpha \leq 3). \]

Then
\[ (S_1 - iI)A = -S_2 L, \]
where $I$ is the $3 \times 3$ unit matrix, and
\[ -iLA^{-1} = S_2^{-1} + iS_2^{-1}S_1. \]

Therefore
\[ (\text{Im} \{LA^{-1}\})^{-1} = S_2. \]

Combining (3.5) and Theorem 3.1 we obtain Theorem 2.1.

The rest of this section is devoted to the proof of Lemma 3.1. According to the degeneracy of Stroh's eigenvalue problem, we have to consider the following six cases.

\[
\begin{array}{ccc}
\text{case 1} & p_\alpha \ (1 \leq \alpha \leq 3) \text{ are distinct,} & \\
\text{case 2} & p_2 = p_3 & p_1 \neq p_2 \quad \text{dim Ker} \ (N - p_2 I) = 2, \\
\text{case 3} & p_1 = p_2 = p_3 & \quad \text{dim Ker} \ (N - p_2 I) = 3, \\
\text{case 4} & p_2 = p_3 & p_1 \neq p_2 \quad \text{dim Ker} \ (N - p_2 I) = 1, \\
\text{case 5} & p_1 = p_2 = p_3 & \quad \text{dim Ker} \ (N - p_1 I) = 2 \\
\text{case 6} & p_1 = p_2 = p_3 & \quad \text{dim Ker} \ (N - p_1 I)^2 = 3, \\
\end{array}
\]

(3.7)

Here for an $n \times n$ matrix $M$, $\text{Ker} \ M = \{ u \in \mathbb{C}^n : Mu = 0 \}$. 
For (3.6), partial proofs are given in (9, 20, 21). However, we have not seen any complete proof of (3.6) dealing with all six cases. We give a mathematically rigorous and consistent proof of (3.6) which can be applied to all six cases. The following basic lemma is necessary for the proof.

**Lemma 3.2** \( N(\psi) = \begin{bmatrix} N_1(\psi) & N_2(\psi) \\ N_3(\psi) & N_7(\psi) \end{bmatrix} \) satisfies

\[
N'(\psi) = -I - N^2(\psi),
\]

where \( N'(\psi) = dN(\psi)/d\psi \) and \( I \) is the \( 6 \times 6 \) unit matrix.

**Proof.** From (3.2) and (3.3) it follows that

\[
Q(\psi) = \cos^2 \psi \langle m^0, m^0 \rangle + \cos \psi \sin \psi \left( \langle m^0, n^0 \rangle + \langle n^0, m^0 \rangle \right) + \sin^2 \psi \langle n^0, n^0 \rangle,
\]

\[
R(\psi) = \cos \psi \sin \psi \left( \langle n^0, n^0 \rangle - \langle m^0, m^0 \rangle \right) + \cos^2 \psi \langle m^0, n^0 \rangle - \sin^2 \psi \langle n^0, m^0 \rangle
\]

and

\[
T(\psi) = \cos^2 \psi \langle n^0, n^0 \rangle - \cos \psi \sin \psi \left( \langle m^0, n^0 \rangle + \langle n^0, m^0 \rangle \right) + \sin^2 \psi \langle m^0, m^0 \rangle.
\]

Differentiating with respect to \( \psi \) we have

\[
Q'(\psi) = 2 \cos \psi \sin \psi \left( \langle n^0, n^0 \rangle - \langle m^0, m^0 \rangle \right)
\]

\[
+ \left( \cos^2 \psi - \sin^2 \psi \right) \left( \langle m^0, n^0 \rangle + \langle n^0, m^0 \rangle \right) = R(\psi) + R^T(\psi),
\]

\[
R'(\psi) = \left( \cos^2 \psi - \sin^2 \psi \right) \left( \langle n^0, n^0 \rangle - \langle m^0, m^0 \rangle \right)
\]

\[
- 2 \cos \psi \sin \psi \left( \langle m^0, n^0 \rangle + \langle n^0, m^0 \rangle \right)
\]

\[
= T(\psi) - Q(\psi) = (R^T)'(\psi),
\]

and similarly we have

\[
T'(\psi) = -(R(\psi) + R^T(\psi)).
\]

Moreover, since \( T^{-1}(\psi)T(\psi) = I \) (the \( 3 \times 3 \) unit matrix) implies that

\[
(T^{-1})'(\psi)T(\psi) + T^{-1}(\psi)T'(\psi) = 0,
\]

we have

\[
(T^{-1})'(\psi) = -T^{-1}(\psi)T'(\psi)T^{-1}(\psi) = T^{-1}(\psi)(R(\psi) + R^T(\psi))T^{-1}(\psi).
\]
Differentiating (3.4) and using the above relations leads to

\[ N'_1(\psi) = -(T^{-1})'(\psi)R^T(\psi) + T^{-1}(\psi)(R^T)'(\psi) \]
\[ = -T^{-1}(\psi)(R(\psi) + R^T(\psi))T^{-1}(\psi)R^T(\psi) + T^{-1}(\psi)Q(\psi) - I, \]

\[ N'_2(\psi) = T^{-1}(\psi)(R(\psi) + R^T(\psi))T^{-1}(\psi), \]

\[ N'_3(\psi) = R'(\psi)T^{-1}(\psi)R^T(\psi) + R(\psi)(T^{-1})'(\psi)R^T(\psi) \]
\[ + R(\psi)T^{-1}(\psi)(R^T)'(\psi) - Q'(\psi) \]
\[ = R(\psi)T^{-1}(\psi)(R(\psi) + R^T(\psi))T^{-1}(\psi)R^T(\psi) \]
\[ - Q(\psi)T^{-1}(\psi)R^T(\psi) - R(\psi)T^{-1}(\psi)Q(\psi), \]

and

\[ (N'_1)'(\psi) = (N'_1(\psi))^T \]
\[ = -R(\psi)T^{-1}(\psi)(R(\psi) + R^T(\psi))T^{-1}(\psi) + Q(\psi)T^{-1}(\psi) - I. \]

Thus we obtain each component of \( N'(\psi) \). Now using (3.4) we compute \(-I - N^2(\psi)\) directly. Then we easily check that each component of \(-I - N^2(\psi)\) is equal to that of \( N'(\psi) \). (For an alternative proof, refer to (9, p. 314).)

Associated with the above six cases, we state several lemmas.

**Lemma 3.3** (cf. (9, 20, 21)) Let \( p(\psi) \) be the solution to the Cauchy problem for the Riccati equation:

\[ p'(\psi) = -1 - p^2(\psi), \quad p(0) = p^0 \]  \hspace{1cm} (3.9)

with \( \text{Im} \ p^0 > 0 \). Define \( K(\psi) \) and \( m(\psi) \) by

\[ K(\psi) = 2 \int_{0}^{\psi} p(\psi') \, d\psi' \]

and

\[ m(\psi) = \int_{0}^{\psi} \exp[-K(\psi')] \, d\psi' \]

respectively. Then we have

\[ \int_{0}^{2\pi} p(\psi) \, d\psi = 2\pi i, \]  \hspace{1cm} (3.10)

\[ \int_{0}^{2\pi} \exp[-K(\psi)] \, d\psi = 0 \]  \hspace{1cm} (3.11)

and

\[ \int_{0}^{2\pi} \exp[-K(\psi)]m(\psi) \, d\psi = 0. \]  \hspace{1cm} (3.12)
Proof. From (3.9) we have \( p(\psi) = \tan (\psi_0 - \psi) \), where \( \psi_0 \) is a complex constant such that \( \tan \psi_0 = p^0 \), or we have \( p(\psi) = i \) with \( p^0 = i \). For the latter case, we can immediately obtain (3.10), (3.11) and (3.12). So we prove them for the former case. Put

\[ \psi_0 = \psi_1 + i\psi_2 \quad (\psi_1, \psi_2 \in \mathbb{R}). \]

Then we have

\[ p^0 = p(0) = \sin 2\psi_1 + i \sinh 2\psi_2 \]
\[ \quad \cos 2\psi_1 + \cosh 2\psi_2. \]

Noting that \( \text{Im} \psi^0 > 0 \) implies that \( \psi_2 > 0 \), we obtain

\[ \int_0^{2\pi} p(\psi) \, d\psi = \left[ \log \cos (\psi_0 - \psi) \right]_{\psi=0}^{\psi=2\pi} \]
\[ = \left[ \log \left\{ \cosh \psi_2 \cos (\psi - \psi_1) + i \sinh \psi_2 \sin (\psi - \psi_1) \right\} \right]_{\psi=0}^{\psi=2\pi} \]
\[ = 2\pi i. \]

This proves (3.10). Moreover we have

\[ \int_0^\phi p(\psi') \, d\psi' = \left[ \log \cos (\psi_0 - \psi') \right]_{\psi=0}^{\phi} = \log \frac{\cos (\psi_0 - \psi)}{\cos \psi_0} \]

and

\[ K(\psi) = \log \left( \frac{\cos (\psi_0 - \psi)}{\cos \psi_0} \right)^2. \]

Hence we obtain

\[ \int_0^{2\pi} \exp [-K(\psi)] \, d\psi = \int_0^{2\pi} \left[ \frac{\cos \psi_0}{\cos (\psi_0 - \psi)} \right]^2 \, d\psi \]
\[ = -\cos^2 \psi_0 \tan (\psi_0 - \psi) \]
\[ = 0. \]

This proves (3.11). Similarly we have

\[ m(\psi) = \int_0^\phi \exp [-K(\psi')] \, d\psi' = -\cos^2 \psi_0 \{ \tan (\psi_0 - \psi) - \tan \psi_0 \} \]

and

\[ \int_0^{2\pi} \exp [-K(\psi)] m(\psi) \, d\psi = \int_0^{\phi} \frac{\{m(\psi)\}^2}{2} \, d\psi = 0. \]

This proves (3.12).

**Lemma 3.4** Let \( \xi \) be an eigenvector of \( N(0) \) with eigenvalue \( p^0 \) satisfying \( \text{Im} \ p^0 > 0 \). Let \( p(\psi) \) be the unique solution to the Cauchy problem (3.9). Then

\[ [N(\psi) - p(\psi)I] \xi = 0. \]
Proof. Put \( \mathbf{h}(\psi) = [\mathbf{N}(\psi) - p(\psi)\mathbf{I}]\xi \). From (3.8) and (3.9),
\[
\mathbf{h}'(\psi) = -[\mathbf{N}(\psi) + p(\psi)\mathbf{I}]\mathbf{h}(\psi).
\]  
(3.13)

Hence \( \mathbf{h}(0) = [\mathbf{N}(0) - p^0\mathbf{I}]\xi = 0 \) and the uniqueness of the solution to the Cauchy problem for the equation (3.13) implies that \( \mathbf{h}(\psi) = 0 \).

From (3.10) and Lemma 3.4, (3.6) holds for each eigenvector associated with one of the eigenvalues of case 1 to case 6. Note that in cases 1, 2, 3 there are no generalized eigenvectors.

**Lemma 3.5** (cf. (20)) Let \( p^0 \) be an eigenvalue of \( \mathbf{N}(0) \) satisfying \( \text{Im} p^0 > 0 \). Let \( \xi_1, \xi_2 \) be vectors of the Jordan chain of height 2 with eigenvalue \( p^0 \). That is,
\[
[\mathbf{N}(0) - p^0\mathbf{I}]\xi_1 = 0, \quad [\mathbf{N}(0) - p^0\mathbf{I}]\xi_2 = \xi_1.
\]

Define \( \xi_2(\psi) \) as the unique solution to the Cauchy problem
\[
\xi_2'(\psi) = 2p(\psi)\xi_2(\psi), \quad \xi_2(0) = \xi_2,
\]  
(3.14)

where \( p(\psi) \) is the unique solution to the Cauchy problem (3.9). Then
\[
[\mathbf{N}(\psi) - p(\psi)\mathbf{I}]\xi_2(\psi) = \xi_1.
\]

**Proof.** Put \( \mathbf{m}(\psi) = [\mathbf{N}(\psi) - p(\psi)\mathbf{I}]\xi_2(\psi) - \xi_1 \). From (3.8), (3.9) and (3.14),
\[
\mathbf{m}'(\psi) = [-\mathbf{N}^2(\psi) - p^2(\psi)\mathbf{I} + 2p(\psi)\mathbf{N}(\psi)]\xi_2(\psi) = -[\mathbf{N}(\psi) - p(\psi)\mathbf{I}]^2\xi_2(\psi)
\]
\[
= -[\mathbf{N}(\psi) - p(\psi)\mathbf{I}][\mathbf{m}(\psi) + \xi_1],
\]
and from Lemma 3.4,
\[
\mathbf{m}'(\psi) = -[\mathbf{N}(\psi) - p(\psi)\mathbf{I}]\mathbf{m}(\psi).
\]

Hence \( \mathbf{m}(0) = [\mathbf{N}(0) - p^0\mathbf{I}]\xi_2 - \xi_1 = 0 \) and the uniqueness of the solution to the Cauchy problem for the above equation of \( \mathbf{m}(\psi) \) implies that \( \mathbf{m}(\psi) = 0 \).

**Lemma 3.6** (cf. (20)) Equation (3.6) holds for \( \xi_2 \) given in Lemma 3.5.

**Proof.** From (3.14) it follows that \( \xi_2(\psi) = \xi_2 \exp K(\psi) \). Hence from the above and Lemma 3.5 we have
\[
\mathbf{N}(\psi)\xi_2 = \exp [-K(\psi)]\mathbf{N}(\psi)\xi_2(\psi) = \exp [-K(\psi)](p(\psi)\xi_2(\psi) + \xi_1)
\]
\[
= p(\psi)\xi_2 + \exp [-K(\psi)]\xi_1.
\]

Thus making angular averages over the interval \([0, 2\pi]\) and combining with (3.10) and (3.11) we obtain (3.6).

From Lemma 3.6, (3.6) holds for each generalized eigenvector \( \xi_2 \) when
\{\xi_1, \xi_2\} is the Jordan chain of height 2 for case 4 and case 5. Note that these are the only cases for which a Jordan chain of height 2 exists.

**Lemma 3.7** (cf. (21)) Let \( p^0 \) be an eigenvalue of \( N(0) \) satisfying \( \text{Im} \ p^0 > 0 \). Let \( \xi_1, \xi_2, \xi_3 \) be vectors of the Jordan chain of height 3 with eigenvalue \( p^0 \). That is,

\[
[N(0) - p^0 I]\xi_1 = 0, \quad [N(0) - p^0 I]\xi_2 = \xi_1, \quad [N(0) - p^0 I]\xi_3 = \xi_2.
\]

Moreover let \( p(\psi) \) be the unique solution of the Cauchy problem (3.9). Define \( \xi_2(\psi) \) and \( \xi_3(\psi) \) as the unique solutions of the Cauchy problems

\[
\xi_2'(\psi) = 2p(\psi)\xi_2(\psi) - \xi_1, \quad \xi_2(0) = \xi_2
\]

and

\[
\xi_3'(\psi) = 4p(\psi)\xi_3(\psi), \quad \xi_3(0) = \xi_3
\]

respectively. Then we have

\[
[N(\psi) - p(\psi) I]\xi_2(\psi) = \xi_1
\]

and

\[
[N(\psi) - p(\psi) I]\xi_3(\psi) = \xi_2(\xi).
\]

**Proof.** Put

\[
m(\psi) = [N(\psi) - p(\psi) I]\xi_2(\psi) - \xi_1.
\]

From (3.8), (3.9) and (3.15),

\[
m'(\psi) = [-N^2(\psi) - p^2(\psi) I + 2p(\psi)N(\psi)]\xi_2(\psi) - [N(\psi) - p(\psi) I]\xi_1
\]

\[
= -[N(\psi) - p(\psi) I]^2\xi_2(\psi) - [N(\psi) - p(\psi) I]\xi_1,
\]

from (3.19),

\[
m'(\psi) = -[N(\psi) - p(\psi) I]m(\psi) - 2[N(\psi) - p(\psi) I]\xi_1,
\]

and from Lemma 3.4,

\[
m'(\psi) = -[N(\psi) - p(\psi) I]m(\psi).
\]

Hence \( m(0) = [N(0) - p^0 I]\xi_2 - \xi_1 = 0 \) and the uniqueness of the solution to the Cauchy problem for the above equation in \( m(\psi) \) implies that \( m(\psi) = 0 \). This proves (3.17). Now put

\[
m(\psi) = [N(\psi) - p(\psi) I]\xi_3(\psi) - \xi_2(\psi).
\]
From (3.8), (3.9), (3.15) and (3.16),
\[ n'(\psi) = [-N^2(\psi) + 4p(\psi)N(\psi) - 3p^2(\psi)I]\xi_3(\psi) - 2p(\psi)\xi_2(\psi) + \xi_1 \]
\[ = -[N(\psi) - 3p(\psi)I][N(\psi) - p(\psi)I]\xi_3(\psi) - 2p(\psi)\xi_2(\psi) + \xi_1, \]
from (3.20),
\[ n'(\psi) = -[N(\psi) - 3p(\psi)I]n(\psi) - [N(\psi) - p(\psi)I]\xi_2(\psi) + \xi_1, \]
and from (3.17),
\[ n'(\psi) = -[N(\psi) - 3p(\psi)I]n(\psi). \]

Hence \( n(0) = [N(0) - p^0I]\xi_3 - \xi_2 = 0 \) and the uniqueness of the solution to the Cauchy problem for the above equation of \( n(\psi) \) implies that \( n(\psi) = 0 \). This proves (3.18).

**Lemma 3.8 (cf. (21)).** Equation (3.6) holds for \( \xi_2, \xi_3 \) given in Lemma 3.7.

**Proof.** From (3.15) and (3.16) it follows that
\[ \xi_2(\psi) = [-\xi_1m(\psi) + \xi_2]\exp K(\psi) \quad (3.21) \]
and
\[ \xi_3(\psi) = \xi_3\exp 2K(\psi). \quad (3.22) \]

Hence from (3.21) we have
\[ N(\psi)\xi_2 = N(\psi)[\exp\{-K(\psi)\}\xi_2(\psi) + m(\psi)\xi_1], \]
from (3.17) and Lemma 3.4,
\[ N(\psi)\xi_2 = \exp\{-K(\psi)\}[p(\psi)\xi_2(\psi) + \xi_1] + m(\psi)p(\psi)\xi_1, \]
and from (3.21),
\[ N(\psi)\xi_2 = p(\psi)\xi_2 + \exp\{-K(\psi)\}\xi_1. \]

Then as in the proof of Lemma 3.6, combining with (3.10) and (3.11) we obtain (3.6) for \( \xi_2 \). Now from (3.22) and (3.18) we have
\[ N(\psi)\xi_3 = \exp\{-2K(\psi)\}N(\psi)\xi_3(\psi) = \exp\{-2K(\psi)\}[p(\psi)\xi_3(\psi) + \xi_2(\psi)] \]
\[ = p(\psi)\xi_3 + \exp\{-2K(\psi)\}\xi_2(\psi), \]
and from (3.21),
\[ N(\psi)\xi_3 = p(\psi)\xi_3 + \exp\{-K(\psi)\}[-m(\psi)\xi_1 + \xi_2]. \]

Then making angular averages over the interval \([0, 2\pi]\) and combining with (3.10), (3.11) and (3.12) we obtain (3.6) for \( \xi_3 \).
The Jordan chain of height 3 appears only in case 6. So (3.6) holds for each of the generalised eigenvectors \( \xi_2, \xi_3 \) if \( \{\xi_1, \xi_2, \xi_3\} \) is the Jordan chain of height 3 for case 6.

Therefore we have proved Lemma 3.1 for all possible eigenvectors and generalized eigenvectors in the Jordan chain arising in case 1 to case 6.

As a final remark, we point out that the above arguments show that the structure of the Jordan chains remains invariant while \( \psi \) changes.

4. Fundamental solution for transversely isotropic elasticity

The explicit closed formula for the fundamental solution

\[
G(x) = (G_{ij}(x) : i \downarrow 1, 2, 3; j \rightarrow 1, 2, 3)
\]

for transversely isotropic media is given in this section. Assume that (A.1') and (A.2) hold. Let the \( x_3 \)-axis be the axis of rotational symmetry and put

\[
A = C_{1111} = C_{2222}, \quad C = C_{3333}, \quad F = C_{1133} = C_{2333},
\]

\[
L = C_{1313} = C_{2323}, \quad N = C_{1122}, \quad C_{1212} = \frac{1}{2}(L - C_{1111} - C_{1122}).
\]

Let \( x \neq 0 \) and let

\[
x / |x| = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)
\]

in terms of the polar coordinates \((r, \varphi, \theta)(0 \leq \varphi \leq \pi, 0 \leq \theta < 2\pi)\). Define \( K, G, H, D, \Delta, D' \) by

\[
K = \left( \cos^2 \varphi + \frac{2L}{A - N} \sin^2 \varphi \right)^{\frac{1}{2}},
\]

\[
G = \frac{\left(2AL \cos^2 \varphi + (AC - F^2 - 2FL) \sin^2 \varphi + 2\sqrt{A(L)}, G = \frac{1}{2} \Delta \right)^{\frac{1}{2}}}{AL},
\]

\[
H = \left( \frac{\Delta}{AL} \right)^{\frac{1}{2}}, \quad D = (A \cos^2 \varphi + L \sin^2 \varphi + AH)K - AG \cos^2 \varphi,
\]

\[
\Delta = AL \cos^4 \varphi + (AC - F^2 - 2FL) \cos^2 \varphi \sin^2 \varphi + CL \sin^4 \varphi,
\]

\[
D' = \{AGHKL - (AC - F^2 - 2FL) \cos^2 \varphi \sin^2 \varphi \} (\cos^2 \varphi + H)
\]

\[
- AL \cos^2 \varphi (\cos^2 \varphi + H)^2 + L \sin^2 \varphi (GHKL - C \cos^2 \varphi \sin^2 \varphi).
\]

Then we have

\[
G(x) = \frac{1}{4\pi |x|} S_2(\varphi, \theta),
\]

\[
S_2(\varphi, \theta) = S_2^T(\varphi, \theta) = (S_{ij}(\varphi, \theta) : i \downarrow 1, 2, 3; j \rightarrow 1, 2, 3),
\]
where
\[
S_{11} = \left[ \frac{\sin^2 \theta}{AGKL \sin^2 \phi} + \frac{\cos^2 \theta}{ALD' \sin^2 \phi} \right] \left\{ AL(GHK - \cos^4 \phi - H \cos^2 \phi) \\
- (AC - (F + L)^2) \cos^2 \phi \sin^2 \phi \right\} D,
\]
\[
S_{12} = \left[ \frac{-1}{AGKL \sin^2 \phi} + \frac{1}{ALD' \sin^2 \phi} \right] \left\{ AL(GHK - \cos^4 \phi - H \cos^2 \phi) \\
- (AC - (F + L)^2) \cos^2 \phi \sin^2 \phi \right\} D \cos \theta \sin \theta,
\]
\[
S_{13} = \frac{(F + L)D}{AD'} \cos \phi \sin \phi \cos \theta,
\]
\[
S_{22} = \left[ \frac{\cos^2 \theta}{AGKL \sin^2 \phi} + \frac{\sin^2 \theta}{ALD' \sin^2 \phi} \right] \left\{ AL(GHK - \cos^4 \phi - H \cos^2 \phi) \\
- (AC - (F + L)^2) \cos^2 \phi \sin^2 \phi \right\} D,
\]
\[
S_{23} = \frac{(F + L)D}{AD'} \cos \phi \sin \phi \sin \theta,
\]
\[
S_{33} = \frac{D}{AD'} (A \cos^2 \phi + L \sin^2 \phi + AH).
\]

In the case of \( \sin \phi = 0 \), \( S_{ij} \) \( (1 \leq i, j \leq 3) \) are obtained by taking the limit as \( \sin \phi \to 0 \) in the above formulae.

According to Tanuma (22), cases 1, 2, 4, 5 in (3.7) are the only cases which appear for transversely isotropic media, and the conditions of these cases are given respectively by

- case 5: \( \phi = 0, \pi \) or \( \sqrt{(AC) - F - 2L} = 0 \) and \( \frac{2L}{A - N} = \left( \frac{C}{A} \right)^{\frac{1}{2}} \),
- case 4: \( \phi \neq 0, \pi \) and \( \sqrt{(AC) - F - 2L} = 0 \) and \( \frac{2L}{A - N} \neq \left( \frac{C}{A} \right)^{\frac{1}{2}} \),
- case 2: \( \phi \neq 0, \pi \) and \( \sqrt{(AC) - F - 2L} \neq 0 \) and
  \[
  AL\left( \frac{2L}{A - N} \right)^2 - (AC - F^2 - 2FL) \frac{2L}{A - N} + CL = 0,
  \]
- case 1: all other conditions.

The eigenvectors and the generalized eigenvectors of the above cases are given explicitly and the surface impedance tensor \(-iLA^{-1}\) can be obtained...
in a unified expression (see (22)). Hence the result in this section follows from Theorem 2.1.

For transversely isotropic media, several efforts have been made to obtain an explicit form of \( G(x) \). The direct evaluation of the integral formula (3.1) was attempted by (1, 17, 19). The formulae in (1, 17) are written in closed forms. However, they are not written in a unified form, that is, they are expressed in different forms depending on the conditions of the elastic tensor, and no consideration is given to the degeneracy depending on the conditions of the direction of \( x \). Willis (19) obtained the formula after reducing the integral (3.1) to the contour integral in the complex plane and computing its residue. However, the cases when the integrand has the multiple poles are not thoroughly discussed. Kröner (16) gave a formula for \( G(x) \) which includes the roots of a certain quadratic equation but the cases where this quadratic equation has multiple roots are not considered. Pan and Chou (18) constructed the potential functions which give the displacements vanishing at infinity like \( O(|x|^{-1}) \) and blowing up at the origin like \( O(|x|^{-1}) \) and give the stresses whose resultant force on an infinitesimally small sphere centred at the origin is equal to the point force at the origin. However, the displacement field in (18) is expressed in different forms depending on whether the quantity \( V(AC) - F - 2L \) becomes zero or not. Moreover, the method in (18) is applicable only for transversely isotropic media, because the forms of the potential functions are typical of these media. It will be difficult to extend their method to universal treatment for general anisotropic elasticity. We also see the potential-functions approach in (15, 16).

Our computations using the formula (2.2) are straightforward and the result in this section is given in a unified expression which is generally valid regardless of whether the eigenvalues are distinct or not. Note that the multiplicity of the poles in the integrand in (19) and the multiplicity of the roots of the quadratic equation in (16) are equivalent to the multiplicity of the eigenvalues of Stroh's eigenvalue problem.

Note. Since writing this paper we have received a preprint from T. C. T. Ting and V. G. Lee where the formula for the fundamental solution of anisotropic elasticity is given explicitly in terms of Stroh's eigenvalues.

REFERENCES

7. —— and ——, ibid. 52 (1972) 45–54.
In this Appendix we prove Theorem 3.1 for its importance and for the sake of completeness of our argument. Put

\[ M(\xi) = \left( (\xi, \xi)_k = \sum_{j=1}^{3} C_{ijk}\xi_1\xi_2\xi_3 : i = 1, 2, 3; k \to 1, 2, 3 \right) \]

for \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \). From (1.1) the Fourier transform \( \hat{G}(\xi) \) of the tempered distribution \( G(x) \) satisfies

\[ -M(\xi)\hat{G}(\xi) + I = 0, \]

where \( I \) is the \( 3 \times 3 \) unit matrix. Then by the Fourier inversion formula, \( G(x) \) is given as the inverse Fourier transform of \( \hat{G}(\xi) \). Since \( M(\xi)^{-1} \in \mathcal{C}^0(\mathbb{R}^3 - \{0\}) \) is homogeneous of degree \(-2\), it is well known that its inverse Fourier transform

\[ G(x) = \tilde{F}[M(\xi)^{-1}](x) \in \mathcal{C}^0(\mathbb{R}^3 - \{0\}) \]

is homogeneous of degree \(-1\) (see Mizohata (23)). Also it is easy to see that

\[ G(x) = Os - (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\xi \cdot x} M(\xi)^{-1} d\xi, \quad (A.1) \]

where \( Os - (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\xi \cdot x} M(\xi)^{-1} d\xi \) is given by

\[ Os - (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\xi \cdot x} M(\xi)^{-1} d\xi = \lim_{\epsilon \to 0} (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\xi \cdot x} \chi(\epsilon \xi) M(\xi)^{-1} d\xi \]

which is independent of the choice of \( \chi \in \mathcal{S}(\mathbb{R}^3) \) with \( \chi(0) = 1 \) and converges in \( \mathcal{S}'(\mathbb{R}^3) \).

In order to compute \( G(x) \) we take \( \chi \in \mathcal{S}(\mathbb{R}^3) \) such that \( \chi(x) = h(|x|) \) with an even function \( h \in \mathcal{S}(\mathbb{R}) \) satisfying

\[ h(0) = 1, \quad \tilde{h} = \tilde{F}[h] \in \mathcal{C}_0^\infty(\mathbb{R}) \]
and $\tilde{h}$ is real-valued. Fix $x \neq 0$ and make a change of variables $(x, \xi) \to (z, \eta)$ in (A.1) given by $x = |x| z$, $\eta = |x| \xi$. Then we have

$$G(x) = (2\pi)^{-3} \frac{1}{|x|} \lim_{|\eta| \to 0} \int_{A_0} e^{i(\xi \cdot \eta)^2} \chi(\eta |x|^{-1} \eta) |\eta|^{-2} M(\eta^0)^{-1} d\eta,$$

where $\eta^0 = |\eta|^{-1} \eta$. Introduce polar coordinates $(|\eta|, \sigma, \psi)(0 \leq \sigma \leq \pi, 0 \leq \psi < 2\pi)$ such that $z \cdot \eta^0 = \cos \sigma$. Then we have

$$G(x) = (2\pi)^{-3} \frac{1}{|x|} \lim \int_{A_0} d\sigma \int_{0}^{\infty} d|\eta| e^{i|\eta| \cos \sigma} h(\eta |x|^{-1} \eta) M(\eta^0)^{-1} \sin \sigma.$$

Put

$$H_\epsilon(x, \psi) = \int_{0}^{\infty} d|\eta| e^{i|\eta| \cos \sigma} h(\epsilon |x|^{-1} \eta) M(\eta^0)^{-1} \sin \sigma.$$

Clearly

$$H_\epsilon(x, \psi) = \frac{1}{2} \text{Re} \left\{ \int_{0}^{\infty} d|\eta| e^{i|\eta| \cos \sigma} h(\epsilon |x|^{-1} \eta) M(\eta^0)^{-1} \sin \sigma \right\}.$$

Make the change of variable $\sigma \to \lambda$ given by $\cos \sigma = \lambda$. Then we have

$$H_\epsilon(x, \psi) = \frac{1}{2} \text{Re} \left\{ \int_{-1}^{1} d\lambda \int_{-\infty}^{\infty} d\tau e^{i\tau \lambda} h(\epsilon |x|^{-1} \tau) M(\lambda) \right\}.$$

Put $\delta = \epsilon |x|^{-1}$ and observe that

$$\text{Re} \left\{ (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\tau \lambda} h(\epsilon |x|^{-1} \tau) d\tau \right\} = \frac{1}{\delta} \tilde{h} \left( \frac{\lambda}{\delta} \right)$$

is a mollifier. Hence

$$H_\epsilon(x, \psi) = \frac{1}{\delta} \int_{-1}^{1} h \left( \frac{\lambda}{\delta} \right) M(\eta^0)^{-1} d\lambda$$

converges uniformly to

$$\pi M(\eta^0)^{-1} \big|_{\sigma = \frac{1}{2\pi}} = \pi M(\eta^0(\psi))^{-1} = \pi(\eta^0(\psi), \eta^0(\psi))^{-1}$$

as $\epsilon \downarrow 0$. Therefore

$$G(x) = \frac{1}{8\pi^2 |x|} \int_{0}^{2\pi} \langle \eta^0(\psi), \eta^0(\psi) \rangle^{-1} d\psi.$$