Dispersion Approach to the Renormalization Group Equations

Kazuhiko NISHIJIMA

Department of Physics, University of Tokyo, Tokyo

(Received November 16, 1973)

There are various derivations of the renormalization group equations. In most of them, however, unrenormalized expressions or regulator fields appear in the course of the derivations. In this paper a new derivation of the renormalization group equations is presented on the basis of dispersion theory using only renormalized finite expressions from the start.

§ 1. Introduction

In considering the problem of studying the ultraviolet asymptotic behavior of Green's functions and scaling, the renormalization group equations (RGE) play a fundamental rôle. These equations have been derived by Ovsiannikov,\(^1\) Callan,\(^2\) Symanzik\(^3\) and many others.\(^4\) These derivations are based essentially on the renormalization group and the dilatation operator. All these methods start from a Lagrangian involving cutoff-dependent renormalization constants, and it is not a trivial task to verify that various resulting expressions are finite and meaningful and that they are independent of the choice of the regularization procedures.

Thus it would be useful to look for a new derivation which is free from any cutoff-dependent expressions, and this is the motivation of the present investigation. In pursuing this program one is reminded of the dispersion approach to the S matrix which deals only with renormalized expressions. In dispersion theory divergences could occur only in the dispersive parts of the S-matrix elements, but they can easily be eliminated by introducing subtracted dispersion relations. In deriving the RGE, however, one has to handle Green's functions rather than the S-matrix elements so that one needs dispersion relations for Green's functions with all the four-momenta off the mass shell. Such dispersion relations have been known to us for some time\(^5\)\(^6\) and they will be utilized for the derivation of the RGE.

In the next section a brief summary of the dispersion approach to Green's functions is given, and in § 3 this approach is extended so that one can evaluate the matrix elements of renormalized Heisenberg operators. Then in § 4 the Ward-Takahashi identities for the energy-momentum tensor\(^7\)\(^8\) are introduced and some of the consequences of these identities relevant to our purpose are derived. Finally in § 5, the generalized unitarity condition is combined with
the results of the preceding section in order to derive the RGE.

§ 2. Dispersion approach

The problem of formulating field theories in terms of unitarity and dispersion relations has been discussed by the present author.\(^5\) The essence of this formulation will be recapitulated in what follows for the purpose of introducing the appropriate notation.

For simplicity we shall consider only the neutral scalar field \(\phi\), and define the \(\tau\) functions by

\[
\tau(x_1, \cdots, x_n) = (-i)^n K_{x_1} \cdots K_{x_n} \langle 0 | T[\phi(x_1) \cdots \phi(x_n)] | 0 \rangle. \tag{2.1}
\]

The Fourier transforms of the \(\tau\) functions give the S-matrix elements when all the four-momenta are on the mass shell. These \(\tau\) functions satisfy the unitarity condition of the following form as a consequence of the LSZ reduction formula:\(^{11}\)

\[
\tau(x_1, \cdots, x_n) + \tau^*(x_1, \cdots, x_n)
\]

\[
+ \sum' \sum^{m} \frac{i^l}{l!} \int (du) (dv) \tau(x_1', \cdots, x_l', u_1, \cdots, u_l)
\]

\[
\times \mathcal{A}^{(+)}(u_1 - v_1) \cdots \mathcal{A}^{(+)}(u_l - v_l) \tau^*(x_{l+1}', \cdots, x_n', v_1, \cdots, v_l)
\]

\[
= 0, \tag{2.2}
\]

where \(\sum'\) denotes that the sum should be taken over all the possible ways of dividing \(x_1, \cdots, x_n\) into two groups, one entering into \(\tau\) and the other into \(\tau^*\) excluding \(k=0\) and \(n\). The contraction function \(\mathcal{A}^{(+)}\) is defined by

\[
i \mathcal{A}^{(+)}(u - v) = \langle 0 | \phi^{in}(u) \phi^{in}(v) | 0 \rangle. \tag{2.3}
\]

The \(\tau\) functions defined above correspond generally to disconnected Feynman graphs, and it is convenient to define the \(\rho\) functions by collecting, among contributions to the \(\tau\) functions, only those corresponding to connected graphs:

\[
\rho(x_1, \cdots, x_n) = \tau(x_1, \cdots, x_n)_{\text{conn}}. \tag{2.4}
\]

The \(\tau\) and \(\rho\) functions are related to each other through the following recursion relationship:

\[
\tau(x, x_1, \cdots, x_n) = \rho(x, x_1, \cdots, x_n)
\]

\[
+ \sum_{k+n} \rho(x, x', \cdots, x') \tau(x_{k+1}, \cdots, x_n'). \tag{2.5}
\]

Then the Fourier transforms of the \(\rho\) functions are introduced by

\[
\rho(x_1, \cdots, x_n) = \frac{-i}{(2\pi)^{4(n-1)}} \int (d\rho) \delta^4(\rho_1 + \cdots + \rho_n)
\]

\[
\times \mathcal{G}(\rho_1, \cdots, \rho_n) \exp \left[ i(\rho_1 x_1 + \cdots + \rho_n x_n) \right]. \tag{2.6}
\]
These $\mathcal{G}$ functions are functions of scalar products of four-momenta and may be eventually denoted as $\mathcal{G}(p_\alpha p_\beta)$, and they are known to satisfy the following dispersion relations:

$$\text{Re } \mathcal{G}(p_\alpha p_\beta \cdot \xi) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \xi} \varepsilon(\xi') \text{Im } \mathcal{G}(p_\alpha p_\beta \cdot \xi'),$$

(2.7)

where $\xi$ is a common scaling parameter to be multiplied into all the scalar products of the form $p_\alpha p_\beta$. Dispersion relations of this type are called parametric dispersion relations.

The coupling constants are introduced through subtractions. As an example, let us consider a theory corresponding to the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2} [\left( \partial \phi \right)^2 + m^2 \phi^2] - \frac{\lambda}{24} \phi^4.$$  

(2.8)

Then the two-point function needs two subtractions and the four-point function needs one subtraction. If we set $\mathcal{G}(p, -p) = \mathcal{G}_2(p)$, the two subtraction conditions are given by

$$\left[ \mathcal{G}_2(p) \right]_{p^2 + m^2 = 0} = 0,$$

$$\left[ \frac{\partial \mathcal{G}_2(p)}{\partial p^2} \right]_{p^2 + m^2 = 0} = -1.$$  

(2.9)

(2.10)

An explicit form of $\mathcal{G}_2(p)$ is given in terms of the Lehmann spectral function $\sigma^\text{th}$ by

$$\mathcal{G}_2(p) = -(p^2 + m^2) \left[ 1 + (p^2 + m^2) \int \frac{d\xi'}{p^2 + \xi'^2 - i\epsilon} \right].$$

(2.11)

The four-point function $\mathcal{G}_4$ needs one subtraction and satisfies

$$\text{Re } \mathcal{G}_4(p_\alpha p_\beta \cdot \xi) = \mathcal{G}_4(0) + \frac{\xi}{\pi} \int \frac{d\xi'}{\xi' - \xi} \varepsilon(\xi') \text{Im } \mathcal{G}_4(p_\alpha p_\beta \cdot \xi').$$

(2.12)

The subtraction constant is determined subject to the condition

$$\left[ \mathcal{G}_4 \right]_{\text{subtraction point}} = \lambda.$$  

(2.13)

As the subtraction point we usually choose either the zero point corresponding to $p_j = 0$ ($j=1, 2, 3, 4$) or the symmetrical point characterized by

$$p_1 p_4 = \frac{m^2}{3} (1 - 4\delta_{14}).$$

(2.14)

The former choice is convenient for the parametric dispersion relations, while the latter is employed in the $S$-matrix theory.

When a $\mathcal{G}$ function is given, it is a rather subtle question to answer whether it needs a subtraction. This problem has been studied in perturbation theory, and we shall quote the results obtained in Ref. 8). For large values of $\xi$, the
ultraviolet asymptotic behavior of \( \mathcal{G}(p_a p_b \cdot \xi) \) is assumed to obey a power law,
\[
\mathcal{G}(p_a p_b \cdot \xi) \sim \xi^{a/2},
\]
provided that all the scalar products are real. The proportionality factor may depend on the \( p \)'s, but the power \( \alpha/2 \) should be independent of the \( p \)'s in almost all the configurations of the \( p \)'s. This statement has been postulated in Ref. 8). The precise meaning of (2·15) is that for an arbitrary positive \( \varepsilon \) we have
\[
\lim_{t \to \infty} \frac{\mathcal{G}(p_a p_b \cdot \xi)}{\xi^{a/2 + \varepsilon}} = 0, \quad \lim_{t \to \infty} \frac{\mathcal{G}(p_a p_b \cdot \xi)}{\xi^{a/2 - 1}} = \infty.
\] (2·16)

The powers \( \alpha \) have been determined in perturbation theory by combining unitarity with the assumed subtracted dispersion relations. In this way we have reached at the conclusion that the powers are given by the canonical dimensions in the renormalizable field theories. To be more precise, the power \( \alpha \) corresponding to the \( n \)-point Green's function in the theory described by (2·8) and denoted by \( c(n) \) is given by
\[
c(n) = 4 - n.
\] (2·17)

These powers are subject to modifications when a non-perturbative approach is employed. Indeed, the RGE indicate that the canonical dimensions should be replaced by the anomalous dimensions.

§ 3. Heisenberg operators in the dispersion approach

In the preceding section we have studied Green's functions \( \mathcal{G} \) whose Fourier transforms represent the \( S \)-matrix elements when all the four-momenta are on the mass shell. In order to study the matrix elements of a local Heisenberg operator \( A(x) \), however, we have to consider the following set of functions:
\[
\tau_A(x; x_1, \cdots, x_n) = (-i)^{n+1} K x_1 \cdots K x_n \langle 0 | T \{ A(x) \phi(x_1) \cdots \phi(x_n) \} | 0 \rangle.
\] (3·1)

An arbitrary matrix element of \( A(x) \) is obtained from (3·1) by taking its Fourier transform with the help of the LSZ reduction formula.\(^{11}\) Just as we have derived the \( \rho \) functions from the \( \tau \) functions, the functions \( \rho_A \) are defined as the connected part of the functions \( \tau_A \).
\[
\rho_A(x; x_1, \cdots, x_n) = \tau_A(x; x_1, \cdots, x_n)_\text{conn}.
\] (3·2)

The recursion formula connecting them is given by
\[
\tau_A(x; x_1, \cdots, x_n) = \rho_A(x; x_1, \cdots, x_n) + \sum_{\text{comb}} \rho_A(x; x_1', \cdots, x_n') \tau(x_{k+1}', \cdots, x_n').
\] (3·3)
The Fourier transforms of the functions $\rho_A$ are introduced by

$$
\rho_A(x; x_1, \cdots, x_n) = \frac{-i}{(2\pi)^n} \int (dp) d^4 q \delta^4(q + p_1 + \cdots + p_n) \times \bar{\mathcal{A}}(q; p_1, \cdots, p_n) \exp \left[ i(qx + p_1 x_1 + \cdots + p_n x_n) \right]. \quad (3.4)
$$

The functions $\bar{\mathcal{A}}$ are considered to be functions of the scalar products of the form $p_a p_{\beta'}$.

The unitarity condition for $\tau_A$ reads

$$
\tau_A(x; x_1, \cdots, x_n) + \tau_A^*(x; x_1, \cdots, x_n)
+ \sum_{\text{comb}} \sum_{l=0}^{\infty} \frac{i^l}{l!} \int (du) (dv) \left[ \tau_A(x; x_1', \cdots, x_k', u_1, \cdots, u_l)
\times \mathcal{A}^{(+)}(u_1 - v_1) \cdots \mathcal{A}^{(+)}(u_l - v_l) \tau_A^*(x_{k+1}', \cdots, x_n', v_1, \cdots, v_l)
+ \tau_A(x_1', \cdots, x_k', u_1, \cdots, u_l)
\times \mathcal{A}^{(+)}(u_1 - v_1) \cdots \mathcal{A}^{(+)}(u_l - v_l) \tau_A^*(x; x_{k+1}', \cdots, x_n', v_1, \cdots, v_l) \right]
= 0. \quad (3.5)
$$

When this unitarity condition is expressed in terms of $\rho_A$ by taking its connected part, we find that the resulting relationship is linear in $\rho_A$ and $\rho_A^*$ so that we shall refer to (3.5) as the linear unitarity condition as contrasted to the non-linear unitarity condition (2.2) for the $\rho$ functions.

The parametric dispersion relations for the $\mathcal{U}$ functions are easily generalized and applied to the $\bar{\mathcal{A}}$ functions.

$$
\text{Re} \bar{\mathcal{A}}(p_a p_{\beta'}; \xi) = \frac{P}{\pi} \int_0^\infty \frac{d\xi'}{\xi' - \xi} \xi'(\xi') \text{Im} \bar{\mathcal{A}}(p_a p_{\beta'}; \xi'). \quad (3.6)
$$

In order to normalize the $\bar{\mathcal{A}}$ functions we have to introduce subtractions for appropriate $n$. The problem of subtractions is apparently related to the asymptotic ultraviolet behavior of the $\mathcal{A}$ functions. For this purpose we shall briefly recall how the canonical power law (2.17) has been obtained in perturbation theory.

Since the power of an absorptive part of a $\mathcal{U}$ function cannot exceed that of the full $\mathcal{U}$ function, we obtain from the non-linear unitarity condition for $\mathcal{U}$ the following inequalities:

$$
d(n) \geq \text{Max}_{k+l \geq 2} \left[ d(k+l) + d(n-k+l) \right], \quad (3.7)
$$

where

$$
d(n) = c(n) + n - 4. \quad (3.8)
$$

Now combining (3.7) with the subtraction conditions, namely, two subtractions...
for $n=2$ and one subtraction for $n=4$, leading to
\[ c(2) \geq 2, \quad c(4) \geq 0, \quad (3.9) \]
we arrive at (3.7) or
\[ d(n) = 0. \quad (3.10) \]

We shall extend this argument to the $\vec{A}$ functions, and let us first assume the asymptotic power law for the $n$-point $\vec{A}$ function as
\[ \vec{A}_n(p_1, \ldots, p_n) \sim \xi^n. \quad (3.11) \]
If we define $b(n)$ by
\[ b(n) = a(n) + n - 4, \quad (3.12) \]
then the linear unitarity condition leads to the inequalities
\[ b(n) \geq \text{Max}[b(k + l) + d(n-k+l)]. \quad (3.13) \]
However, since $d = 0$, the general solution is given by
\[ b(n) = b_0 \]
or
\[ a(n) = b_0 + 4 - n, \quad (3.14) \]
where the $n$-independent constant $b_0$ should be determined subject to the subtraction conditions. We shall illustrate determination of the constant $b_0$ by simple and typical examples.

**Example 1.** $A(x) = \frac{1}{2} \phi^2(x)$

This choice poses two problems. The first problem stems from the fact that the product of two field operators at the same space-time point is singular and we have to establish the regularization procedures. The second problem is concerned with the question of subtractions. In what follows we shall solve these two problems simultaneously by a heuristic method. In order to solve the subtraction problem, let us calculate the $\rho_\phi$ function in the lowest order perturbation theory. Then we find
\[ \rho_\phi(x; x_1, x_2) = -i \delta^4(x-x_1) \delta^4(x-x_2) \quad (3.15) \]
and all others vanish in the free field approximation. In the momentum space Eq. (3.15) reads
\[ \vec{\rho}(q; p_1, p_2) = 1. \quad (3.16) \]
Thus it is clear that we should introduce a subtraction for $n=2$ implying $a(2) = 0$ and $b_0 = -2$, so that
\[ a(n) = 2 - n. \quad (3.17) \]
Dispersion Approach to the Renormalization Group Equations

Hence $\mathcal{A}(q; p_1, p_2)$ corresponding to $n=2$ is the only member of the $\mathcal{A}$ functions that needs a subtraction. For $n=0$ and 1, $\mathcal{A}$ functions can be set equal to zero.

As the most convenient subtraction condition we may employ the following one:

$$\mathcal{A}(q; p_1, p_2) = 1 \text{ for } q=0, \quad p_1^* + m^1 = p_2^* + m^2 = 0. \quad (3.18)$$

**Example 2.**

$$A(x) = \frac{1}{24} \phi^4(x) \quad (3.19)$$

We can proceed in the same way as in the preceding example, and the only non-vanishing $\rho_4$ function in the free field approximation is given by

$$\rho_4(x; x_1, x_2, x_3, x_4) = -i \delta^4(x - x_1) \delta^4(x - x_2) \delta^4(x - x_3) \delta^4(x - x_4) \quad (3.20)$$

or

$$\mathcal{A}(q; p_1, p_2, p_3, p_4) = 1. \quad (3.21)$$

In this case we have $b_0 = 0$ and hence

$$a(n) = 4 - n, \quad (3.22)$$

which implies two subtractions for $n=2$ in addition to one subtraction for $n=4$.

A difficulty with this example consists in the point that we have no guiding principle to fix the two subtraction constants for $n=2$. This is still a simple example, but in the next section we shall discuss how to fix the subtractions when the operator $A$ represents the energy-momentum tensor.

§ 4. Ward-Takahashi identities for the energy-momentum tensor

In deriving the RGE the energy-momentum tensor plays an important rôle. In this section we shall discuss the properties of the energy-momentum tensor.

The canonical energy-momentum tensor is defined by

$$T_{\mu\nu} = -\frac{\partial L}{\partial \phi_{;\mu}} + \delta_{\mu\nu} L, \quad (4.1)$$

where $\phi_{;\mu} = \partial_{\mu} \phi$. In what follows we shall confine ourselves to the scalar model characterized by the Lagrangian (2.8), then (4.1) is also the symmetric energy-momentum tensor.

The Ward-Takahashi (W-T) identities are given by

$$\frac{\partial}{\partial x_\mu} T^* \left[ T_{\mu\nu}(x) \phi(x_1), \cdots, \phi(x_n) \right]$$

$$= i \sum \delta^4(x - x_j) \frac{\partial}{\partial x_{\lambda^*}} T[\phi(x_1) \cdots \phi(x_n)]. \quad (4.2)$$

In the following it is our policy to employ (4.2) as the definition of the energy-
momentum tensor rather than the dangerous explicit formula (4·1). Although this leaves a certain arbitrariness in the definition of the energy-momentum tensor, it is harmless for our purpose of deriving the RGE.

In order to study to what extent the energy-momentum tensor is determined, let us first write down the W-T identities in the momentum space.

\[ q_\mu T_{\mu \nu} (q; p_1, \ldots, p_n) = \sum \frac{p_j^2 + m^2}{(p_j + q)^2 + m^2 - i\varepsilon} (p_j + q)_\nu \]
\[ \times G (p_1, \ldots, p_j + q, \ldots, p_n) , \quad (4·3) \]

where \( p + q_1 + \cdots + p_n = 0 \). \( T_{\mu \nu} \) are defined from \( T_{\mu \nu} \) just as \( \mathcal{A} \) were defined from \( \mathcal{A} \) in (3·4). When all the momenta \( p_j \) are on the mass shell, the right-hand side of Eq. (4·3) vanishes. With the help of the LSZ reduction formula this amounts to the conservation law

\[ \partial_\nu T_{\mu \nu} = 0. \quad (4·4) \]

Furthermore, Eq. (4·3) dictates the canonical asymptotic ultraviolet behavior of \( T_{\mu \nu} \).

\[ c (\mathcal{T}_{\mu \nu} (n)) = c (\mathcal{L} (n)) = 4 - n. \quad (4·5) \]

Thus, when we consider a set of functions \( \{ T_{\mu \nu} (q; p_1, \ldots, p_n) \} \), two subtractions are generally required for \( n=2 \) and at most one subtraction is required for \( n=4 \).

In this paper we shall not work out the subtraction conditions explicitly, but instead we shall employ the W-T identities as the substitute. In quantum electrodynamics it is known that the subtraction condition is completely contained in the W-T identities,7 but this is not the case for the energy-momentum tensor. The W-T identities determine the subtraction conditions only incompletely as we shall see below.

For this purpose we shall first solve (4·3) for \( n=2 \) by making use of the spectral representation (2·11). The equation for the two-point \( \mathcal{T}_{\mu \nu} \) function is inhomogeneous so that its general solution is given by the sum of a special solution of the inhomogeneous equation and the general solution of the homogeneous equation.

\[ \mathcal{T}_{\mu \nu} = (p_1 p_2 - m^2) \delta_{\mu \nu} - (p_1 \mu p_{2\nu} + p_{2\mu} p_{1\nu}) \]
\[ + \frac{1}{2} (p_1^2 + m^2) (p_2^2 + m^2) \left[ \delta_{\mu \nu} \int d\kappa \sigma (\kappa) \left( \frac{1}{(p_1^{2} + \kappa^2 - i\varepsilon)} + \frac{1}{(p_2^{2} + \kappa^2 - i\varepsilon)} \right) \right. \]
\[ \left. + k_\mu k_\nu \int d\kappa \frac{\sigma (\kappa)}{(p_1^{2} + \kappa^2 - i\varepsilon) (p_2^{2} + \kappa^2 - i\varepsilon)} \right] + S_{\mu \nu}^{(L)} + S_{\mu \nu}^{(O)} G, \quad (4·6) \]

where \( q = -(p_1 + p_2) \), \( k = p_1 - p_2 \) and

\[ S_{\mu \nu}^{(O)} = q^2 \delta_{\mu \nu} - q_\mu q_\nu , \]
Dispersion Approach to the Renormalization Group Equations

\[ S_{\mu \nu}^{(2)} = (qk)^2 q_{\mu} q_{\nu} + (q^2)^2 k_{\mu} k_{\nu} - q^2 (qk) (qk_{\mu} + q_{\mu} k) \]  

(4.7)

Both \( S_{\mu \nu}^{(2)} \) and \( S_{\mu \nu}^{(3)} \) are symmetric in \( \mu \) and \( \nu \) and satisfy

\[ q_{\mu} S_{\mu \nu}^{(j)} = 0. \quad (j = 1, 2) \]

(4.8)

Hence we can read off the asymptotic behavior of the form factors as given by

\[ F(\xi) \sim \xi^0 \quad \text{and} \quad G(\xi) \sim \xi^{-2}. \]

This means that we have to fix one subtraction constant for \( F \), and this is the only arbitrariness in determining the energy-momentum tensor.

Now we shall proceed to \( n = 4 \). Then it is clear that covariants symmetric in \( \mu \) and \( \nu \) and satisfying (4.8) cannot stay constant in the ultraviolet asymptotic region. Besides, we have \( c(\mathcal{T}_{\mu \nu}(4)) = 0 \) so that form factors, coefficients of the covariants mentioned above, need no subtractions.

Thus we may conclude that the only arbitrariness left for the functions \( \mathcal{T}_{\mu \nu} \) is the subtraction constant for the form factor \( F \). Needless to say, this arbitrariness propagates from \( n = 2 \) to many-point functions through unitarity introducing an infinite set of \( F \) functions. In order to work out this problem further we shall introduce a set of functions

\[ \mathcal{T}(q; p_1, \ldots, p_n) \]

(4.9)

corresponding to the operator \(-m^2 \partial^4(x)\). This set of functions is essentially identical with the one considered in the example 1 of the preceding section. The only difference consists in the subtraction condition.

\[ \mathcal{T}(q; p_1, p_k) = -2m^2 \quad \text{for} \quad q = 0, \quad p_1^2 + m^2 = p_k^2 + m^2 = 0. \]

(4.10)

Both sets of functions, \( \mathcal{T} \) and \( (q^2 \delta_{\mu \nu} - q_{\mu} q_{\nu}) F \), satisfy the same linear unitarity condition, and in both cases one subtraction is required for \( n = 2 \). Hence they should be proportional, namely,

\[ F(q; p_1, \ldots, p_n) = a \mathcal{T}(q; p_1, \ldots, p_n). \]

(4.11)

Thus, if a set \( \{\mathcal{T}_{\mu \nu}\} \) represents an acceptable solution, then

\[ \{\mathcal{T}_{\mu \nu}(q; p_1, \ldots, p_n) + a(q^2 \delta_{\mu \nu} - q_{\mu} q_{\nu}) \mathcal{T}(q; p_1, \ldots, p_n)\} \]

(4.12)

is also an acceptable solution. This settles the nature of the arbitrariness of the energy-momentum tensor completely. From this result it is clear that \( \{\mathcal{T}_{\mu \nu}\} \) is uniquely determined when \( q = 0 \). In what follows we shall explicitly express \( \mathcal{T}_{\mu \nu}(0; p_1, \ldots, p_n) \) in terms of the \( \mathcal{G} \) functions.

Let us introduce ordinary Green's functions by

\[ G(p_1, \ldots, p_n) = \frac{\mathcal{G}(p_1, \ldots, p_n)}{\Pi_j (-p_j^2 - m^2 + i\epsilon)}, \]

(4.13a)

\[ T_{\mu \nu}(q; p_1, \ldots, p_n) = \frac{\mathcal{T}_{\mu \nu}(q; p_1, \ldots, p_n)}{\Pi_j (-p_j^2 - m^2 + i\epsilon)}, \]

(4.13b)
\[ T(q; p_1, \cdots, p_n) = \frac{T(q; p_1, \cdots, p_n)}{\prod_j (-p_j^2 - m^2 + i\varepsilon)}. \] (4.13c)

Then the W.T identities (4.3) takes the form
\[ q_\mu T_{\mu \nu}(q; p_1, \cdots, p_n) = \sum_j (p_j + q)_\mu G(p_1, \cdots, p_j + q, \cdots, p_n). \] (4.14)

By differentiating (4.14) with respect to \( q \) and then setting \( q = 0 \), we find
\[ T_{\mu \nu}(0; p_1, \cdots, p_n) = \left[ (n - 1) \delta_{\mu \nu} + \sum_j p_j \frac{\partial}{\partial p_j} \right] G(p_1, \cdots, p_n). \] (4.15)

This formula explicitly shows that \( T_{\mu \nu} \) is uniquely determined and can be expressed in terms of \( G \) when \( q = 0 \). We also notice that \( G_n \) is a homogeneous function of \( p_1, \cdots, p_n \) and \( m \) with the dimension of \( m_{4-s} \), and Euler's theorem on homogeneous functions leads to
\[ \left( \sum_j p_j \frac{\partial}{\partial p_j} + m \frac{\partial}{\partial m} + 3n - 4 \right) G(p_1, \cdots, p_n) = 0. \] (4.16)

Taking the trace of (4.15) and then combining it with (4.16) we obtain
\[ T_{\mu \nu}(0; p_1, \cdots, p_n) = \left[ 4(n - 1) + \sum_j p_j \frac{\partial}{\partial p_j} \right] G(p_1, \cdots, p_n) = \left( n - m \frac{\partial}{\partial m} \right) G(p_1, \cdots, p_n). \] (4.17)

§ 5. Derivation of the renormalization group equations

As we have seen so far it is clear that sets of functions such as
\[ \{ T_{\mu \nu}(q; p_1, \cdots, p_n) \} \] (5.1)
and
\[ \{ T(q; p_1, \cdots, p_n) \} \] (5.2)
satisfy the linear unitarity condition. Similarly, the following set satisfies the linear unitarity condition:
\[ \sum_j \frac{p_j^2 + m^2}{(p_j + q)^2 + m^2 - i\varepsilon} G(p_1, \cdots, p_j + q, \cdots, p_n). \] (5.3)

In verifying unitarity it is important that the \( j \)-th term in the sum vanishes when \( p_j^2 + m^2 = 0 \). When all the \( p \)'s are on the mass shell, the expression (5.3) vanishes.

Now a question is raised if these sets satisfy the linear unitarity condition even in the limit \( q \to 0 \). It so happens, however, that these functions develop discontinuities at \( q = 0 \). For instance, the expression (5.3) reduces, in the limit
Dispersion Approach to the Renormalization Group Equations

$q \to 0$, to

\[ n \mathcal{G}(p_1, \cdots, p_n), \quad (5.4) \]

which neither vanishes when all the momenta are on the mass shell, nor satisfies
the linear unitarity condition. A similar situation occurs to the other sets. The
cause of all such troubles is the presence of poles of the form

\[ \frac{1}{(p_j + q)^2 + m^2 - i\epsilon}. \quad (5.5) \]

In the case of the sets (5.1) and (5.2) such a pole arises from diagrams in
which the two-point function

\[ \mathcal{I}_{\mu
u}(q; p_j, -p_j - q) \text{ or } \mathcal{I}(q; p_j, -p_j - q) \]

is connected to the rest by a propagator of the form (5.5). Thus, in order to
make them satisfy the linear unitarity condition it is necessary to cancel such
poles in the limit $q \to 0$.

This goal is achieved if the two-point functions at $q = 0$ do not contribute,
and in order to realize it they have to vanish on the mass shell even after they
are divided by $p_j^2 + m^2$. It is impossible to satisfy this condition for each set,
but it is possible to satisfy it by taking an appropriate linear combination of
the two-point functions. Take the combination

\[ \mathcal{I}_{\mu
u}(p) - \mathcal{I}(p) - 2d_\phi \mathcal{G}(p), \quad (5.6) \]

where each term represents the two-point function at $q = 0$ from each set. On
the mass shell we have, on the basis of Eqs. (4.6) and (4.10),

\[ \mathcal{I}_{\mu
u}(p) - \mathcal{I}(p) = -2m^2 + 2m^2 = 0, \]

\[ \mathcal{G}(p) = 0. \]

Hence we can write the following relationship in the neighborhood of the mass
shell:

\[ \mathcal{I}_{\mu
u}(p) - \mathcal{I}(p) = O(p^4 + m^4), \]

\[ \mathcal{G}(p) = O(p^4 + m^4), \]

and by choosing the parameter $d_\phi$ appropriately we may require

\[ \mathcal{I}_{\mu
u}(p) - \mathcal{I}(p) - 2d_\phi \mathcal{G}(p) = O((p^4 + m^4)^2). \quad (5.7) \]

On the other hand we know from (4.6) and (2.11) that

\[ \mathcal{I}_{\mu
u}(p) = -2m^2 - 2(p^4 + m^4) + O((p^4 + m^4)^2), \]

\[ \mathcal{G}(p) = -(p^4 + m^4) + O((p^4 + m^4)^2), \]

so that $d_\phi = 1 + \gamma_\phi$ should be identified with the following expansion coefficient
of $\mathcal{G}(p)$ in powers of $(p^4 + m^4)$:
Thus, if $d_\phi$ is chosen as indicated above, the set

\[
\{ \mathcal{I}(p_1, \cdots, p_n) \}
\]

with

\[
\mathcal{I}(p_1, \cdots, p_n) = \mathcal{I}_{\rho \rho}(0; p_1, \cdots, p_n) - \mathcal{I}(0; p_1, \cdots, p_n) - n d_\phi \mathcal{G}(p_1, \cdots, p_n)
\]

satisfies the linear unitarity condition. The subtraction conditions for this set are as follows:

For $n=2$; $\mathcal{I}_2(p) = 0$, $\frac{\partial \mathcal{I}_2(p)}{\partial p^2} = 0$ when $p^2 + m^2 = 0$. (5.10)

For $n=4$; $\mathcal{I}_4(p) = \beta(\lambda)$ at the subtraction point. (5.11)

Next consider a new set

\[
\left\{ \frac{\partial}{\partial \lambda} \mathcal{G}(p_1, \cdots, p_n) \right\}
\]

Then it is clear, by differentiating the non-linear unitarity condition for $\mathcal{G}$ with respect to $\lambda$, that this set satisfies the linear unitarity condition. From the representation (2.11) it is also clear that

for $n=2$; $\frac{\partial}{\partial \lambda} \mathcal{I}_2(p) = 0$, $\frac{\partial}{\partial p^2} \left( \frac{\partial}{\partial \lambda} \mathcal{I}_2(p) \right) = 0$ when $p^2 + m^2 = 0$, (5.13)

for $n=4$; $\frac{\partial}{\partial \lambda} \mathcal{I}_4(p) = 1$ at the subtraction point. (5.14)

Thus we may conclude that

\[
\mathcal{I}(p_1, \cdots, p_n) = \beta(\lambda) \frac{\partial}{\partial \lambda} \mathcal{G}(p_1, \cdots, p_n)
\]

or

\[
T_{\rho \rho}(0; p_1, \cdots, p_n) - T(0; p_1, \cdots, p_n) - n d_\phi \mathcal{G}(p_1, \cdots, p_n)
\]

\[
= \beta(\lambda) \frac{\partial}{\partial \lambda} \mathcal{G}(p_1, \cdots, p_n).
\]

These are precisely the anomalous trace identities derived by Ukawa on the continuous dimensional method. By combining (5.16) with (4.17) we finally arrive at the RGE for Green's functions.

\[
\left( m - \frac{\partial}{\partial m} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n_d(\phi) \right) G(p_1, \cdots, p_n) = \Delta G(p_1, \cdots, p_n),
\]

where

\[
\Delta G(p_1, \cdots, p_n) = -T(0; p_1, \cdots, p_n).
\]

This completes the derivation of the RGE.
Dispersion Approach to the Renormalization Group Equations

References

4) A. Ukawa, University of Tokyo preprint, UT-215.


The author thanks Prof. C. H. Woo for pointing out the presence of this paper.