Higgs' Theorem in the Yang-Feldman Formalism

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Higgs' theorem forms the basis for the unified model of electromagnetic and weak interactions proposed by Weinberg and is closely related to unitarity. In the functional formulation of the gauge field theory, the unitarity of the $S$ matrix in the renormalizable gauge is proved indirectly on the argument that the gauge-invariant $S$ matrix should be common to the renormalizable and unitary gauges. A direct proof of Higgs' theorem and of unitarity in the renormalizable gauge has been given by Nakanishi on the basis of the indefinite metric quantum field theory. The present article is aimed at presentation of its modified version exploiting the Yang-Feldman formalism.

§ 1. Introduction

Elimination of the Goldstone bosons$^3$ associated with a spontaneous breakdown of a symmetry is imperative in view of the absence of massless scalar particles in nature. Higgs$^2$ has found a mechanism of eliminating them by introducing gauge fields, and soon, later this idea has been extended by Kibble$^8$ so as to include non-abelian gauge fields. This discovery of the so-called Higgs-Kibble mechanism has been exploited by Weinberg$^4$ to formulate a unified renormalizable model of electromagnetic and weak interactions of leptons.

In introducing gauge fields it is customary to start from the functional integration method to quantize the theory as has been initiated by Faddeev and Popov.$^5$ Depending on the choice of the gauge condition one can introduce a variety of gauges, and the most common ones among them are the R (renormalizable) and U (unitary) gauges. In the former the renormalizability is manifest, whereas in the latter the unitarity is apparent. Therefore, as long as the $S$ matrix is gauge-invariant, it should enjoy both conditions of unitarity and of renormalizability.$^6$ The gauge transformation connecting these two gauges, however, seems to be rather singular, and the proof of the equivalence of the two gauges is of rather classical nature. Indeed, practical calculations indicate that they might not be equivalent in a delicate manner.$^7$

For these reasons, it is desirable to present a direct proof in the R gauge of the Higgs-Kibble theorem closely related to unitarity. Recently this program has been carried out by Nakanishi$^9$ on the basis of the indefinite metric quantum field theory in the limited case of the Higgs mechanism. The present article is aimed at presentation of a modified and more intuitive version of Nakanishi's proof on the basis of the Yang-Feldman formalism.$^9$
In the next section the essence of the Yang-Feldman formalism is briefly reviewed. In § 3 the original Higgs model is introduced, in § 4 this model is quantized according to the prescription presented in § 2 and Higgs’ theorem is proved accordingly.

§ 2. Quantization in the Yang-Feldman formalism

In this article the Higgs model is quantized according to the Yang-Feldman formalism, and the essence of its quantization procedure is recapitulated in what follows.

The total Lagrangian of a system of interacting fields is divided into the free and interaction Lagrangians as

$$L = L_f + L_{int},$$  \hspace{1cm} (2.1)

where $L_f$ is bilinear in field operators. The field operators are denoted as $\phi_a$, where the subscript $a$ represents either the internal quantum number or the Lorentz index, or both. In this paper $\phi_a$ are chosen to represent Bose fields.

Let us define a matrix $D_{ab}(\partial)$, being a differential operator, by

$$[L_f]_{ab} = D_{ab}(\partial) \phi_b,$$  \hspace{1cm} (2.2)

where the left-hand side denotes the Euler derivative of $L_f$ with respect to $\phi_a$. The retarded and advanced Green's functions satisfy

$$D_{ab}(\partial) A^a_b(x) = D_{ab}(\partial) A^a_b(x) = -\delta_{ab} \delta^t(x).$$  \hspace{1cm} (2.3)

By the same token the propagation function satisfies

$$D_{ab}(\partial) A^a_b(x) = i\delta_{ab} \delta^t(x).$$  \hspace{1cm} (2.4)

In terms of the asymptotic fields $\phi^a_n$ and $\phi^a_n$, the Heisenberg operators $\phi_a$ are expressed as

$$\phi_a(x) = \phi^a_n(x) - \int d^4x' D_{ab}^a(x-x') D_{bc}(\partial') \phi_b(x'),$$  \hspace{1cm} (2.5)

where use of the Euler-Lagrange equations allows us to rewrite

$$D_{ab}(\partial) \phi_b = -[L_{int}]_{ab}.$$  \hspace{1cm} (2.6)

These relationships are called the Yang-Feldman equations. The asymptotic fields fulfill the following conditions:

$$D_{ab}(\partial) \phi^a_n(x) = D_{ab}(\partial) \phi^a_n(x) = 0,$$  \hspace{1cm} (2.7)

$$\langle T^*[\phi^a_n(x), \phi^a_n(y)] \rangle_b = \langle T^*[\phi^a_n(x), \phi^a_n(y)] \rangle = \delta^x_y.$$  \hspace{1cm} (2.8)

With these preliminaries quantization of the Higgs model will be discussed in the following sections.
§ 3. The Higgs model

The Lagrangian of the Higgs model is identical in form with that of a self-interacting charged scalar field interacting with the electromagnetic field. It is given in the Pauli metric by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \phi + \frac{1}{4} m^2 \phi \phi - \frac{f^2}{8} (\phi \phi \phi)^2, \]  
(3.1)

where

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \]
\[ \nabla_{\mu} \phi_1 = \partial_{\mu} \phi_1 - e A_{\mu} \phi_1, \]
\[ \nabla_{\mu} \phi_2 = \partial_{\mu} \phi_2 + e A_{\mu} \phi_1. \]

(3.2)

The relation of \( \phi_1 \) and \( \phi_2 \) to the conventional complex fields \( \phi \) and \( \phi^* \) is given by

\[ \phi = \frac{1}{\sqrt{2}} (\phi_1 - i \phi_2), \quad \phi^* = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2). \]

(3.3)

Since the Lagrangian (3.1) is gauge-invariant, one has to introduce a gauge condition to quantize it. For this purpose reference is made to the Faddeev-Popov method of expressing \( \mathcal{T} \) in terms of a functional integral.

\[ \mathcal{T} = \int \mathcal{D} \phi_a(x) \mathcal{D} A_\mu(x) \prod_x \mathcal{D} (\partial_\mu A_\mu(x)) \exp \left[ i \int d^4 x (\mathcal{L}(x) - \phi_a(x) J_a(x)) \right], \]

(3.4)

where a special gauge condition \( \partial_\mu A_\mu(x) = 0 \) has been adopted. \( \mathcal{T} \) is the generating functional of the time-ordered Green's functions and is given in terms of Heisenberg operators by

\[ \mathcal{T} = \left< T^\ast \exp \left( -i \int d^4 x \phi_a(x) J_a(x) \right) \right>_0, \]

(3.5)

and \( J_a(x) \) are the c-number external sources.

The functional integral (3.4) may also be written as

\[ \mathcal{T} = \int \mathcal{D} \phi_a(x) \mathcal{D} A_\mu(x) \mathcal{D} B(x) \exp \left[ i \int d^4 x (\mathcal{L}(x) + B(x) \partial_\mu A_\mu(x) - \phi_a(x) J_a(x)) \right], \]

(3.6)

where \( B \) may be regarded as a Lagrange multiplier. Thus, corresponding to the gauge employed in (3.4), the original Lagrangian (3.1) may be modified as

\[ \mathcal{L} \rightarrow \mathcal{L} + B \partial_\mu A_\mu. \]

(3.7)

This modified Lagrangian is indeed the starting point of Nakanishi's proof in the Landau gauge.

Now a spontaneous symmetry breakdown is introduced through
where

\[ \langle \Phi \rangle_0 = \langle \chi \rangle_0 = 0, \quad \langle \phi_2 \rangle_0 = \eta. \quad (3.9) \]

The total Lagrangian is then decomposed into three parts:

\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\text{int}}, \quad (3.10) \]

where

\[ \begin{align*}
\mathcal{L}_1 &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu \phi - m_1 A_\mu) \left( \partial_\mu \phi - m_1 A_\mu \right) + B \partial_\mu A_\mu, \\
\mathcal{L}_2 &= -\frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{4} m_2 \chi^2, \\
\mathcal{L}_{\text{int}} &= e A_\mu \left( \chi \partial_\mu \phi - \phi \partial_\mu \chi \right) - e m_1 \chi A_\mu A_\rho - \frac{1}{2} f m_2 (\phi^2 + \chi^2) \\
&\quad - \frac{1}{2} e^2 A_\mu A_\rho (\phi^2 + \chi^2) - \frac{f^2}{8} (\phi^2 + \chi^2)^2
\end{align*} \quad (3.11) \]

and

\[ m_1 = \eta, \quad m_2 = \eta. \quad (3.12) \]

The free Lagrangian \( \mathcal{L}_f \) of this model is given by \( \mathcal{L}_1 + \mathcal{L}_2 \). The quantization prescription mentioned in § 2 will be applied to \( \mathcal{L}_1 \) in the next section.

In the present article all the arguments are given in the unrenormalized form. Renormalization prescription implies a different division of the total Lagrangian into free and interaction parts from the unrenormalized one in (3.11), but as long as the free part maintains a form similar to (3.11), the essence of the proof remains unchanged.

§ 4. Quantization of the Higgs model

As is obvious from (3.11) quantization of the massive scalar \( \chi \) field is rather trivial so that our main concern is quantization of the first piece \( \mathcal{L}_1 \). The \( D \) matrix is obtained from the Euler derivatives of \( \mathcal{L}_1 \).

\[ \begin{align*}
[\mathcal{L}_1]_\rho &= \left[ \partial_\rho (\Box - m_1^2) - \partial_\rho \partial_\sigma \right] A_\sigma + m_1 \partial_\rho \phi - \partial_\rho B, \\
[\mathcal{L}_1]_B &= \partial_\mu A_\mu, \\
[\mathcal{L}_1]_\phi &= \Box \phi - m_1 \partial_\rho A_\rho.
\end{align*} \quad (4.1) \]

The matrix elements of \( D \) are then given by

\[ \begin{align*}
D_{\rho \sigma} &= \delta_{\rho \sigma} (\Box - m_1^2) - \delta_{\rho \sigma} \partial_\sigma, \\
D_{\rho \rho} &= \partial_\rho, \\
D_{\rho B} &= -\partial_\rho, \\
D_{B B} &= 0, \\
D_{\rho \phi} &= -m_1 \partial_\rho, \\
D_{B \phi} &= 0, \\
D_{\phi \phi} &= \Box.
\end{align*} \quad (4.2) \]

In order to find the propagation functions satisfying Eq. (2.4) the inverse matrix
of $D$ has to be found, and its matrix elements are readily derived from (4.2) as

$$D^{-1}_{\mu\nu} = \frac{1}{\Box - m^2_1} \left( \partial_{\mu} \partial_{\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box} \right), \quad D^{-1}_{\mu\nu} = -\frac{\partial_{\mu}}{\Box}, \quad D^{-1}_{\nu\mu} = 0,$$

$$D^{-1}_{\mu\nu} = \frac{\partial_{\mu}}{\Box}, \quad D^{-1}_{\nu\mu} = 0, \quad D^{-1}_{\nu\nu} = \frac{m^2_1}{\Box}, \quad (4.3)$$

From these expressions the propagation functions are easily found. For instance, one finds

$$\langle T^* [A_{\mu}^{in}(x), A_{\nu}^{in}(y)] \rangle_0 = \frac{1}{\Box - m^2_1} \left( \partial_{\mu} \partial_{\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box} \right) \delta^4(x-y)$$

$$= \left( \partial_{\mu} \partial_{\nu} - \frac{\partial_{\mu} \partial_{\nu}}{m^2_1} \right) D_F(x-y, m^2_1) + \frac{1}{m^2_1} \partial_{\mu} \partial_{\nu} D_F(x-y),$$

where $D_F$ is the propagation function for the massless field. In this way all the propagation functions for the asymptotic fields are completely determined.

Our next task is to find a representation of the asymptotic fields satisfying (4.4) in terms of irreducible free fields. The solution reads

$$A_{\mu}^{in} = U_{\mu}^{in} - \frac{1}{m^2_1} \partial_{\mu} \Phi^{in}_{(-)},$$

$$B^{in} = m^2_1 (\Phi^{in}_{(+)} + \Phi^{in}_{(-)}),$$

$$\Phi^{in} = \Phi^{in}_{(+)}.$$  \hspace{1cm} (4.5)

These irreducible fields satisfy

$$\Box U_{\mu}^{in} = 0, \quad \partial_{\mu} U_{\mu}^{in} = 0,$$

$$\Box \Phi^{in}_{(+)} = \Box \Phi^{in}_{(-)} = 0$$  \hspace{1cm} (4.6)

and

$$\langle T^* [U_{\mu}^{in}(x), U_{\nu}^{in}(y)] \rangle_0 = \left( \partial_{\mu} \partial_{\nu} - \frac{\partial_{\mu} \partial_{\nu}}{m^2_1} \right) D_F(x-y, m^2_1),$$

$$\langle T [\Phi^{in}_{(+)}(x), \Phi^{in}_{(+)}(y)] \rangle_0 = \pm D_F(x-y),$$

$$\text{all others} = 0.$$  \hspace{1cm} (4.7)

From these expressions one can see that $U_{\mu}^{in}$ describes a massive Proca field, that $\Phi^{in}_{(\pm)}$ represents the Goldstone boson and that $\Phi^{in}_{(-)}$ corresponds to a massless scalar boson of indefinite metric.

After the quantization of the asymptotic fields the interactions have to be introduced. From the Lagrangian (3.10) it follows immediately that

$$\Box B = 0.$$  \hspace{1cm} (4.8)
Since, however, one has

\[ B(x) = B^{in}(x) - \int d^4x' \mathcal{A}^\mu(x-x') \Box B(x') \]

\[ = B^{out}(x) - \int d^4x' \mathcal{A}^\mu(x-x') \Box B(x'), \]

one may readily draw a conclusion that

\[ B(x) = B^{in}(x) = B^{out}(x) \]  \hspace{1cm} (4.9)

and hence

\[ \phi^{in} \phi^{\dagger} = \phi^{out} \phi^{\dagger} . \]  \hspace{1cm} (4.10)

It should be emphasized that, since \( \phi \) does not satisfy a free field equation, one has

\[ \phi^{in} \neq \phi^{out} , \]  \hspace{1cm} (4.11)

so that

\[ \phi^{in} \neq \phi^{out} . \]  \hspace{1cm} (4.12)

The \( S \) matrix in the Yang-Feldman formalism is introduced through

\[ S^{-1} U^{in}_\mu S = U^{out}_\mu , \]

\[ S^{-1} \phi^{in}_{\phi} S = \phi^{out}_{\phi} , \]

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\[ S^{-1} \phi^{in}_{\phi} S = \phi^{out}_{\phi} , \]

and Eq. (4.9) implies

\[ S^{-1} B S = B \]

or

\[ [S, B] = 0 . \]  \hspace{1cm} (4.14)

Thus, physical states can be defined, as done by Nakanishi, by the following subsidiary condition:

\[ B^{(+)}(x) |\text{phys}\rangle = 0 . \]  \hspace{1cm} (4.15)

Then, the \( S \) matrix transforms a physical state into another physical state as is clear from Eq. (4.14). The consistency between constraints (4.15) at two distinct space-time points is guaranteed by the commutation relation

\[ [B(x), B(y)] = 0 \]  \hspace{1cm} (4.16)

that follows from Eq. (4.5).

The subsidiary condition (4.15) is very similar to the Lorentz condition, and as has been remarked by Nakanishi, the physical states are generated from the vacuum by applying the Hermitian conjugates of
By combining (4.14) and (4.15), however, it immediately follows that, if no massless bosons are present in the initial physical state, the same is true with the final physical state, reached at by applying the $S$ matrix to the former. Thus one can introduce a submatrix of $S$ defined for the vacuum of the massless bosons and denoted by $S_{\text{vac}}$. Then $S_{\text{vac}}$ is a functional of the massive asymptotic fields $U_{\mu}^{\text{in}}$ and $\chi^{\text{in}}$ alone and satisfies

$$S_{\text{vac}}^{-1} U_{\mu}^{\text{in}} S_{\text{vac}} = U_{\mu}^{\text{out}},$$

$$S_{\text{vac}}^{-1} \chi^{\text{in}} S_{\text{vac}} = \chi^{\text{out}}.$$  (4.18)

The unitarity of $S_{\text{vac}}$ is then evident. Also, Higgs' theorem apparently holds, since the Goldstone boson has been completely eliminated in the matrix $S_{\text{vac}}$ and at the same time the gauge field acquired a finite mass.

References

   A. Salam and J. Strathdee, ICTP/71/145.
10) The quantization procedure in the Yang-Feldman formalism presented in this article is essentially based on Peierls' method.
    See also, K. Nishijima, Fields and Particles (W. A. Benjamin Inc., New York, 1969).