Kinetic Equation for a Classical Nonideal Plasma

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The Balescu-Lenard theory of plasma kinetic equation is reformulated in such a way that the resulting collision term now conserves the sum of the kinetic and potential energy to the first order in the plasma parameter. The new collision term is expressed in terms of the instantaneous values of the single-particle distribution function and its first derivative in time.

§ 1. Introduction

The collision term of the kinetic equation for a classical plasma, obtained originally by Balescu and Lenard, takes explicit account of the dynamic screening action of the plasma in the interaction processes of the charged particles; this has been a major improvement over the classical collision term of charged particles due to Landau. It is well known that the Balescu-Lenard collision term conserves the kinetic energy density as well as the number density and the mean velocity. This property implies that the Balescu-Lenard collision term is relevant to describing the relaxation processes of an ideal-gas plasma only; in an isolated plasma containing a significant amount of interaction energy, the sum of the kinetic and potential energies should be conserved. It thus becomes important to reformulate the Balescu-Lenard theory so that the conservation law of energy may be properly taken into account in the relaxation processes of a nonideal plasma.

Recently, a remarkable progress has been made in these directions by Klimontovich. He first analyzed the Boltzmann equation for an imperfect neutral gas and obtained a kinetic equation in which interaction is completely taken into account within the framework of the pair collision approximation. He then proceeded to consider a similar kinetic equation for a plasma in a first approximation with respect to the plasma parameter; the parameter, \( g = 1/n\lambda_D^3 \), where \( \lambda_D \) is the Debye distance, measures the discreteness of the particles contained in the plasma. Because the effects of polarization brought about by the long-range Coulomb interaction play an important part in the plasma, this problem is more difficult than that considered for the case of a neutral system. Klimontovich treated the problem through calculation of time evolution of electric-field fluctuations in the plasma.

We consider this problem of formulating the kinetic equation of a classical
nonideal plasma along a similar line of approach. We thus calculate the collision term with inclusion of the effects of the interaction energy to the first order in the plasma parameter. As any kinetic equation may indicate, the interaction between particles produces temporal variation or approach to equilibrium of a single-particle distribution function. By taking explicit account of such a temporal change of the distribution function in the calculation of the collision term, one should be able to include the effects of finite correlations in the collisional processes. Mathematically, we note the existence of a time scale $\tau$ such that $\tau_1 \gg \tau \gg \tau_2$, where $\tau_1$ is the relaxation time for the single-particle distribution function and $\tau_2$ is the characteristic time associated with evolution of the pair correlation function. During the time interval $\tau$, the single-particle distribution changes by an amount proportional to $\tau/\tau_1$, while we may still regard the pair correlation function as having relaxed to its own equilibrium values. Inclusion of the effects to the first order in $\tau/\tau_1$ thus enables us to express the collision term in terms of the single-particle distribution functions and its time derivatives. The result so obtained no longer contains the time interval $\tau$ explicitly. It will be shown that the collision term now conserves the sum of the kinetic and potential energies in the plasma.

§ 2. Mathematical formulation of the problem

We consider a spatially homogeneous system containing $N$ identical particles of charge $q$ and mass $m$ in a volume $V$; $n=N/V$ is the average number density. We assume a smeared out background of opposite charge such that the average space-charge field of the system vanishes. The first two equations of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy for such a system read\(^6\)

\[ \frac{\partial}{\partial t} f_i(1, \cdot\cdot\cdot, s; t) = \int d2 \mathcal{C}V(1, 2) f_{i+1}(1, 2; t) \]

\[ \left[ \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial r_1} + v_2 \cdot \frac{\partial}{\partial r_2} - \frac{1}{n} \mathcal{C}V(1, 2) \right] f_i(1, 2; t) \]

\[ = \int d3 [\mathcal{C}V(1, 3) + \mathcal{C}V(2, 3)] f_s(1, 2, 3; t). \]  

(1)

Here, $i=(r_i, v_i)$ represents the position and the velocity of the $i$-th particle; $f_i(1, \cdot\cdot\cdot, s; t)$ is the $s$-particle distribution function normalized so that

\[ V^{-1} \int d(s+1) f_{i+1}(1, \cdot\cdot\cdot, s+1; t) = f_i(1, \cdot\cdot\cdot, s; t), \]

with $f_0=1$; and $\mathcal{C}V(i, j)$ is a two-particle operator,

\[ \mathcal{C}V(i, j) = \frac{q^2 n}{m} \left[ \frac{\partial}{\partial r_i} \frac{1}{|r_i-r_j|} \right] \cdot \frac{\partial}{\partial v_i} \]

which represents the effects of Coulomb interaction.
We find it convenient to introduce the pair and the ternary correlation functions along the line analogous to the Mayer cluster expansion. Thus,

\[ G(1, 2; t) = f_i(1, 2; t) - f(v_1; t)f(v_2; t), \]

\[ H(1, 2, 3; t) = f_i(1, 2, 3; t) - f(v_1; t)f(v_2; t)f(v_3; t) \]

\[ - f(v_1; t)G(2, 3; t) - f(v_2; t)G(3, 1; t) - f(v_3; t)G(1, 2; t). \]

(5)

Physically, these correlation functions represent the totally correlated parts of the respective distribution functions. Because of the presence of the neutralizing background, Eq. (1) now reduces to

\[ \frac{\partial}{\partial t} f(v_i; t) = \int d2C\nu(1, 2) G(1, 2; t). \]

(6)

Thus the principal problem in formulating a kinetic theory of the plasma is to find an appropriate expression for the pair correlation function \( G(1, 2; t) \) in terms of the single-particle distribution function so that the resulting equation becomes a closed equation involving the single-particle distribution function alone.

The expression for the pair correlation function is obtained from a solution of Eq. (2). In solving this equation, we assume that relative magnitudes of the correlation functions obey the ordering with respect to the plasma parameter,

\[ \frac{G(1, 2; t)}{f(v_1; t)f(v_2; t)} \sim g^4, \]

(7)

\[ \frac{H(1, 2, 3; t)}{f(v_1; t)f(v_2; t)f(v_3; t)} \sim g^4. \]

(8)

This assumption implies that the short-range effects at distances of the order of \((n\lambda_d)^{-1}\), the average distance of the closest approach, are neglected. As is well known, the short-range correlations begin to affect the calculation of the correlation energy of the plasma only in the second-order terms in the plasma parameter expansion, while the long-range correlations give rise to the first-order effects. Hence, as long as we limit ourselves to a consideration of the effects of the interaction energy to the first order in \(g\), the assumptions of Eqs. (7) and (8), appropriate for long-range correlations, may be justified.

Substituting Eqs. (4) and (5) into Eq. (2) and retaining only the first order terms in the plasma parameter, we obtain

\[ \left[ \frac{\partial}{\partial t} + v_i \cdot \frac{\partial}{\partial r_i} + v_j \cdot \frac{\partial}{\partial r_j} \right] G(1, 2; t) \]

\[ - \int d3C\nu(1, 3)f(v_1; t) G(2, 3; t) - \int d3C\nu(2, 3)f(v_2; t) G(3, 1; t) \]

\[ = \frac{1}{n} [C\nu(1, 2) + C\nu(2, 1)] f(v_1; t)f(v_2; t). \]

(9)
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A formal solution of this equation can be expressed as

\[ G(1, 2; t) = \int d1' \int d2' \int_{-\infty}^{t} dt' G_0(1', 2'; t') U(1, 1'; t, t') U(2, 2'; t, t'), \tag{10} \]

where

\[ G_0(1, 2; t) = \frac{1}{n} [C(1, 2) + C(2, 1)] f(v_1; t) f(v_2; t) \tag{11} \]

and the propagator \( U(1, 1'; t, t') \) satisfies the linearized Vlasov equation

\[ \left[ \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial r_1} \right] U(1, 1'; t, t') - \int d2C(1, 2) f(v_1; t) U(2, 1', t, t') = 0 \tag{12} \]

with the initial condition

\[ U(1, 1'; t, t) = \delta(1 - 1'). \tag{13} \]

Our problem, formulated mathematically, is therefore to find a solution of Eq. (12) so that the pair correlation function, Eq. (10), may be expressed in terms of the single-particle distribution function; in the process of this solution, we must take into account the first-order effects of relaxation brought about by the particle interaction.

§ 3. Propagator

Since the function \( U(1, 1'; t, t') \) is determined from a solution of the linearized Vlasov equation, its characteristic time of evolution is the plasma time, \( 1/\omega_p \), where

\[ \omega_p = (4\pi n q^2 / m)^{1/2} \]

is the plasma frequency. As Eq. (10) indicates, this function plays the part of a propagator and describes the space-time developments of the pair correlation function. Hence, we may take

\[ \tau \approx 1 / \omega_p . \]

The rate of relaxation for the single-particle distribution function is characterized by the time scale \( \tau_1 \), that is

\[ \frac{\partial}{\partial t} f(v_1; t) \approx - \frac{f(v_1; t) - f(v_1; \infty)}{\tau_1} . \]

The relative magnitudes of these time scales are estimated to be \( \tau / \tau_1 \approx g \ll 1 \); we may thus choose a time scale, \( \tau = t - t' \), in such a way that

\[ \tau / \tau_1 \gg 1. \tag{14} \]

We now wish to solve Eq. (10) in a power-series expansion with respect to the small parameter \( \tau / \tau_1 \); we thus write
The function \( U^0 \) satisfies the equation
\[
\left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \right] U^0(1, 1'; t, t') - \int d^2 \mathbf{C} V(1, 2) f(\mathbf{v}_1; t') U^0(2, 1'; t, t') = 0 \tag{15}
\]
with the initial condition (13). After a solution of this equation is obtained, the function \( U^1 \) may be determined from
\[
\left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \right] U^1(1, 1'; t, t') - \int d^3 \mathbf{C} V(1, 3) f(\mathbf{v}_1; t) U^1(3, 1'; t, t') = \tau \int d^2 \mathbf{C} V(1, 2) \frac{\partial}{\partial t} f(\mathbf{v}_1; t) U^0(2, 1'; t, t') \tag{17}
\]
with the initial condition
\[
U^1(1, 1'; t, t) = 0. \tag{18}
\]
These equations may be solved through the technique of Fourier-Laplace transformations.\(^{10}\) We thus find from Eq. (16)
\[
U_k^0(\mathbf{v}_1, \mathbf{v}_1'; t, t') = \int d(\mathbf{r}_1 - \mathbf{r}_1') U^0(1, 1'; t, t') \exp[-i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_1')] = \delta(\mathbf{v}_1 - \mathbf{v}_1') \exp(-i\mathbf{k} \cdot \mathbf{v}_1 t)
\]
\[
+ \frac{\omega_p^2}{2\pi i k^2} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}_1} f(\mathbf{v}_1; t') \int_0^\infty d\omega \frac{\exp(-i\omega \tau)}{(\omega - \mathbf{k} \cdot \mathbf{v}_1 + i0)(\omega - \mathbf{k} \cdot \mathbf{v}_1' + i0)} \mathbf{e}(\mathbf{k}, \omega, t'), \tag{19}
\]
where \( \mathbf{e}(\mathbf{k}, \omega, t) \) is the dielectric response function of the plasma defined by
\[
\mathbf{e}(\mathbf{k}, \omega, t) = 1 + \frac{\omega_p^2}{k^2} \int d\mathbf{v} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}; t) \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v} + i0}. \tag{20}
\]
The positive infinitesimals involved in the imaginary parts of the denominators in Eqs. (19) and (20) serve as a reminder that the functions are calculated with the retarded boundary conditions. Note that the single-particle distribution function and the dielectric response function contained in Eq. (19) are those evaluated at time \( t' \); this time is different from time \( t \) at which the kinetic evolution of the distribution function is considered. In order to reduce the time variable from \( t' \) to \( t \), we carry out the Taylor expansion of these functions around \( t \). To the first order in \( \tau/r_n \), Eq. (19) may thus be written as
\[
U_k^0(\mathbf{v}_1, \mathbf{v}_1'; t, t') = U_k^{(0)}(\mathbf{v}_1, \mathbf{v}_1'; t, t') + U_k^{(1)}(\mathbf{v}_1, \mathbf{v}_1'; t, t'), \tag{21}
\]
where
\[ U_k^{(0)}(v_1, v_1'; t, t') = \delta(v_1 - v_1') \exp(-i k \cdot v_1 \tau) \]
\[ + \frac{\omega_p^2}{2\pi k^2} k \cdot \frac{\partial}{\partial v_1} f(v_1; t) \int_{-\infty}^{\infty} d\omega \exp(-i \omega \tau) \frac{\exp(-i \omega \tau)}{(\omega - k \cdot v_1 + i0)(\omega - k \cdot v_1' + i0) \epsilon(k, \omega, t)}, \]
\[ (22) \]
\[ U_k^{(1)}(v_1, v_1'; t, t') = \frac{i \omega_p^2}{2\pi k^2} \int_{-\infty}^{\infty} d\omega \frac{\exp(-i \omega \tau)}{(\omega - k \cdot v_1 + i0)(\omega - k \cdot v_1' + i0) \epsilon(k, \omega, t)} \frac{\partial}{\partial t} \left[ k \cdot \frac{\partial}{\partial v_1} f(v_1; t) / \epsilon(k, \omega, t) \right]. \]
\[ (23) \]
Equation (17) may similarly be solved with the initial condition (18). Since we are interested in calculating the propagator to the accuracy of the first order in \( \tau/\tau_1 \), we may substitute Eq. (22) in place of \( U^0 \) on the right-hand side of Eq. (17). After a lengthy but straightforward calculation, we obtain
\[ U_k^{(1)}(v_1, v_1'; t, t') = \frac{-i \omega_p^2}{2\pi k^2} \int_{-\infty}^{\infty} d\omega \frac{\exp(-i \omega \tau)}{(\omega - k \cdot v_1 + i0)(\omega - k \cdot v_1' + i0) \epsilon(k, \omega, t)} \frac{\partial}{\partial t} \left[ k \cdot \frac{\partial}{\partial v_1} f(v_1; t) / \epsilon(k, \omega, t) \right] \]
\[ + \frac{\omega_p^2}{2\pi k^2} \int_{-\infty}^{\infty} d\omega \frac{\exp(-i \omega \tau)}{(\omega - k \cdot v_1 + i0) \epsilon(k, \omega, t)} \frac{\partial}{\partial \omega} \left[ \frac{\epsilon(k, \omega, t)}{\omega - k \cdot v_1 + i0} \frac{\partial}{\partial v_1} \left( k \cdot \frac{\partial}{\partial v_1} f(v_1; t) / \epsilon(k, \omega, t) \right) \right]. \]
\[ (24) \]
Summation of Eqs. (23) and (24) yields that part of the propagator which is of the first order in \( \tau/\tau_1 \),
\[ U_k^{(1)}(v_1, v_1'; t, t') = U_k^{(0)}(v_1, v_1'; t, t') + U_k^{(1)}(v_1, v_1'; t, t') \]
\[ = \frac{\omega_p^2}{2\pi k^2} \int_{-\infty}^{\infty} d\omega \frac{\exp(-i \omega \tau)}{(\omega - k \cdot v_1 + i0) \epsilon(k, \omega, t)} \frac{\partial}{\partial \omega} \left[ \frac{\epsilon(k, \omega, t)}{\omega - k \cdot v_1 + i0} \frac{\partial}{\partial v_1} \left( k \cdot \frac{\partial}{\partial v_1} f(v_1; t) / \epsilon(k, \omega, t) \right) \right]. \]
\[ (25) \]
Consequently, the expression for the propagator appropriate to the present problem is given by
\[ U_k(v_1, v_1'; t, t') = U_k^{(0)}(v_1, v_1'; t, t') + U_k^{(1)}(v_1, v_1'; t, t'). \]
\[ (26) \]

§ 4. Collision term

The collision term given by the right-hand side of Eq. (6) can be expressed in terms of the Fourier components of the propagator and the source function (11) as
where

$$G_{0k}(v_1, v_2; t) = \frac{i\omega_p}{nk^2} \cdot [\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}] f(v_1; t) f(v_2; t).$$

As we have done in the calculation of the propagator in Eq. (26), we expand Eq. (28) with respect to $\tau_1$; we have

$$G_{0k}(v_1, v_2; t') = G_{0k}^{(0)}(v_1, v_2; t) + G_{0k}^{(1)}(v_1, v_2; t, t'),$$

where

$$G_{0k}^{(0)}(v_1, v_2; t) = \frac{i\omega_p}{nk^2} k \cdot [\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}] f(v_1; t) f(v_2; t),$$

$$G_{0k}^{(1)}(v_1, v_2; t, t') = \frac{i\omega_p}{nk^2} \cdot [\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}] \frac{\partial}{\partial t} (f(v_1; t) f(v_2; t)).$$

We may similarly expand the collision term as

$$\frac{\partial f(v_1; t)}{\partial t} = \frac{\partial f(v_1; t)}{\partial t} \bigg|_0 + \frac{\partial f(v_1; t)}{\partial t} \bigg|^{(1)}.$$

With the aid of Eqs. (22) and (30), the zeroth order term may then be calculated as

$$\frac{\partial f(v_1; t)}{\partial t} \bigg|_0 = \frac{-i\omega_p}{N} \sum \frac{\partial}{\partial v_1} \int dv_1' \int dv_1' \int dv_2 \int_0^\infty d\tau \times G_{0k}^{(0)}(v_1, v_1'; t, t') U_k^{(0)}(v_2, v_2'; t, t') U_k^{(0)}(v_2, v_2') \delta(k \cdot v_1 - k \cdot v_1').$$

This is the Balescu-Lenard collision term. Our new result arises from the first-order calculations; these may be obtained as

$$\frac{\partial f(v_1; t)}{\partial t} \bigg|^{(1)} = \frac{-i\omega_p}{N} \sum \frac{\partial}{\partial v_1} \int dv_1' \int dv_1' \int dv_2 \int_0^\infty d\tau \times \{ G_{0k}^{(0)}(v_1', v_1'; t, t') U_k^{(0)}(v_1, v_1', t, t') U_k^{(0)}(v_2, v_2'; t, t') + G_{0k}^{(0)}(v_1', v_1'; t) U_k^{(0)}(v_1, v_1', t, t') U_k^{(0)}(v_2, v_2'; t, t') + G_{0k}^{(0)}(v_1', v_1'; t) U_k^{(0)}(v_1, v_1', t, t') U_k^{(0)}(v_2, v_2; t, t') \}.$$

Carrying out the integrations with respect to $\tau$ and $v_3$, we find
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\[
\frac{\partial f(v_1; t)}{\partial t} = \frac{\omega_p^4}{2\pi iN} \sum_{k} \frac{k}{k^2} \frac{\partial}{\partial v_1} \int dv_1' \int dv_2' \left\{ \frac{k \cdot \left( \frac{\partial}{\partial v_1'} - \frac{\partial}{\partial v_2'} \right) \frac{\partial}{\partial v_1}}{2\pi i \delta(v_1 - v_1')} \right\} \left[ \frac{1}{\epsilon(k, -k \cdot v_1 + i0) (\omega - k \cdot v_2' + i0) (\omega - k \cdot v_2 - i0) \epsilon(k, -k \cdot v_1, t)} \right]
\]

\[
\times \left[ \frac{\partial}{\partial \omega} \left( \frac{\epsilon(k, \omega, t)}{\omega - k \cdot v_1 + i0} \frac{\partial}{\partial \omega} \left( \frac{k \cdot \frac{\partial}{\partial v_1} f(v_1; t)}{\epsilon(k, \omega, t)} \right) + \frac{\omega_p^2}{k^2} \frac{k \cdot \frac{\partial}{\partial v_1} f(v_1; t)}{\epsilon(k, \omega, t)} \right) \right]
\]

\[
\times \left[ \frac{\partial^2}{\partial \omega \partial t} \ln \epsilon(-k, -k \cdot v_1 + i0) \epsilon(k, -k \cdot v_1', t) \right]
\]

\[
\times \left[ \frac{\partial^2}{\partial \omega \partial t} \ln \epsilon(-k, -k \cdot v_1, t) \right]
\]

Finally we obtain

\[
\frac{\partial f(v_1; t)}{\partial t} = \frac{\omega_p^4}{N} \sum_{k} \frac{k}{k^2} \frac{\partial}{\partial v_1} \left[ f(v_1; t) \frac{\partial}{\partial (k \cdot v_1)} \ln \epsilon(-k, -k \cdot v_1, t) \right]
\]

\[
+ 2\pi i \frac{k \cdot \frac{\partial}{\partial v_1} f(v_1; t)}{\epsilon(k, k \cdot v_1, t)} \left[ \frac{\partial^2}{\partial t \partial \omega (k \cdot v_1)} \ln \epsilon(-k, -k \cdot v_1, t) \right]
\]

\[
\times \int dv_1 f(v_1; t) (k \cdot v_1 - k \cdot v_2) + \frac{k}{k^2} \frac{\partial^2}{\partial v_1 \partial t} f(v_1; t)
\]

\[
\times \int dv_1 f(v_1; t) \frac{k \cdot v_1}{(k \cdot v_1 - k \cdot v_2 - i0) \epsilon(k, k \cdot v_1, t)} + \frac{k}{k^2} \frac{\partial}{\partial v_1} f(v_1; t)
\]

\[
\times \int dv_1 f(v_1; t) \frac{k \cdot v_1}{(k \cdot v_1 - k \cdot v_2 - i0) \epsilon(k, -k \cdot v_1, t)} + \frac{k}{k^2} \frac{\partial}{\partial v_1} f(v_1; t)
\]
§ 5. Conservation laws

We now wish to investigate the conservation properties associated with the kinetic equation with the new collision term (32). It is well known that the Balescu-Lenard collision term, Eq. (33), conserves the number density of the particles, the mean velocity, and the density of kinetic energy. Written in mathematical terms, these properties derive from the calculations,\textsuperscript{11}

\[ \int dv_1 \frac{\partial f(v_1; t)}{\partial t} \bigg|_e = 0, \quad \int dv_1 v_1 \frac{\partial f(v_1; t)}{\partial t} \bigg|_e = 0, \quad \int dv_1 \frac{1}{2} m v_1^2 \frac{\partial f(v_1; t)}{\partial t} \bigg|_e = 0. \]

One can prove that Eq. (35) is generally satisfied by the collision term in the form of Eq. (6). We also note that (36) is a consequence of the uniformity of the system. For such a plasma, the collision term is written as

\[ \frac{\partial f(v_1; t)}{\partial t} = \frac{-i \omega_p^2}{V} \sum_k \frac{k}{k^2} \frac{\partial}{\partial v_1} \int dv_1' G_k(v_1, v_1'), \]

whence one finds

\[ \int dv_1 v_1 \frac{\partial f(v_1; t)}{\partial t} \bigg|_e = \frac{i \omega_p^2}{V} \sum_k \frac{k}{k^2} \int dv_1 \int dv_1' G_k(v_1, v_1'), \]

\[ = i \frac{\omega_p^2}{N} \sum_k \frac{k}{k^2} [S(k) - 1]. \]

The function \( S(k) \) is the static structure factor of the plasma defined in terms of the statistical average of the fluctuating density field \( n(r, t) \) as

\[ S(k) = \frac{1}{n} \int dr \langle n(r + r, t) n(r', t) \rangle \exp(-ik \cdot r). \]

This function has the symmetry \( S(-k) = S(k) \); hence, Eq. (38) vanishes in a uniform plasma. In fact, it can be shown without difficulty that our particular calculation of the collision term in Eq. (32) likewise satisfies the conservation properties, (35) and (36).

The most significant improvement of the present calculation over the Balescu-
Lenard formula arises in the conservation property of energy. In order to investigate this aspect in detail, we multiply Eq. (34) by \((1/2)mv_1^2\) and integrate it over \(v_i\). After a series of calculations described in the Appendix, we find

\[
\int dv_i \frac{1}{2} mv_i^2 \frac{\partial f(v_i; t)}{\partial t} \bigg|^{(1)} e = -\frac{\partial}{\partial t} \left\{ \frac{1}{2V} \sum_k \frac{4\pi a^2}{k^3} \left[ \int_{-\infty}^{\infty} d\omega \int dv_i f(v_i; t) \delta(\omega - k \cdot v_i) \right] - 1 \right\}.
\]

Combining this result with Eq. (37), we obtain a conservation law from the moment calculation of Eq. (6) with respect to \((1/2)mv_i^2\); the result is

\[
\int dv_i \frac{1}{2} mv_i^2 \left\{ \frac{\partial f(v_i; t)}{\partial t} \bigg|^{(0)} e + \frac{\partial f(v_i; t)}{\partial t} \bigg|^{(1)} e \right\} = -\frac{\partial}{\partial t} \left\{ \frac{n}{2V} \sum_k \frac{4\pi a^2}{k^3} \left[ \int_{-\infty}^{\infty} d\omega \int dv_i f(v_i; t) \delta(\omega - k \cdot v_i) \right] - 1 \right\}.
\]

Generally, the density of the interaction energy in the plasma is given by

\[
E_{\text{int}} = \frac{n}{2V} \sum_k \frac{4\pi a^2}{k^3} [S(k) - 1].
\]

To the lowest order in the plasma-parameter expansion, the structure factor of a nonequilibrium plasma may be calculated through the technique of superposing the dressed test particles\(^{11,12}\); one then obtains

\[
S(k) = \int_{-\infty}^{\infty} d\omega \int dv_i f(v_i; t) \delta(\omega - k \cdot v_i) |\epsilon(k, \omega, t)|^3.
\]

It is now clear that Eq. (41) expresses the conservation law of the sum of the kinetic and potential energies in the plasma. The collision term as given by Eq. (32) conserves the total energy of the plasma up to the first-order terms in the plasma parameter.

§ 6. Concluding remark

We have thus obtained a kinetic equation for a uniform plasma which conserves the total energy up to the first order in the plasma parameter. The collision term is now written in terms of the instantaneous values of the single-particle distribution function and its first derivative in time. In particular, the additional term (34) is proportional to the time derivative of the single-particle distribution function. Combining this fact with Lenard's proof of the \(H\) theorem for the collision term (33), we can show likewise that Maxwellian distributions are the only stationary solutions of our kinetic equation.

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Appendix

Derivation of Eq. (40)

We multiply the expression (34) by \(1/2mv_1^2\) and integrate over \(v_1\) by parts; we obtain

\[
\int dv_1 \frac{1}{2} mv_1^2 \frac{\partial f(v_1; t)}{\partial t} \bigg|^{(1)}_0 = \frac{\omega_x^2 m}{2\pi iN} \sum_k \frac{1}{k^2} \int dv_1 \int dv_1' \left( \frac{k}{k^2} \left( \frac{\partial}{\partial v_1'} - \frac{\partial}{\partial v_1} \right) \frac{\partial}{\partial t} (f(v_1'; t)f(v_1'; t)) \right) 
\]

\[
\times \left[ \int_0^\infty d\omega \frac{\omega}{\epsilon(k, \omega, t)(\omega - k \cdot v_1' + i0)} \frac{\partial}{\partial \omega} \left( \frac{1}{(\omega - k \cdot v_1' + i0)(\omega - k \cdot v_1' + i0)\epsilon(k, \omega, t)} \right) \right.
\]

\[
\left. + \left[ \frac{k}{k^2} \left( \frac{\partial}{\partial v_1'} - \frac{\partial}{\partial v_1} \right) f(v_1'; t)f(v_1'; t) \right] \right]
\]

\[
\times \left[ \int_0^\infty d\omega \frac{1}{(\omega - k \cdot v_1' + i0)(\omega - k \cdot v_1' + i0)\epsilon(k, \omega, t)^2} \left[ \frac{-\frac{\partial}{\partial \omega} \ln \epsilon(k, \omega, t)}{\partial \omega} + \frac{1}{\partial \omega} \ln \epsilon(-k, -\omega, t) \right] \right].
\]

(A.1)

We may carry out partial integration with respect to \(\omega\) for the first term in the curly bracket; after a symmetrization involving the change of variables, we obtain

\[
\frac{1}{2} \int_0^\infty d\omega \frac{\omega}{\epsilon(k, \omega, t)(\omega - k \cdot v_1' + i0)} \frac{\partial}{\partial \omega} \left( \frac{1}{(\omega - k \cdot v_1' + i0)(\omega - k \cdot v_1' + i0)\epsilon(k, \omega, t)} \right)
\]

\[
= -\frac{1}{2} \int_0^\infty d\omega \frac{1}{(\omega - k \cdot v_1' + i0)(\omega - k \cdot v_1' + i0)\epsilon(k, \omega, t)^2}.
\]

Similarly, we symmetrize the third term in the curly bracket of (A.1). As a result we find

\[
\int dv_1 \frac{1}{2} mv_1^2 \frac{\partial f(v_1; t)}{\partial t} \bigg|^{(1)}_0 = \frac{i\omega_x^2 m}{2\pi N} \sum_k \frac{1}{k^2} \int dv_1' \int dv_1' \left( \frac{k}{k^2} \left( \frac{\partial}{\partial v_1'} - \frac{\partial}{\partial v_1} \right) \frac{\partial}{\partial t} (f(v_1'; t)f(v_1'; t)) \right)
\]

\[
\times \left[ \frac{1}{2} \left( \frac{k}{k^2} \left( \frac{\partial}{\partial v_1'} - \frac{\partial}{\partial v_1} \right) \frac{\partial}{\partial t} (f(v_1'; t)f(v_1'; t)) \right) \right.
\]

\[
\times \left[ \int_0^\infty d\omega \frac{1}{|\epsilon(k, \omega, t)|^2(\omega - k \cdot v_1' + i0)(\omega - k \cdot v_1' + i0)} \right.
\]

\[
\left. + \left[ \frac{k}{k^2} \left( \frac{\partial}{\partial v_1'} - \frac{\partial}{\partial v_1} \right) f(v_1'; t)f(v_1'; t) \right] \right]
\]

\[
\times \left[ \int_0^\infty d\omega \frac{1}{|\epsilon(k, \omega, t)|^2(\omega - k \cdot v_1' + i0)(\omega - k \cdot v_1' - i0)\frac{\partial}{\partial t} \ln \epsilon(k, \omega, t)} \right].
\]

(A.2)
Integrating over $v_1'$ and $v_i'$, we obtain

$$
\int dv_1 \frac{1}{2} m v_1^2 \frac{\partial f(v_1; t)}{\partial t} (x) = \frac{-1}{2V} \sum_k \frac{\omega_k^2}{nk^2} \sum_k \frac{\omega_k^2}{nk^2} \frac{1}{|\epsilon(k, \omega, t)|^2} \frac{1}{\epsilon(k, \omega, t)} \int dv_1 f(v_1'; t) \delta(\omega - k \cdot v_1')
$$

(A.3)

where we have used the Dirac identities

$$
\frac{1}{x \pm i0} = \frac{P}{x} \mp i\pi \delta(x).
$$

The integration over $\omega$ in the second term on the right-hand side of (A.3) may be carried out by closing the contour with an infinite semicircle in the lower half-plane; thus,

$$
\frac{i}{\pi} \int_{-\infty}^{\infty} d\omega \frac{1}{(\omega, -\omega, t)(\omega - k \cdot v_1' - i0)} = -1.
$$

Consequently, we obtain

$$
\int dv_1 \frac{1}{2} m v_1^2 \frac{\partial f(v_1; t)}{\partial t} (x) = \frac{-1}{2V} \sum_k \frac{4\pi q^2}{k^2} \left[ \int d\omega \int dv_1 f(v_1'; t) \delta(\omega - k \cdot v_1') \right] - 1 \right)
$$

(A.4)

We have thus shown Eq. (40).

References

2) A. Lenard, Ann. of Phys. 10 (1960), 390.
10) S. Ichimaru, Basic Principles of Plasma Physics (W. A. Benjamin Inc., Reading, Massachusetts, 1973), Ch. II.