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Gauge Transformations of Propagators in the Spontaneously Broken Gauge Theory

Tadashi WATANABE*

Department of Physics, University of Tokyo, Tokyo

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The effect of gauge transformations on Green's functions is studied systematically for the spontaneously broken gauge theory—the Abelian Higgs model. Starting from the quantization of massive vector field in the Coulomb gauge, it is shown that the propagator of photon type is transformed into all kinds of relativistic gauges. Next, by recomposing field variables, we obtain Green's functions in the unitary and/or the renormalizable gauges which depend on a gauge function $M(x-y)$ introduced.

§ 1. Introduction

The renormalizable theory of massive vector meson has recently been applied to unifying the electromagnetic and the weak interactions.\(^1\),\(^2\) The theory contains a phenomenon that the massless vector meson of the Yang-Mills\(^3\) type guaranteeing the gauge invariance of original Lagrangian acquires the mass through the breaking of the symmetry involved. The $S$ matrices and Green's functions of the theory are obtained usually in two gauges, the unitary and the renormalizable ones, by making use of the functional integration technique.\(^3\) It may safely be said, however, not only that the technique is rather transcendental but that it does not sufficiently clarify the connection between the two gauges.

As is well-known, the longitudinal component of vector meson plays an important role in the gauge theory. Roughly speaking, the difference between various gauges depends on the way how to deal with that component. In the Coulomb gauge the longitudinal part is an arbitrary $c$-number, while in relativistic gauges that part is connected closely with the scalar component. Then, in the unitary gauge the Goldstone boson is absorbed into the longitudinal state of massive vector meson, without disturbing the transverse state, so as to disappear from the scene. On the other hand we are forced to treat the two bosons as independent in the renormalizable gauge.

In this paper the gauge transformation and its effect on propagators are studied for the Higgs model\(^4\) on the basis of the canonical formalism by employing the method of functional derivatives. The method makes use of a generating functional $\mathcal{T}$ from which all propagators, i.e., Green's functions, can be obtained by functional differentiation. A set of differential equations satisfied by the func-

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* The present address is Department of Physics, Kobe University, Kobe.
tional is investigated exhaustively. The gauge ambiguity of the propagators will be shown to arise from the ambiguity in the set of equations. This is an approach, used first by Zumino and Bialynicki-Birula, to study the problem of gauge invariance in the spinor electrodynamics.

The Heisenberg equations in the Coulomb gauge provide a complete basis for the gauge theory. The canonical quantization of the Higgs model in this gauge is described in § 2. It is also reviewed in what manner the gauge ambiguity is related to the conservation of generalized current. In § 3 a set of differential equations for the generating functional \( \mathcal{F} \) is derived in the Lorentz frame chosen to define the Coulomb gauge. Then, following Zumino, we extend the definition of the functional to more general gauges, characterized by an operator four-vector \( a_\mu \). A suitable choice of \( a_\mu \) gives the Landau gauge, in which the propagator of vector meson is written down in the lowest order,

\[
    D_{\mu \nu} = \left[ \delta_{\mu \nu} - \partial_\mu \partial_\nu / \partial^2 \right] D_F
\]

(1.1)

with the Feynman function \( D_F(x-y) = -i(\partial^2 - iz)^{-1}\partial^4(x-y) \). A further generalization is possible by introducing gauges depending in addition upon a function \( M \). We shall obtain in this manner all gauges where the zero-order propagator has the form

\[
    D_{\mu \nu} + i\partial_\rho \partial_\nu M.
\]

(1.2)

The propagators of photon type (1.1) and (1.2) appear because of the fact that the bilinear term of interaction is still left. In § 4 we recompose field variables in order to decouple the bilinear term above. Such a procedure is performed through the transformation of the external sources. The set of differential equations reformed thus is nothing but one obtained directly by using the Stueckelberg formalism for Higgs' modified Lagrangian (Eq. (21) in Ref. 4)), which is known as a formulation in the unitary gauge. The vector and scalar fields of Stueckelberg are combined to produce the following propagator in the lowest order:

\[
    D_{\mu \nu}(m_1^2) = \left[ \delta_{\mu \nu} - \partial_\mu \partial_\nu / m_1^2 \right] D_F(m_1^2).
\]

(1.3)

Next the generating functional is extended to more general gauges including the renormalizable one, in which the vector field has the zero-order propagator of the form

\[
    \delta_\mu \nu D_F(m_1^2) + i\partial_\rho \partial_\nu M
\]

(1.4)

with a gauge function \( M \) introduced after the way of the preceding section. On the other hand, Stueckelberg's scalar field is treated separately and its propagator is written as

\[
    D_F(m_1^2) - im_1^2 M.
\]

(1.5)
Gauge Transformations of Propagators

Whether the propagator (1.5) is of the massless field depends on what function \( M \) is chosen.

\[ \text{§ 2. The Higgs model in the Coulomb gauge} \]

The Lagrangian density from which we shall work is of the Higgs model,\(^9\)
\[ \mathcal{L} = -\frac{1}{4} (F_{\mu\nu})^2 - \frac{1}{2} (D_\mu \phi_a)^2 + \frac{1}{2} m_0^2 (\phi_a)^2 - \frac{1}{2} f^2 (\Phi_a)^2, \tag{2.1} \]
where
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]
\[ D_\mu \phi_a = \partial_\mu \phi_a - e A_\mu \varepsilon_{ab} \Phi_b, \quad \textasteriskcentered a, b = 1, 2. \]

This dynamics is enough for us to show how to study the effect of gauge transformations on propagators, though the symmetry breaking down here is the simplest \( U(1) \)-group. The method exhibited in this paper would be applied straightforwardly to gauge theories with non-Abelian groups and, if necessary, to theories in which spinor fields take part moreover.

We shall use only the Heisenberg operators. The symmetry involved in the Lagrangian (2.1) breaks down so that \( \phi_a = \phi \) and \( \Phi_a = \chi + \eta \) with \( \langle \phi \rangle_0 = \eta = m_0/f \). The Hamiltonian in the Coulomb gauge is written as
\[ \mathcal{H} = \frac{1}{2} (\mathbf{F}^2 + \mathbf{H}^2) + \frac{1}{2} (\Pi_\phi)^2 + \frac{1}{2} (\Pi_\chi + \eta)^2 + \frac{1}{2} m_0^2 \chi^2 + e A \{ \phi \cdot (\mathbf{F} \chi) - \mathbf{F} \phi \cdot (\chi + \eta)\} + \frac{1}{2} e^2 A^2 \{ \phi^3 + (\chi + \eta)^3 \} + F(\phi, \chi), \tag{2.2} \]
where
\[ F(\phi, \chi) = \frac{1}{2} f m_0 \chi (\phi^2 + \chi^2) + \frac{1}{8} f^2 \phi^3 + \frac{1}{8} \chi^3 - \frac{1}{8} m_0^2. \]

In analogy to the electrodynamics the field strengths \( E_j = i F_{jk} \) and \( H_j = \frac{1}{2} \varepsilon_{jkl} F_{kl} \) are introduced. The canonical conjugate momenta of charged scalar fields are given by
\[ \Pi_\phi = \partial + e A \cdot (\chi + \eta) \]
and
\[ \Pi_\chi = \partial - e A \cdot \phi. \tag{2.3} \]

From the canonical commutation relations the commutators of fields \( A_\mu, \phi \) and \( \chi \) can be deduced for the equal time as follows:
\[ [E_\mu(x), A_\nu(y)] = i (\partial_\mu - \partial_\nu \partial_\lambda F^{-1}) \delta^\lambda(x - y), \tag{2.4} \]
\[ [\phi(x), \phi(y)] = i \delta^3(x - y) - ie^2 (\chi + \eta) F^{-1} \{ (\chi + \eta) \delta^3(x - y) \}, \tag{2.5a} \]
\[ [\chi(x), \chi(y)] = i \delta^3(x - y) - ie^3 \phi F^{-1} \{ \phi \delta^3(x - y) \}, \tag{2.5b} \]

\( \text{\footnotesize *) Here, } \varepsilon_{ab} \text{ is the totally antisymmetric 2nd rank tensor.} \)
\[ [\phi(x), \chi(y)] = -ie^2(\chi + \eta)e^{-1}\{\phi\delta^a(x - y)\}, \quad (2.6a) \]

\[ [\psi(x), \phi(y)] = -ie^2\psi e^{-1}\{(\chi + \eta)\delta^a(x - y)\}, \quad (2.6b) \]

\[ [\phi(x), A_\theta(y)] = ie^{-1}\{(\chi + \eta)\psi(x - y)\}, \quad (2.7a) \]

\[ [\chi(x), A_\theta(y)] = -ie^{-1}\{\phi\psi(x - y)\}, \quad (2.7b) \]

\[ [\phi(x), A_\theta(y)] = -ie^{-1}\{i\lambda\delta^a(x - y)\} + ie^2A_\theta e^{-1}\{\phi\delta^a(x - y)\}, \quad (2.8a) \]

and

\[ [\chi(x), A_\theta(y)] = -ie^{-1}\{i\lambda\delta^a(x - y)\} + ie^2A_\theta e^{-1}\{(\chi + \eta)\delta^a(x - y)\}, \quad (2.8b) \]

while other commutators are trivial or unnecessary.\(^4\)

Since the longitudinal part of \( A \) commutes with all other operators, it can be taken as an arbitrary \( c \)-number \( \lambda(x) \). Therefore it follows that

\[ A = A^{tr} + \psi \lambda, \quad (2.9) \]

\[ E = E^{tr} - \psi V, \quad (2.10a) \]

\[ V = -\psi^{-1} \rho, \quad (2.10b) \]

\[ \rho = -e\{\Pi_\phi(\chi + \eta) - \phi \cdot \Pi_\lambda\} \quad (2.10c) \]

and

\[ j_\lambda = e\phi\bar{\psi}(\chi + \eta) + e^2A_\lambda\{\phi + (\chi + \eta)\}. \quad (2.11) \]

The foregoing Hamiltonian and commutation relations give rise to equations of motion

\[ \frac{\partial}{\partial t} E^{tr} = \text{curl } H + J^{tr}, \quad (2.12) \]

\[ \frac{\partial}{\partial t} A^{tr} = -E^{tr}, \quad (2.13) \]

\[ \Delta \phi - e^2A_\mu^2\phi - eA_\mu \partial_\mu \chi - e\partial_\mu \{A_\mu \cdot (\chi + \eta)\} - \frac{\partial F}{\partial \phi} = 0 \quad (2.14a) \]

and

\[ (\Delta - m^2) \chi - e^2A_\mu^2(\chi + \eta) + eA_\mu \partial_\mu \phi + e\partial_\mu \{A_\mu \cdot \phi\} - \frac{\partial F}{\partial \chi} = 0. \quad (2.14b) \]

Next we proceed to study the invariance of the theory in the Coulomb gauge. By setting

\[ A_\theta = V - \frac{\partial \lambda}{\partial t}, \quad (2.15) \]

\(^4\) By identifying \( m \) with \( \epsilon \eta \), one can easily see that our commutators coincide with those given by Higgs in the interaction representation, namely Eqs. (9), (10) and (11) in Ref. 4.
we can write as
\[ E = -\frac{\partial}{\partial t} A - \nabla A_\alpha. \quad (2.16) \]

The set of equations (2.12) ~ (2.14b) is found to be invariant under the c-number gauge transformation,
\[
\begin{align*}
\lambda(x) &\rightarrow \lambda(x) + \tilde{\lambda}(x), \\
A_\mu &\rightarrow A_\mu + \partial_\mu \tilde{\lambda}, \\
\phi &\rightarrow \phi \cos(e\tilde{\lambda}) + (\chi + \eta) \sin(e\tilde{\lambda}), \\
\chi + \eta &\rightarrow -\phi \sin(e\tilde{\lambda}) + (\chi + \eta) \cos(e\tilde{\lambda}).
\end{align*}
\quad (2.17)
\]

The choice \(\lambda = 0\), for instance, fixes the gauge to be transverse Coulomb one. In order to define those operators that cause the gauge transformation, however, it is imperative to leave the gauge function \(\lambda(x)\) arbitrary.

We consider the generating functional, with \(J_\mu\), \(K_1\) and \(K_2\) external sources,
\[
\mathcal{T}[J_\mu, K_1, K_2] = \left\langle T \exp \left[ -i \int \{ A_\mu J_\mu + \phi K_1 + (\chi + \eta) K_2 \} d^4x \right] \right\rangle \\
\times \exp \frac{-i}{2} \int J_\mu \nabla^\alpha J_\alpha d^4x. \\
\quad (2.18)
\]

Actually this expression of \(\mathcal{T}\) is somewhat redundant. First, Green's functions are indirectly obtained through the recurrent procedure when the field \(\chi\) takes part in. Second, \(A_4\) is expressed in terms of charged scalar fields through the relations (2.10b), (2.10c) and (2.15). The form of (2.18), however, is more convenient for the purpose of studying the transformation properties. It is worth while to mention that the gauge ambiguity of \(\mathcal{T}\) and, therefore, of propagators is closely connected with the necessity of extending the definition of \(\mathcal{T}\) to unphysical values of \(J_\mu\) not satisfying a continuity equation like \(\partial_\mu J_\mu = 0\).

The dependence of \(\mathcal{T}\) on the gauge function \(\lambda\) is illustrated due to Eq. (2.17) as
\[
\mathcal{T}_\lambda[J_\mu, K_1, K_2] = \mathcal{T}_{\lambda=0}[J_\mu, \cos(e\lambda) K_1 - \sin(e\lambda) K_2, \sin(e\lambda) K_1 + \cos(e\lambda) K_2] \\
+ \cos(e\lambda) K_2] \times \exp -i \int J_\mu \partial_\mu \lambda d^4x, \\
\quad (2.19a)
\]

of which differential form is
\[
-i \frac{\partial}{\partial \lambda} \mathcal{T}_\lambda = \left\{ \partial_\mu J_\mu + e \left( K_\mu i \frac{\partial}{\partial K_\mu} - K_\mu i \frac{\partial}{\partial K_\mu} \right) \right\} \mathcal{T}_\lambda. \\
\quad (2.19b)
\]

§ 3. Transformation properties of the generating functional

In this section we shall investigate how to extend the functional \(\mathcal{T}\) into relativistic gauges in a most general manner, which is in parallel with that of
Zumino\textsuperscript{6} or Bialynicki-Birula\textsuperscript{7} on spinor electrodynamics. The contents would be very instructive when we consider massive gauges in the next section.

By using the definition of $\mathcal{T}$, (2.18), the commutation relations and the equation of motion for the Heisenberg operators, it is shown that the generating functional satisfies the following differential equations:

$$\left\{ \partial_{\nu} \left( \partial_{\mu} - \partial_{\nu} \frac{\partial}{\partial J_{\nu}} \right) - (\partial_{\mu} + a_{\nu} \partial_{\nu}) (i e + J_{\nu}) \right\} \mathcal{T} = 0 , \quad (3.1)$$

$$\left\{ \partial_{\nu} \left( \partial_{\mu} - e^{2} (i \frac{\partial}{\partial J_{\mu}}) \frac{\partial}{\partial K_{1}} - e \frac{\partial}{\partial J_{\mu}} \frac{\partial}{\partial K_{1}} - e \frac{\partial}{\partial J_{\mu}} \frac{\partial}{\partial K_{1}} \right) - \left[ \frac{\partial}{\partial \varphi} - K_{1} \right] \mathcal{T} = 0 , \quad (3.2)$$

$$\left\{ \partial_{\nu} \left( \frac{\partial}{\partial J_{\mu}} - e^{2} (i \frac{\partial}{\partial J_{\mu}}) \frac{\partial}{\partial K_{1}} + e \frac{\partial}{\partial J_{\mu}} \frac{\partial}{\partial K_{1}} + e \frac{\partial}{\partial J_{\mu}} \frac{\partial}{\partial K_{1}} \right) - \left[ \frac{\partial}{\partial (\chi + \eta)} - K_{1} \right] \mathcal{T} = 0 \quad (3.3)$$

and

$$\left\{ a_{\mu} \frac{\partial}{\partial J_{\mu}} + \lambda \right\} \mathcal{T} = 0 , \quad (3.4)$$

where

$$j_{\mu} = e i \frac{\delta}{\delta K_{1}} i \frac{\partial}{\partial J_{\mu}} + e i \frac{\partial}{\partial J_{\mu}} \left( i \frac{\partial}{\partial K_{1}} \right)^{2} \quad (3.5)$$

and

$$\tilde{F}(\phi, \chi + \eta) = F(\phi, \chi) + \frac{1}{2} m_{0}^{2} \chi^{2}$$

$$= \frac{\epsilon^{2}}{8} (\phi^{2} + (\chi + \eta)^{2}) - \frac{m_{0}^{2}}{4} \left( \phi^{2} + (\chi + \eta)^{2} \right) . \quad (3.6)$$

Moreover the $A_{\mu}$, $\phi$ and $\chi + \eta$ in the square brackets of Eqs. (3.2) and (3.3) should be replaced respectively by $i \phi/\partial J_{\mu}$, $i \phi/\partial K_{1}$ and $i \phi/\partial K_{1}$. In the above a vector $a_{\mu}$ is also introduced, which stands for an operator,

$$a_{\mu} = - \partial_{\mu} F^{-1} , \quad a_{4} = 0 . \quad (3.7)$$

The form of $a_{\mu}$ in (3.7) results from the Lorentz frame chosen to define the Coulomb gauge. More generally we can consider operator vectors $a_{\mu}$ satisfying

$$\partial_{\mu} a_{\mu} = -1 . \quad (3.8)$$

An alternative form of interest, which leads to the Landau gauge, is the limit as $\varepsilon$ tends to zero of

$$a_{\mu} = \partial_{\mu} [(\partial^{2} - i \varepsilon)^{-1}] . \quad (3.9)$$
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The different choices of $a_\mu$ give rise to different functionals $\mathcal{T}$, which are not obtained by changes in the gauge function $\lambda$. Thus, a change $\delta a_\mu$ of $a_\mu$, which preserves Eq. (3.8),

$$\partial_\mu (\delta a_\mu) = 0 \quad (3.10)$$

can be considered as a generalized type of gauge transformation and the various choices of $a_\mu$ result in physically equivalent formulations of the theory.

The change in $\mathcal{T}$ caused by an infinitesimal change $\delta a_\mu$ in $a_\mu$ is written as

$$\delta \mathcal{T} = \int \delta a_\mu(x) \frac{\delta}{\delta J_\mu(x)} \frac{\delta}{\delta \lambda(x)} \delta^4 x \cdot \mathcal{T}. \quad (3.11)$$

With this variation (3.11) it can be shown that the functional differential equations (3.1) ~ (3.4) remain invariant. When the functional $\mathcal{T}$ does not change for the change of the gauge function $\lambda$, it is not affected either by the change $\delta a_\mu$ of $a_\mu$. Now, from Eqs. (3.2) and (3.3) the following equation is derived:

$$\left\{ \partial_\mu j_\mu + e \left( K_i \partial_\mu K_i - K_i \partial K_i \right) \right\} \mathcal{T} = 0. \quad (3.12)$$

By combining Eqs. (3.12) and (2.19b), we can prove that the variation with respect to the gauge function $\lambda$ is simply related to the generalized current conservation,

$$-i \frac{\delta \mathcal{T}}{\delta \lambda} = (\partial_\mu j_\mu + \partial_\mu a_\mu) \mathcal{T}. \quad (3.13)$$

So far we have considered the generating functional in various gauges specified by different choices of $\lambda$ and $a_\mu$. More general types of gauges can also be considered. By making use of $\delta M(x-y)$ which is an arbitrary infinitesimal function even in $x-y$, a new type of gauge transformation is realized as

$$\delta \mathcal{T} = -\frac{i}{2} \int \frac{\delta}{\delta \lambda(x)} \delta M(x-y) \frac{\delta}{\delta \lambda(y)} \delta^4 x \delta^4 y \mathcal{T}. \quad (3.14a)$$

Or, instead of the differential form, we can relate the new functional to the old one through a unitary transformation,

$$\mathcal{T}' = U \mathcal{T}, \quad (3.14b)$$

where

$$U = \exp \left( -\frac{i}{2} \int \frac{\delta}{\delta \lambda} (\delta M) \frac{\delta}{\delta \lambda} \delta^4 x \delta^4 y. \right) \quad (3.14c)$$

The generating functional is now considered as dependent upon a new function $M(x-y)$. A functional $\mathcal{O}$, which is invariant under the original c-number gauge transformations (2.17), will not also be affected by the change (3.14).

Then, we exhibit the effect of the change (3.14) on propagators of photon type. By putting $K_i = K_1 = 0$, we see from Eq. (2.20) that in a finite form
$$T = \exp \frac{i}{2} \int \int \partial_\mu J_\mu (\delta M) \partial_\nu J_\nu d^4 x d^4 y \cdot \Delta.$$  
(3.15)

From this modified functional the exact propagator for vector meson is derived as

$$\Pi_{\mu \nu} = G_{\mu \nu} + i \partial_\mu \partial_\nu (\delta M),$$  
(3.16)

where

$$G_{\mu \nu} = i \frac{\partial}{\partial J_\nu (y)} \frac{\partial}{\partial J_\mu (x)} \ln \Delta.$$  
(3.17)

Thus the gauge transformations (3.14) considered above permit us to operate the transition from the Landau form of the propagator to the Feynman and the Yennie forms.\(^*\) To achieve this, we have only to choose

$$\delta M = -\gamma (-\partial^2 - i\epsilon)^{-2} \delta^4 (x - y)$$  
(3.18)

with a suitable constant \(\gamma\).\(^*\) As pointed out in §1, the propagator of photon type is possible because the bilinear term of interaction is still left.

Before proceeding to the next section we shall refer to the fact that the differential equation with respect to vector meson (3.1) reduces, especially in the Landau gauge (3.9), to such a simple form as

$$\left\{ \square i \frac{\partial}{\partial J_\nu} - (j_\nu + J_\nu) \right\} \Delta = 0,$$  
(3.19)

while the other equations for charged scalar mesons are unaltered.

\section{§ 4. The unitary and the renormalizable gauges}

In the preceding sections we have quantized the fields so that the vector meson has a propagator of photon type in the lowest order even after the symmetry of the theory breaks down. However, by recomposing the field variables, we are able to do with the dynamical system in a more natural way, where the coupling of vector meson with matter fields does not appear in the bilinear form by any means.

Let us make a transformation of external sources as follows:

$$i \frac{\partial}{\partial J_\mu} = i \frac{\partial}{\partial J_\mu} \cos \left( i \frac{\partial}{\partial \lambda} \right),$$

$$i \frac{\partial}{\partial K_1} = i \frac{\partial}{\partial K} \sin \left( i \frac{\partial}{\partial \lambda} \right) \right\}$$  
(4.1)

\(^*\) Landau and Khalatnikov\(^*\) first discussed the photon propagator by dividing it into the divergenceless part and the remainder.
where

\[
\begin{align*}
J_\mu &= J_\mu^*, \\
K_1 &= \cos \left(i \frac{\delta}{\delta A}\right) K - \left(i \frac{\delta}{\delta K}\right)^{-1} A \sin \left(i \frac{\delta}{\delta A}\right), \\
K_2 &= \sin \left(i \frac{\delta}{\delta A}\right) K + \left(i \frac{\delta}{\delta K}\right)^{-1} A \cos \left(i \frac{\delta}{\delta A}\right).
\end{align*}
\]

(4.2)

With these relations we can rewrite the set of Eqs. (3.2), (3.3) and (3.19) as

\[
\begin{align*}
\left[ \left( \Box - m^2 \right) i \frac{\delta}{\delta \mathcal{A}_\mu} - \epsilon^2 \left\{ i \frac{\delta}{\delta K} + \frac{1}{m_1} \partial_\mu i \frac{\delta}{\delta I} \right\} - \mathcal{A}_\mu - \partial_\mu \int \left( \Box - m^2 \right) M(x-y) d^4 y \mathcal{Q}(y) \right] T &= 0, \\
\left[ \left( \Box - m^2 \right) i \frac{\delta}{\delta I} - \epsilon^2 \left\{ i \frac{\delta}{\delta K} + \frac{1}{m_1} \partial_\mu i \frac{\delta}{\delta I} \right\} - \mathcal{Q}(x) \right] T &= 0
\end{align*}
\]

(4.3)

(4.4)

and

\[
\begin{align*}
\left[ \left( \Box - m^2 \right) i \frac{\delta}{\delta K} - \epsilon^2 \left\{ i \frac{\delta}{\delta K} + \frac{1}{m_1} \partial_\mu i \frac{\delta}{\delta I} \right\} - f^4 \left\{ i \frac{\delta}{\delta K} \right\} - \frac{i}{2} \left( i \frac{\delta}{\delta K} \right)^{-1} - K \right] T &= 0
\end{align*}
\]

(4.5)

with \( m_1 = \epsilon \eta \) and \( A = (m_1/e) \mathcal{I} \). In the above, \( M(x-y) \) gives the gauge-dependent part of the vector meson propagator and \( \mathcal{Q}(x) \) is an operator related to the current conservation given by

\[
\mathcal{Q}(x) T = -i \frac{\delta}{\delta \lambda(x)} T = \left\{ \partial_\mu \mathcal{A}_\mu + m_1 \mathcal{I} \right\} T.
\]

(4.6)

The gauge transformation of \( T \) is described as, using a unitary operator \( U \),

\[
\widetilde{T} = U T = \exp \left( i \int \mathcal{Q}(x) (\partial M) \mathcal{Q}(y) d^4 x d^4 y \right) T
\]

(4.7)

with the infinitesimal function \( \delta M = \tilde{M}(x-y) - M(x-y) \). It is easily proved that the set of differential equations (4.3) ~ (4.5) is invariant under the transformation (4.7).

Now, when the gauge function \( M = 0 \), the set of equations for \( T \) is nothing but that derived\(^{10}\) in the unitary gauge. In that gauge we can quantize the system.
with the help of the Stueckelberg formalism,\textsuperscript{7} where a vector and a scalar fields, \( CV_\mu \) and \( \phi \) respectively, are introduced in place of the original vector field \( V_\mu \) in such a way as

\[
V_\mu \rightarrow CV_\mu + \frac{1}{m_1} \partial_\mu \phi.
\]

This replacement implies that the vector field has intrinsically a kind of gauge invariance for the transformation

\[
\begin{align*}
CV_\mu &\rightarrow CV_\mu + \partial_\mu \lambda, \\
\phi &\rightarrow \phi - m_1 \lambda.
\end{align*}
\]

(4.9)

In the above derivation of Eqs. (4.3) \(~(4.5)\) we have also changed the definition of the physical state from one satisfying the Lorentz condition to one satisfying the following subsidiary condition:

\[
(\partial_\mu CV_\mu + m_1 \phi)| \rangle = 0
\]

(4.10)

in accordance with Stueckelberg's formula. This is a reason why we have not designated what states sandwich the time-ordered product of fields in the definition of \( \mathcal{D} \) (2.18).

We have seen previously\textsuperscript{10} that the generating functional \( \mathcal{D} \) is given in terms of the Heisenberg operators

\[
\mathcal{D} = \langle T \exp \left[ -i \int \{ CV_\mu \delta_\mu + \phi I + (\chi + \eta) K \} d^4 x \right] \rangle,
\]

(4.11)

or in the interaction picture

\[
\mathcal{D} = \langle T^* \exp \left[ -i \int \{ V_\mu^{\text{int}} \delta_\mu + (\chi^{\text{int}} + \eta) K_\text{int}(V_\mu^{\text{int}}, \chi^{\text{int}}) \\
+ i \delta^*(0) \ln \left( 1 + \frac{e}{m_1} \chi^{\text{int}} \right) \} d^4 x \right] \rangle,
\]

(4.12)

where

\[
\mathcal{L}_{\text{int}} = - em_1 V_\mu^{\text{int}} \chi - \frac{e^2}{2} V_\mu^{\text{int}} \chi^2 - \frac{1}{2} f m_1 \chi^2 - \frac{1}{8} f^2 \chi^4.
\]

(4.13)

This interaction Lagrangian agrees with the one which Higgs derived after redefining the field variables, Eq. (2.1) in Ref. 5. The vector and scalar fields of Stueckelberg are prescribed to have propagators in the lowest order, respectively,

\[
\langle T^* CV_\mu(x) CV_\nu(y) \rangle_0 = \delta_{\mu \nu} D_F(m_1^2)
\]

(4.14a)

and

\[
\langle T^* \phi(x) \phi(y) \rangle_0 = D_F(m_1^2).
\]

(4.14b)

Furthermore, they are combined to represent the original vector field,\textsuperscript{10} which
has been assumed in advance in Eq. (4.12). Thus the zero-order propagator of vector meson reads

\[
D_{\mu\nu}(m^2_i) = \langle T^* V_\mu(x) V_\nu(y) \rangle_0 \\
= \langle T^* \mathcal{C}V_\mu(x) \mathcal{C}V_\nu(y) \rangle_0 + \frac{1}{m^2_i} \langle T^* \partial_\mu \phi(x), \partial_\nu \phi(y) \rangle_0 \\
= [\delta_{\mu\nu} - \delta_\mu \delta_\nu/m^2_i] D_F(m^2_i). \tag{4.15}
\]

Next, let the function \( \delta M \) in Eq. (4.7) be

\[
- (-\partial^2 + m^2_i - i\varepsilon)^{-1} (-\partial^2 + i\varepsilon)\delta^4(x-y), \tag{4.16}
\]
then the set of Eqs. (4.3) ~ (4.5) corresponds to the formulation in the renormalizable gauge. The functional \( \mathcal{I} \) is obtained by virtue of (4.7) from the form of (4.11). In this gauge the fields \( \mathcal{C}V_\mu \) and \( \phi \) are not combined but treated separately throughout. The zero-order propagator of \( \mathcal{C}V_\mu \) has a form to be divergenceless, using (4.6),

\[
\tilde{D}_{\mu\nu}(m^2_i) = \delta_{\mu\nu} D_F(m^2_i) + i\partial_\mu \partial_\nu \delta M \\
= [\delta_{\mu\nu} - \partial_\mu \partial_\nu/m^2_i] D_F(m^2_i), \tag{4.17}
\]

while the propagator of the Goldstone boson type is made up as follows:

\[
\frac{1}{m^2_i} \partial_\mu \partial_\nu \partial I(x) \partial I(y) + \frac{2}{m^2_i} i \frac{\delta}{\delta \mathcal{G}_\mu(x)} \partial I(y) = \left\{ \frac{2i\partial_\mu \partial_\nu \delta M}{m^2_i} \right\} + \{-2i\partial_\mu \partial_\nu \delta M\}
\]

\[
= \frac{-\partial_\mu \partial_\nu D_F(m^2_i) - 0}{m^2_i}. \tag{4.18}
\]

A more general type of gauge can be considered by choosing\[^{11}\]

\[
\delta M = -\gamma (-\partial^2 + m^2_i - i\varepsilon)^{-1} (-\partial^2 + (1-\gamma) m^2_i - i\varepsilon)\delta^4(x-y) \tag{4.19}
\]
with a suitable constant \( \gamma \). Whatever value of \( \gamma \) is taken, the following holds always good:

\[
\left( i \frac{\delta}{\delta \mathcal{G}_\mu(x)} + \frac{1}{m^2_i} \partial_\mu \frac{\delta}{\delta I(x)} \right) \left( i \frac{\delta}{\delta \mathcal{G}_\nu(y)} + \frac{1}{m^2_i} \partial_\nu \frac{\delta}{\delta I(y)} \right) = [\delta_{\mu\nu} - \partial_\mu \partial_\nu/m^2_i] D_F(m^2_i) \tag{4.20}
\]
in the lowest order.

References

2) A. Salam and J. Strathdee, Nuovo Cim. 11A (1972), 397.
11) As for the application of the Stueckelberg formalism to the massive Yang-Mills field, see