Higher Order Gravitational Potential for Many-Body System

Tadayuki OHTA, Hiroshi OKAMURA,*
Toshiei KIMURA** and Kichiro HIIDA***

Department of Mathematical Physics, College of Technology
Seikei University, Musashino, Tokyo

*Department of Physics, University of Tokyo, Tokyo
and
Department of Physics, Faculty of General Education
Kogakuin University, Hachioji, Tokyo

**Research Institute for Theoretical Physics, Hiroshima University
Takehara, Hiroshima-ken

***Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo

(Received October 8, 1973)

Gravitational potential for many-body system is obtained up to post-post-Newtonian order of approximation from the metric tensor derived previously, which is Minkowskian at spatial infinity. The calculation is based on the Lagrangian of Fokker type. The transverse-traceless part of the metric tensor contributes to the potential of post-post-Newtonian order, even to the $G^{3}$-static part. The fact that the $G^{3}$-static potential includes the contribution from the transverse-traceless part is the manifestation of non-linear nature of the theory of gravity. The gravitational potential obtained here coincides with that calculated in the canonical formalism, but does not coincide with that obtained in the conventional formalism of quantized theory.

§ 1. Introduction

Since Einstein's theory of gravity is invariant under the general coordinate transformation, there exists a degree of freedom of choosing four arbitrary functions. In order to determine the four functions, we must impose a coordinate condition on the metric tensor $g_{ab}$.*) When Einstein's equation for many-body system is solved successively by the approximation method of expanding all quantities in the inverse powers of the velocity of light $c$, the metric tensors up to post-Newtonian order become Minkowskian at a place infinitely remote from the region where the matter exists, as far as known coordinate conditions are used.5) These metric tensors give the right equations of motion up to post-

*) In this paper we use the following conventions. Greek indices run from 0 to 3, while Latin indices $i, j, k$ and $l$ run from 1 to 3. Repetition of these indices implies summation. A comma in a subscript or the symbol $\partial$ denotes a partial derivative. We discriminate the bodies by Latin indices $a, b, c$ and $d$ which take the values 1, 2, ..., $n$. 
Higher Order Gravitational Potential for Many-Body System

Newtonian order. In the post-post-Newtonian and higher orders, however, the metric tensors diverge at spatial infinity when known coordinate conditions are employed. These solutions are physically unacceptable.

In a previous paper (hereafter referred to as [I]), we proposed a method to determine the coordinate condition which leads to the Minkowskian metric tensor at spatial infinity, and obtained the solution up to post-post-Newtonian order. We called it the physically acceptable metric tensor. In this paper we calculate the Hamiltonian for many-body system up to post-post-Newtonian order, using the metric tensor.

Gravitational potential for many-body system is obtained from the Lagrangian of Fokker type:  

\[ L = \int d^4x (L_M + L_G). \]  

The Lagrangian densities \( L_M \) and \( L_G \) are given by  

\[ L_M = -\sum_a m_a c^2 \delta(x - z_a) \sqrt{-g_{\mu\nu}} \frac{dz_a^\mu}{dx^\alpha} \frac{dz_a^\nu}{dx^\beta}, \]  

\[ L_G = -\frac{c^4}{16\pi G} \sqrt{-g} g_{\mu\nu} (\Gamma_{\rho\mu}^\alpha \Gamma_{\rho\nu}^\alpha - \Gamma_{\mu\rho}^\alpha \Gamma_{\nu\rho}^\alpha), \]  

where \( G, g, \Gamma_{\rho\mu}^\alpha, m_a \) and \( z_a \) are Newton's gravitational constant, \( \det (g_{\mu\nu}) \), Christoffel's symbol, the rest mass and the coordinate of \( a \)-th body, respectively, and \( z_a^0 = 1 \). In the expressions (1·2) and (1·3), \( \Gamma_{\rho\mu}^\alpha \) and \( g_{\mu\nu} \) depend on the \( z_a \)'s and \( z_\alpha \)'s, and do not depend on higher time derivatives of the \( z_a \)'s, which can be reduced to the combinations of the \( z_a \)'s and \( z_\alpha \)'s by repeated use of the equations of motion in lower order of approximation.

We calculated in [I] the Lagrangian (1·1) and obtained the most general Hamiltonian for many-body system up to post-Newtonian order. In the post-post-Newtonian and higher orders, if a divergent metric tensor is employed, the spatial integration in the Lagrangian (1·1) diverges and Hamiltonian cannot be determined. The Hamiltonian for many-body system is obtained from (1·1) using the physically acceptable metric tensor. The potential in post-post-Newtonian order consists of three parts; \( G^2 \), \( G^2p^2 \) and \( Gp^4 \)-parts, where \( p \) denotes momentum. We have already obtained in [I] the \( Gp^4 \)-part of the potential, which coincides with the one-graviton-exchange potential calculated in quantized theory. The remaining two parts are calculated in this paper.

In the next section the metric tensor given in [I] is rewritten in a simple form. We calculate the Lagrangian (1·1) in post-post-Newtonian order in § 3 and obtain the potential in this order in § 4. It was pointed out in quantized theory that the so-called "substitution law" exists which relates the \( n \)-body static potential of order \( G^{n-1} \) with \( (n-1)-, (n-2)-, \ldots, 3- \) and 2-body static potentials in the same order of \( G \). The reason why the law holds is clarified in the case of \( n = 4 \) from the viewpoint of this work. Contributions from the
transverse-traceless part of the metric tensor appear first in the post-post-Newtonian potential. In particular the $G^3$-static potential also contains the contributions. This is a characteristic feature of the theory of gravity.

In a separate paper\(^7\) the potential in post-post-Newtonian order will be calculated in the canonical formalism.\(^9\) It will be shown that the potential coincides with that obtained in §4. On the other hand, in a previous paper\(^9\) (hereafter referred to as [II]) the $G^3$-static potential for four-body system was calculated in the conventional formalism of quantized theory. It does not coincide with the static potential given in §4, though the $G^p$-potentials in classical and quantized theories coincide. In §5 the Hamiltonian for two-body system is presented up to post-post-Newtonian order. The final section is devoted to discussion. Various formulas needed to evaluate the Lagrangian (1.1) are collected in Appendices.

§2. Physically acceptable metric tensor

Einstein's equation for gravitational field is

\[
R_{\mu\nu} = \frac{8\pi G}{c^4} (g_{\mu\rho} g_{\nu\sigma} - \frac{1}{2} g_{\mu\sigma} g_{\nu\rho}) T^{\rho\sigma},
\]  
(2·1)

where $R_{\mu\nu}$ and $T^{\rho\sigma}$ are the curvature tensor and the energy-momentum tensor for material system, respectively. For the system of point particles, $T^{\rho\sigma}$ is given by

\[
T^{\rho\sigma} = - \sum \frac{m_a}{\sqrt{-g}} \frac{dx^\rho}{dt} \frac{dx^\sigma}{dt} \frac{dt}{d\tau} \delta(x-x_a),
\]  
(2·2)

where $d\tau = c^{-1} \sqrt{-g} dx^\rho dx^\sigma$. Equation (2·1) was solved in [I] by expanding the metric tensor $g_{\alpha\beta}$ in the inverse powers of the velocity of light:

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)} + h_{\alpha\beta}^{(3)} + h_{\alpha\beta}^{(4)} + \cdots,
\]  
(2·3)

where the numbers in superscripts with parentheses denote the order of $c^{-1}$. We have obtained the metric tensor up to post-post-Newtonian order, which is Minkowskian at spatial infinity. The metric tensor contains the parameter $\gamma$ which corresponds to the gauge-parameter $\frac{1}{2}(1-x)$ in quantized theory.\(^9\) The metric tensor $g_{\alpha\beta}$ takes the most simple form when $\gamma = 0$. The precession of the perihelion for two-body system is independent of $\gamma$, at least, up to post-Newtonian order.\(^9\) For these reasons we put in the following

\[
\gamma = 0.
\]  
(2·4)

Now we write down the solutions:\(^8\)

\[
h_{\alpha\beta}^{(2)} = 2G \sum \frac{m_a}{r_a},
\]  
(2·5)

\(^{8}\) Hereafter we take the unit $c=1$. 

Downloaded from https://academic.oup.com/ptp/article-abstract/51/4/1220/1845509 by guest on 02 March 2019
Higher Order Gravitational Potential for Many-Body System

\( h^{(i)} = \delta_{ij} \ 2G \sum_a \frac{m_a}{r_a}, \)  
\( (2.6) \)

\( h_0^{(i)} = -4G \sum_a \frac{m_a v_a^i}{r_a} + \frac{1}{2} \partial_i \{ G \sum_a m_a (n_a \cdot v_a) \}, \)  
\( (2.7) \)

\( h_0^{(i)} = -2G^2 \sum_a \frac{m_a m_b}{r_a r_b} - 2G^2 \sum_{b \neq a} \frac{m_a m_b}{r_a r_{ab}} + 3G \sum_a \frac{m_a v_a^2}{r_a}, \)  
\( (2.8) \)

\( h_0^{(i)} = h_0^{(i)} = h_0^{(i)} = h_0^{(i)} = 0, \)  
\( (2.9) \)

where \( r_a = |x - z_a|, \) \( r_{ab} = |z_a - z_b|, \) \( n_a = (x - z_a)/|x - z_a|, \) and \( v_a \) is the velocity of \( a \)-th body. At the first glance the expression \((1.2 \cdot 39)\) with \((1.2 \cdot 39)\) is complicated. But it can be decomposed into the form

\[
\begin{align*}
&h^{(i)}_f = \delta_{ij} \left( \frac{3}{2} G^2 \sum_a \frac{m_a m_b}{r_a r_b} - G^2 \sum_{b \neq a} \frac{m_a m_b}{r_{ab}} + G \sum_a \frac{m_a v_a^2}{r_a} \right) \\
&+ \partial_i \partial_j \left( -\frac{1}{2} G^2 \sum_a \frac{m_a m_b}{r_a r_{ab}} + \frac{1}{2} G \sum_a m_a v_a^2 \right) + \{ h^{(i)}_f \}^{TT},
\end{align*}
\]
\( (2.10) \)

where

\[
\{ h^{(i)}_f \}^{TT} = -4 f^{TT}_{ij} + 4G \sum_a \left( \frac{1}{r_a} v_a^i v_a^j \right)^{TT},
\]  
\( (2.11) \)

in which \( f^{TT}_{ij} \) and \( (v^i_a v^j_a/r_a)^{TT} \) represent the transverse-traceless parts of

\[
f_{ij} = G^2 \sum_{b \neq a} \frac{m_a m_b}{r_{ab}} \frac{\partial^2}{\partial z_a^i \partial z_b^j} \ln (r_a + r_b + r_{ab}) \quad \text{and} \quad \left( \frac{1}{r_a} v_a^i v_a^j \right)^{TT},
\]

respectively. Details of the decomposition \((2.10)\) are given in Appendix 1. For the calculation of the potential up to post-post-Newtonian order, the explicit expressions for \( h_0^{(i)} \) and \( h_0^{(i)} \) are not necessary as far as they damp at least like \( 1/|x| \) at infinity.

§ 3. Lagrangian in post-post-Newtonian order

In this section we evaluate the Lagrangian \((1.1)\) in the post-post-Newtonian order, using the solutions \((2.5) \sim (2.10)\). The Lagrangian densities \( L_M^{\text{PPN}} \) and \( L_\varphi^{\text{PPN}} \) in this order, which we denote as \( L_M^{\text{PPN}} \) and \( L_\varphi^{\text{PPN}} \), are expressed in terms of \( h_{ij}^{(0)} \) in Appendix 2. The integration \( \int d^3x L_M^{\text{PPN}} \) is easily performed. The result is

\[
\int d^3x L_M^{\text{PPN}} = -U (\infty) - \frac{1}{2} U (\infty)
\]
+ \frac{1}{16} \sum_a m_a (v_a^a)^4 + \frac{1}{2} \sum_a m_a [h^{(0)}_{00}]_{x-z_a} + \sum_a m_a v_a^a [h^{(0)}_{ij}]_{x-z_a} \nonumber \\
+ \frac{1}{2} \sum_a m_a v_a^a v_a^a [h^{(0)}_{ij}]_{x-z_a} + \frac{1}{4} \sum_a m_a v_a^a [h^{(0)}_{ij}]_{x-z_a}, \quad (3.1)

where \( a_{ab} = (z_a - z_b)/(z_a - z_b) \). The symbols \( U(\ldots) \) and \( U(\ldots) \) denote two types of static potentials and are given by

\begin{align*}
U(\ldots) &= G^a \sum_{b+c+d} \sum_{s \neq c} \sum_{d \neq a} m_a m_b m_c m_d r_{ab} r_{bc} r_{cd}, \quad (3.2) \\
U(\ldots) &= G^a \sum_{b+c+d} \sum_{s \neq c} \sum_{d \neq a} m_a m_b m_c m_d r_{ab} r_{ac} r_{dc}. \quad (3.3)
\end{align*}

These two static potentials include not only four-body potentials but also two- and three-body ones. The four-body parts of potentials \( U(\ldots) \) and \( U(\ldots) \) are equal to \( 2V(\ldots) \) and \( 6V(\ldots) \) defined in [II], respectively. It will be shown later that the contributions from \( h^{(0)}_{00}, h^{(0)}_{ij}, h^{(0)}_{ij}, \) and \( h^{(0)}_{ij} \) in (3.1) are cancelled by corresponding ones in \( \mathcal{L}_{\text{PPN}} \).

The Lagrangian density \( \mathcal{L}_{\text{PPN}} \) is divided into three parts: \( \mathcal{L}_a(h^8) \), \( \mathcal{L}_a(h^8) \) and \( \mathcal{L}_a(h^8) \). The densities \( \mathcal{L}_a(h^8) \) and \( \mathcal{L}_a(h^8) \) consist of seventeen and ten terms, respectively, which we call in turn 1, 2, \ldots, 27, following the order written in (A2.3) and (A2.4). The density \( \mathcal{L}_a(h^8) \) consists of only one term. First we integrate \( \mathcal{L}_a(h^8) \). In the following the contribution from each term is denoted as \( \langle \text{No. 1} \rangle \), etc. Since \( h^{(0)}_{ij} \) vanishes at infinity, the partial integration with respect to the space coordinate can be done freely. Then the contribution from the term (No. 1) becomes

\begin{equation}
\langle \text{No. 1} \rangle = -\frac{1}{2} \sum_a m_a [h^{(0)}_{00}]_{x-z_a}, \quad (3.4)
\end{equation}

which is cancelled by the contribution from \( h^{(0)}_{00} \) in \( \mathcal{L}_{\text{PPN}} \). Summing up the contributions from the terms (No. 2), (No. 3) and (No. 4), we have

\begin{equation}
\langle \text{No. 2} \rangle + \langle \text{No. 3} \rangle + \langle \text{No. 4} \rangle = -\sum_a m_a v_a^a [h^{(0)}_{ij}]_{x-z_a} + \frac{1}{8\pi G} \int d^3x \{ h^{(3)}_{00} h^{(0)}_{ij} \}, \quad (3.5)
\end{equation}

where the first term on the right-hand side is cancelled by the corresponding one in (3.1). The treatment of the second term, which is differentiated with respect to time, will be discussed later.

In the integration \( \int d^3x \mathcal{L}_a(h^8) \), the contribution from the transverse-traceless part of \( h^{(0)}_{ij} \) exists only in \( \langle \text{No. 5} \rangle \). Using the solution (2.10) for \( h^{(0)}_{ij} \) and the integration formulas in Appendix 3, we obtain

\begin{equation}
\langle \text{No. 5} \rangle = -\frac{3}{8} U(\ldots) + \frac{3}{16} U(\ldots) - \frac{1}{4\pi G} \int d^3x \ f^{TT}_{ij,k} f^{TT}_{ij,k}
\end{equation}
Higher Order Gravitational Potential for Many-Body System

\[ \frac{G}{4\pi} \sum_{a} \sum_{b} \sum_{c} m_{a} m_{b} m_{c} \left( n_{ab} \cdot v_{b} \right)^{2} + G \sum_{a} \sum_{b} \sum_{c} m_{a} m_{b} \left( \frac{3}{8} v_{a}^{2} v_{b}^{2} - \frac{1}{4} v_{a}^{2} \left( n_{ab} \cdot v_{b} \right)^{2} \right) \]

\[ + \frac{G}{4\pi} \int d^{4}x \sum_{a} \sum_{b} m_{a} m_{b} \left( \frac{v_{a}^{i} v_{a}^{j}}{r_{a}} \right)_{TT} \left( \frac{v_{b}^{i} v_{b}^{j}}{r_{b}} \right)_{TT} \]

\[ \frac{1}{2} \sum_{a} m_{a} v_{a}^{i} v_{a}^{j} \left[ h_{ij}^{(0)} \right]_{x=x_{a}} \quad (3.6) \]

where the third and the sixth terms on the right-hand side denote the contributions from the transverse-traceless part of \( h_{ij}^{(0)} \). It is very difficult to integrate \( f_{TT}^{TT} f_{ij,k}^{TT} \) for many-body system. The integration is performed in § 5 in the case of two-body system. The integration of the sixth term on the right-hand side of (3.6) is done using the expression (A1.8) and formulas (A3.7) and (A3.9). The result is

\[ \frac{G}{4\pi} \int d^{4}x \sum_{a} \sum_{b} m_{a} m_{b} \left( \frac{1}{16} v_{a}^{z} v_{b}^{z} + \frac{1}{8} \left( v_{a} \cdot v_{b} \right)^{2} - \frac{5}{8} v_{a}^{2} \left( n_{ab} \cdot v_{b} \right)^{2} \right) \]

\[ + \frac{3}{4} \left( v_{a} \cdot v_{b} \right) \left( n_{ab} \cdot v_{a} \right) \left( n_{ab} \cdot v_{b} \right) + \frac{3}{16} \left( n_{ab} \cdot v_{a} \right)^{2} \left( n_{ab} \cdot v_{b} \right)^{2} \].

\[ (3.7) \]

The contributions from the terms (No. 6) ~ (No. 10) are

\[ \langle \text{No. 6} \rangle + \langle \text{No. 7} \rangle = \frac{3}{16} U\left( \phi_{0} \right) + \frac{3}{8} G \sum_{a} \sum_{b} \sum_{c} m_{a} m_{b} m_{c} v_{a}^{2} \]

\[ \langle \text{No. 8} \rangle + \langle \text{No. 9} \rangle = - \frac{1}{2} U\left( \phi_{0} \right) + \frac{5}{4} U\left( \phi_{0} \right) + G \sum_{a} m_{a} m_{b} m_{c} \left( - \frac{9}{8} v_{a}^{2} \right) \]

\[ + \frac{3}{4} v_{b}^{2} \] \[ - \frac{3}{16} \sum_{a} m_{a} v_{a}^{2} \left[ h_{ij}^{(0)} \right]_{x=x_{a}} \]

\[ \langle \text{No. 10} \rangle = \frac{1}{2} U\left( \phi_{0} \right) - \frac{3}{8} U\left( \phi_{0} \right) + G \sum_{a} \sum_{b} \sum_{c} m_{a} m_{b} m_{c} \left( \frac{3}{4} v_{a}^{2} - v_{b}^{2} \right) \]

\[ + \frac{1}{2} G \sum_{a} \sum_{b} m_{a} m_{b} v_{a}^{2} v_{b}^{2} \].

\[ (3.10) \]

The terms (No. 11) ~ (No. 16) include time-derivatives which must be evaluated using the equations of motion in Newtonian order,

\[ m_{a} z_{a,00} = - G \sum_{b \neq a} \frac{m_{a} m_{b}}{r_{ab}^{2}} n_{ab}. \]

\[ (3.11) \]

Since \( h_{ij}^{(0)} \rightarrow 0(1/|x|^{2}) \) as \( |x| \rightarrow \infty \), we can perform the following transformation on the term (No. 11):

\[ \int d^{4}x \ h_{ij}^{(0)} h_{ij}^{(0)} \]
\[ -\int d^2x \, h_{ij}^{(0)} \, \hbar^{(0)} + \int d^2x \{ h_{j0}^{(0)} \, \hbar^{(0)} \}_{,0} \]
\[ = -6 \int d^2x \left\{ G \sum_{a} \frac{m_a}{r_a^2} (n_a \cdot v_a) \right\}_{,0} \left( G \sum_{b} \frac{m_b}{r_b^2} v_b^2 \right) \]
\[ \times \int d^3x \left\{ G \sum_{a} \frac{m_a}{r_a^2} (n_a \cdot v_a) \right\} \left( \frac{9}{2} G^2 \sum_{c} \frac{m_c}{r_c^2} - 6G^2 \sum_{c} \frac{m_c}{r_c} \sum_{e} \frac{m_e}{r_e} \sum_{f} \frac{m_f}{r_f} \right)_{,0} \]
\[ - \int d^3x \left\{ G \sum_{a} \frac{m_a}{r_a^2} (n_a \cdot v_a) \right\} \left( \frac{9}{2} G^2 \sum_{c} \frac{m_c}{r_c^2} - 6G^2 \sum_{c} \frac{m_c}{r_c} \sum_{e} \frac{m_e}{r_e} \sum_{f} \frac{m_f}{r_f} \right) - h_{ij}^{(0)} \hbar^{(0)} \right\}_{,0}. \]

(3.12)

After similar transformations on the terms (No. 12)\( \sim \) (No. 16), we have

\[ \langle \text{No. 11} \rangle + \langle \text{No. 12} \rangle + \langle \text{No. 13} \rangle + \langle \text{No. 14} \rangle + \langle \text{No. 15} \rangle + \langle \text{No. 16} \rangle \]
\[ = \frac{1}{4} G^2 \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_a m_b m_c}{r_ab^2 r_c} \left\{ (v_a \cdot v_b) - (n_{ab} \cdot v_a) (n_{ab} \cdot v_b) \right\} \]
\[ - \frac{1}{4} G^2 \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_a m_b m_c}{r_ab} \left\{ (n_{ab} \cdot (v_a - v_b)) (n_{ac} \cdot v_c) - (n_{ab} \cdot n_{ac}) v_c^2 \right\} \]
\[ - \frac{1}{4} G \sum_{a} \sum_{b \neq a} \frac{m_a m_b}{r_ab} \left\{ v_a^2 v_b^2 - v_a^2 (n_{ab} \cdot v_b) \right\} \]
\[ + \frac{1}{32\pi G} \int d^3x \left[ G \sum_{a} \frac{m_a}{r_a^2} (n_a \cdot v_a) \left\{ \frac{3}{2} G^2 \sum_{c} \frac{m_c}{r_c^2} + 6G^2 \sum_{c} \frac{m_c}{r_c} \sum_{e} \frac{m_e}{r_e} \sum_{f} \frac{m_f}{r_f} \right\} + 13G \sum_{a} \frac{m_a}{r_a^2} v_a^2 - 2h_{ij}^{(0)} \hbar^{(0)} \right\}_{,0}. \]

(3.13)

The last term in \( L_{ij}^{(h^2)} \) gives

\[ \langle \text{No. 17} \rangle = \frac{1}{8\pi G} \int d^2x \left\{ h_{ij}^{(0)} \hbar^{(0)} \right\}_{,0} \left( \text{PN} \right) \]

(3.14)

where the subscript PN means that the equations of motion in post-Newtonian order should be applied for the acceleration \( z_{a,00} \) in \( h_{ij}^{(0)} \).

Next we perform the integration \( \int d^3x L_{ij}(h^2) \). Only the term (No. 18) has contribution from \( \{ h_{ij}^{(0)} \}_{TT}^\text{TT} \):

\[ \langle \text{No. 18} \rangle = -\frac{3}{4} U(\infty \infty \infty) + \frac{5}{8} U(\infty \infty) + \frac{1}{8\pi G} \int d^2x \left\{ f_{ij,k}^{TT} f_{ij,k}^{TT} \right\} \]
\[ + \frac{1}{8\pi} \int d^2x \sum_{a} m_a \left\{ v_a^2 v_a^2 \right\}_{r_a}^{TT} h_{ij}^{(0)} h_{ij}^{(0)} \]
\[ + G^2 \sum_{a} \sum_{b \neq a} \sum_{c \neq a} m_a m_b m_c \left\{ -\frac{3}{8} v_a^2 - \frac{1}{4} (v_a \cdot v_b) + v_b^2 + \frac{1}{4} (n_{ab} \cdot v_b) (n_{ac} \cdot v_c) \right\} \]
\[ + G^2 \sum_{a} \sum_{b \neq a} \sum_{c \neq a} m_a m_b m_c \left\{ \frac{1}{4} (n_{ab} \cdot n_{ac}) v_c^2 + \frac{1}{4} (n_{ac} \cdot v_c) n_{ab} \cdot (v_a - v_b) \right\} \]
where the third term on the right-hand side is obtained using the relation
\[ \int d^3x \ f_{ij}^{TT} \ h_{00,i}^{(3)} \ h_{00,j}^{(3)} = -4 \int d^3x \ f_{ij,k}^{TT} \ f_{ij,k}^{TT}. \] (3.16)

The fourth term on the right-hand side of (3.15) can be calculated using the formulas (A3.3) \(\sim\) (A3.6) and (A3.10) \(\sim\) (A3.13). That is,
\[
\frac{1}{8\pi} \int d^3x \sum_{a} m_{a} \left( \frac{v_{a} \cdot v_{a} \cdot f_{TT}}{r_{a}} \right)^{TT} \ h_{00,i}^{(3)} \ h_{00,j}^{(3)}
\]
\[= G^{2} \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_{a} m_{b} m_{c}}{r_{ab} r_{ac}} \left\{ \frac{1}{4} v_{a}^{2} - \frac{3}{4} v_{b}^{2} + \frac{1}{4} (n_{ab} \cdot v_{b})^{2} \right\}
\]
\[- \frac{5}{8} (n_{ab} \cdot n_{ac}) v_{c}^{2} + \frac{1}{8} (n_{ab} \cdot n_{ac}) (n_{ac} \cdot v_{c})^{2}
\]
\[= \frac{7}{4} (n_{ab} \cdot v_{c}) (n_{ac} \cdot v_{c}) \}
\[+ \frac{G^{2}}{8\pi} \int d^3x \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_{a} m_{b} m_{c}}{r_{ab} r_{ac}} \{ (n_{a} \cdot v_{a}) (n_{b} \cdot v_{b}) + 4 (n_{a} \cdot v_{c}) (n_{b} \cdot v_{c}) \},
\] (3.17)

where the last integral can be performed using the formulas (A3.3), (A3.4) and (A3.5).

The terms (No. 19) \(\sim\) (No. 27) can be integrated similarly. The results are
\[\langle \text{No. 19} \rangle + \langle \text{No. 20} \rangle = \frac{5}{2} U(\phi_1 - \phi_2) - \frac{9}{8} U(\phi_1 - \phi_2),
\] (3.18)
\[\langle \text{No. 21} \rangle = -\frac{1}{4} G^{2} \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_{a} m_{b} m_{c}}{r_{ab} r_{ac}} \{ (v_{a} \cdot v_{b}) - (n_{ab} \cdot v_{a}) (n_{ab} \cdot v_{b}) \}
\]
\[+ \frac{1}{4} G^{2} \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_{a} m_{b} m_{c}}{r_{ab}} n_{ab} \cdot (v_{a} + v_{b}) (n_{ac} \cdot v_{c})
\]
\[= \frac{G^{2}}{8\pi} \int d^3x \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_{a} m_{b} m_{c}}{r_{ab} r_{ac}} \{ 8 (n_{a} \cdot v_{a}) (n_{b} \cdot v_{c}) + (n_{a} \cdot v_{c}) (n_{b} \cdot v_{c}) \},
\] (3.19)
\[\langle \text{No. 22} \rangle + \langle \text{No. 23} \rangle
\]
\[= -\frac{1}{2} G^{2} \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_{a} m_{b} m_{c}}{r_{ab} r_{ac}} \{ (v_{a} \cdot v_{b}) - (n_{ab} \cdot v_{a}) (n_{ab} \cdot v_{b}) \}
\]
\[- \frac{1}{2} G^{2} \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_{a} m_{b} m_{c}}{r_{ab} r_{ac}} n_{ab} \cdot (v_{a} + v_{b}) (n_{ac} \cdot v_{c})
\]
\[ T. \text{ Ohta, H. Okamura, T. Kimura and K. Hiida} \]

\[ -8(n_a \cdot v_a) (n_b \cdot v_a) + (n_a \cdot v_a) (n_b \cdot v_b) \}, \quad (3.20) \]

\[ \langle \text{No. 24} \rangle + \langle \text{No. 25} \rangle = - \frac{1}{32 \pi G} \int d^3 x \left( h_{ab}^{(3)} h_{ab}^{(3)} h_{ab}^{(3)} \right) s, \quad (3.21) \]

\[ \langle \text{No. 26} \rangle + \langle \text{No. 27} \rangle = - U(\varphi) - G^2 \sum_{a} \sum_{b \neq a} \sum_{\alpha \neq a} \frac{m_am_b_m_c}{r_{ab}r_{ac}} v_a \varphi, \quad (3.22) \]

where the third terms on the right-hand side of (3.19) and (3.20) can be calculated using the formulas (A3.3), (A3.4), (A3.5) and (A3.6). Finally we get

\[ \int d^3 x \ L_{ab} (h') = U(\varphi). \quad (3.23) \]

From the above results, we obtain the Lagrangian in post-post-Newtonian order. It is as follows:

\[ L_{\text{PPN}} = \int d^3 x \left( L_{\text{M}}^{\text{PPN}} + L_{\text{G}}^{\text{PPN}} \right) \]

\[ = \frac{1}{16} \sum_{a} m_a (v_a^2)^3 + \frac{3}{8} U(\varphi) \rightarrow \Delta \frac{1}{4} U(\varphi) - U^{\text{TT}} \]

\[ + \frac{1}{4} G^2 \sum_{a} \sum_{b \neq a} \sum_{\alpha \neq a} \frac{m_am_b_m_c}{r_{ab}r_{ac}} \left\{ 9v_a^2 - 7v_b^2 - 17(v_a \cdot v_b) + (n_a \cdot v_b) (n_a \cdot v_b) \right\} \]

\[ + (n_a \cdot v_b)^2 + 16(v_b \cdot v_c) \}

\[ + \frac{1}{8} G^2 \sum_{a} \sum_{b \neq a} \sum_{\alpha \neq a} \frac{m_am_b_m_c}{r_{ab}r_{ac}} \left\{ -5(n_a \cdot n_c) v_a^2 + (n_a \cdot n_c) (n_a \cdot v_c)^2 \right\} \]

\[ - 2(n_a \cdot v_a) (n_c \cdot v_c) - 2(n_a \cdot v_b) (n_a \cdot v_c) + 14(n_a \cdot v_c) (n_a \cdot v_c) \}

\[ - \frac{1}{2} G^2 \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_am_b_m_c}{r_{ab}r_{bc}r_{ca}} \left\{ 3(n_a + n_c) \cdot v_a (n_a - n_b) \cdot v_b \right\}

\[ + (n_a + n_c) \cdot v_A (n_a - n_b) \cdot v_c + 8(n_a + n_c) \cdot v_c (n_a - n_b) \cdot v_e \]

\[ - 16(n_a + n_c) \cdot v_c (n_a - n_b) \cdot v_c + 4(n_a + n_c) \cdot v_c (n_a - n_b) \cdot v_c \}

\[ + \frac{1}{2} G^2 \sum_{a} \sum_{b \neq a} \sum_{\alpha \neq a} \frac{m_am_b_m_c}{r_{ab}(r_{ab} + r_{bc} + r_{ca})} \left\{ 3 \left\{ v_a \cdot v_b \right\} - (n_a \cdot v_a) (n_a \cdot v_b) \right\} \]

\[ + \left\{ v_a^2 - (n_a \cdot v_a)^2 \right\} - 8 \left\{ (v_a \cdot v_c) - (n_a \cdot v_a) (n_a \cdot v_c) \right\} \]

\[ + 4 \left\{ v_c^2 - (n_a \cdot v_c)^2 \right\} \]

\[ - \frac{1}{4} G^2 \sum_{a} \sum_{b \neq a} \frac{m_am_b^2}{r_{ab}} \left\{ v_a^2 + v_b^2 - 2(v_a \cdot v_b) \right\} \]

\[ + \frac{1}{16} G \sum_{a} \sum_{b \neq a} \frac{m_am_b^2}{r_{ab}} \left\{ 14(v_a^2) - 28v_a^2 (v_a \cdot v_b) - 4v_a^2 (n_a \cdot v_a) (n_a \cdot v_b) \right\} \]
Higher Order Gravitational Potential for Many-Body System

\[ +11v_a^2v_b^2 + 2(v_a \cdot v_b)^2 - 10v_a^2(n_{ab} \cdot v_b)^2 + 12(v_a \cdot v_b)(n_{ab} \cdot v_a)(n_{ab} \cdot v_b) \\
+ 3(n_{ab} \cdot v_a)^2(n_{ab} \cdot v_b)^2 \]

where

\[ U^{TT} = - \frac{1}{4\pi G} \int d^3x \, f_{ij,k}^{TT} f_{ij,k}^{TT} \]

\[ M = - \frac{1}{32\pi G} \int d^3x \left[ \left\{ G \sum_{a} \frac{m_a}{r_a^2} (n_a \cdot v_a) \right\} \left( \frac{3}{2} G^2 \sum_{c} \frac{m_{bc}m_{ce}}{r_{bc}r_{ce}} + 6G^2 \sum_{e \neq a} \frac{m_{be}m_{ce}}{r_{be}r_{ce}} \right) \\
- 13G \sum_{b} \frac{m_b}{r_b} \right] + 4h_{ib,i}^{(1)} h_{ib,i}^{(1)} - 2h_{ib,i}^{(2)} h_{ib,i}^{(2)} - h_{ib,i}^{(3)} h_{ib,i}^{(3)} \right] \]

\[ + \frac{1}{8\pi G} \int d^3x \left[ h_{ib,i}^{(1)} h_{id,i}^{(1)} \right] \cdot \left( \gamma_{ib} \right) + \frac{1}{4} G^2 \left\{ \sum_{a} \sum_{b \neq a} \sum_{c \neq a} \frac{m_a m_b m_c}{r_{ab}} (n_{ac} \cdot v_c) \right\} \]

It can be shown that all the contributions from the second terms on the right-hand sides of (2.7) and (2.10) are included in the term \( M \).

It is customary to neglect the terms such as \( N \) defined by (1-3.8) and \( M \), which are differentiated with respect to time. Up to post-Newtonian order, it was verified in [1] that the term \( N \) leads to the same type of potential as the \( y \)-dependent part of the Lagrangian (1-3.7). Taking into account the term \( N \) corresponds to changing the value of the parameter \( y \). The term \( N \) has no effect on the precession of the perihelion.5) We consider that the same is true in post-post-Newtonian order, and neglect the term \( M \) in the following.

§ 4. Potential in post-post-Newtonian order

In this section we calculate the Hamiltonian for many-body system in post-post-Newtonian order. From the Lagrangian (1-3.7), we get the momentum of \( a \)-th body. In the case of \( y = 0 \) it is given by

\[ p_a = \frac{\partial L}{\partial (v_a)} = m_a v_a + \frac{1}{2} m_a v_a^2 + \frac{1}{2} G \sum_{b \neq a} \frac{m_a m_b}{r_{ab}} (6v_a - 7v_b - n_{ab}(n_{ab} \cdot v_b)) \]

(4.1)

Inversely we can represent \( v_a \) in terms of the \( p_a \)'s:

\[ v_a = \frac{p_a}{m_a} - \frac{p_a^2}{2m_a^2} - \frac{1}{2} G \sum_{b \neq a} \frac{m_b}{r_{ab}} \left\{ \frac{6p_a}{m_a} - 7 \frac{p_b}{m_b} - \frac{n_{ab}(n_{ab} \cdot p_b)}{m_b} \right\} + \cdots \]

(4.2)

where the terms in the order of \( c^{-4} \) and higher orders are neglected. In order to calculate the Hamiltonian \( H = \sum_a (p_a \cdot v_a) - L \) in post-post-Newtonian order, it is sufficient to know the relation (4.2). When \( (p_a \cdot v_a) \) and \( L \) are expressed in terms of the \( p_i \)'s and \( r_{ab} \)'s, the terms in the order of \( c^{-4} \) involved in \( (p_a \cdot v_a) \) are
cancelled out by the corresponding terms in \( v_a^2/2m_a \) involved in \( L \).

From the Lagrangian (1-3·7) and the Lagrangian (3·24), we obtain the Hamiltonian in post-post-Newtonian order in the case \( y=0 \). It is

\[
H_{^{\text{PPN}}} = \frac{1}{16} \sum_a m_a (P_a^2)^{\frac{3}{2}} + U(G^2) + U(G^2p^2) + U(Gp^4),
\]

where

\[
U(G^2) = - \frac{3}{8} U \left( \frac{1}{\infty} \right) - \frac{1}{4} U \left( \frac{1}{\infty} \right) + U^{\text{T}},
\]

\[
U(G^2p^2) = \frac{1}{8} G a \sum_{b+c} m_a m_b m_c \left[ 18 \frac{P_a^2}{m_a^2} + 14 \frac{P_b^2}{m_b^2} + 2 \frac{(n_{ab} \cdot P_b)^2}{m_b^2} \\
- 50 \frac{(P_a \cdot P_b)}{m_a m_b} + 17 \frac{(P_b \cdot P_c)}{m_b m_c} - 14 \frac{(n_{ab} \cdot P_a)(n_{ab} \cdot P_b)}{m_a m_b} \\
+ 14 \frac{(n_{ab} \cdot P_b)(n_{ac} \cdot P_c) + (n_{ab} \cdot n_{ac})(n_{ab} \cdot P_b)(n_{ac} \cdot P_c)}{m_b m_c} \\
+ \frac{1}{8} G a \sum_{b+c} m_a m_b m_c \left[ 2 \frac{(n_{ab} \cdot P_a)(n_{ac} \cdot P_c)}{m_a m_c} \\
+ 2 \frac{(n_{ab} \cdot P_b)(n_{ac} \cdot P_c)}{m_b m_c} + 5 \frac{(n_{ab} \cdot n_{ac})(n_{ac} \cdot P_c)}{m_c} \\
- (n_{ab} \cdot n_{ac}) \frac{(n_{ac} \cdot P_c)^2}{m_c^2} - 14 \frac{(n_{ab} \cdot P_b)(n_{ac} \cdot P_c)}{m_c} \right] \\
+ \frac{1}{2} G a \sum_{b+c} m_a m_b m_c \left[ \frac{m_a m_b m_c}{(r_{ab} + r_{bc} + r_{ca})^2} (n_{ab}^2 + n_{ac}^2)(n_{ab}^2 - n_{bc}^2) \\
\times \left( 8 \frac{p_a^4 p_c^4}{m_a m_c} - 16 \frac{p_a^4 p_b^4}{m_a m_b} + 3 \frac{p_b^4 p_c^4}{m_b m_c} + 4 \frac{p_b^4 p_c^4}{m_b m_c} + \frac{p_a^4 p_c^4}{m_a^2} \right) \\
+ \frac{1}{2} G a \sum_{b+c} m_a m_b m_c \left[ \frac{m_a m_b m_c}{r_{ab}^2} \left( 8 \frac{(P_a \cdot P_c)(n_{ab} \cdot P_a)(n_{ab} \cdot P_c)}{m_a m_c} \\
- 3 \frac{(P_a \cdot P_b)(n_{ab} \cdot P_a)(n_{ab} \cdot P_b)}{m_b m_a} - 4 \frac{P_c^2 - (n_{ab} \cdot P_c)^2}{m_a^2} - \frac{P_a^2 - (n_{ab} \cdot P_a)^2}{m_b^2} \right) \\
+ \frac{1}{4} G a \sum_{b+c} \left\{ \frac{m_a m_b m_c}{r_{ab}^2} \left( \frac{P_a^2 + P_b^2}{m_a m_b} - 2 \frac{(P_a \cdot P_b)^2}{m_a m_b} \right) \right\} \right],
\]

\[
U(Gp^4) = \frac{1}{16} G a \sum_{b+c} m_a m_b m_c \left[ 10 \frac{(P_a^2 p_b^2)}{m_a^4} + 11 \frac{p_b^4 p_a^4}{m_a^2 m_b^2} - 2 \frac{(P_a \cdot P_b)^2}{m_a^2 m_b^2} \\
+ 10 \frac{p_a^4 (n_{ab} \cdot P_b)^2}{m_a^4 m_b^2} - 12 \frac{(P_a \cdot P_b)(n_{ab} \cdot P_a)(n_{ab} \cdot P_b)}{m_a^2 m_b^2} - 3 \frac{(n_{ab} \cdot P_a)^2 (n_{ab} \cdot P_b)^2}{m_a^2 m_b^2} \right].
\]
The potential $U(G^3)$ includes not only four-body potential but also two- and three-body potentials. Let us consider the potential

$$G^3 \frac{m_am_bm_cm_d}{r_{ab}r_{bc}r_{cd}}. \quad (4.7)$$

It is a two-body potential when $a = c$ and $b = d$, and is a three-body potential when $a = c, b = d$ or $a = d$. Otherwise it is a four-body potential. The potential $U(\ldots)$ defined by (3.2) contains these two-, three- and four-body potentials in a definite manner. Similar situations are found also for $U(\ldots)$ and $U^{TT}$.

In quantized theory it is easy to calculate four-body potential in the order of $G^3$, because we have only to consider contributions from tree-diagrams for four-body scattering. In order to obtain two- and three-body potentials in the order of $G^3$, we must calculate $S$-matrix elements for diagrams with radiative corrections. This is a hard task. However, there exists a simple substitution law by which we can write down two- and three-body potentials at once, if we know the four-body potential. The relation between the two- and three-body potentials thus obtained and the four-body potential is exactly the same with the relation among two-, three- and four-body potentials in $U(G^3)$.

The transverse-traceless part of $h_{ij}^{ij}$ contributes to the static potential $U^{TT}$. This is the manifestation of non-linear nature of the general theory of relativity. If it were a linear theory, the transverse-traceless part would contribute only to velocity-dependent potential. The four-body part of $U^{TT}$ coincides with $V^{TT}$ in [II]. Inserting the expression (A1.9) into $f_{ij}^{TT}$ in $U^{TT}$ and performing the $x$-integration, we can express the four-body part of $U^{TT}$ as

$$- \frac{G^3}{16\pi^2} \sum_a \sum_{b=a} \sum_{c=b} \sum_{d=ac} \int d^3k_1 \, d^3k_2 \, d^3k_3 \, d^3k_4 \, \delta(k_1 + k_2 + k_3 + k_4)$$

$$\times \left\{ \left( k_1 \cdot k_3 \right) - \frac{k \cdot (k_1 + k_3)}{k^2} \right\} \left\{ \left( k_2 \cdot k_4 \right) - \frac{k \cdot (k_2 + k_4)}{k^2} \right\}$$

$$+ \left\{ \left( k_2 \cdot k_4 \right) - \frac{k \cdot (k_2 + k_4)}{k^2} \right\} \left\{ \left( k_3 \cdot k_1 \right) - \frac{k \cdot (k_3 + k_1)}{k^2} \right\}$$

$$- \left\{ \left( k_3 \cdot k_1 \right) - \frac{k \cdot (k_3 + k_1)}{k^2} \right\} \left\{ \left( k_2 \cdot k_4 \right) - \frac{k \cdot (k_2 + k_4)}{k^2} \right\} \right], 

(4.8)$$

where $k = -(k_3 + k_1) = (k_3 + k_1)$. This is just equal to the contribution from the $S$-matrix element for the diagram in Fig. 1, in which the internal wavy line with momentum $k$ represents pure transverse-traceless graviton. The
detailed derivation of (4·8) in quantized theory has been given in [II].

It will be shown in a subsequent paper that exactly the same potential with (4·3) is obtained in the calculations based on the canonical formalism of gravitation theory. On the other hand, the potential obtained in the conventional formalism of quantized theory is different from (4·3) except for \( U(G\rho) \). The potential \( U(G\rho) \) coincides with that obtained from the contribution of one-graviton-exchange diagram. The static four-body potential obtained in [II] is

\[
-\frac{3}{4} V(\cdots\cdots) + 12 V(\begin{array}{c} 1 \\ -1 \end{array}) + V^{TT},
\]

while the four-body part of the potential \( U(G\rho) \) in (4·4) is

\[
-\frac{3}{4} V(\cdots\cdots) - \frac{3}{2} V(\begin{array}{c} 1 \\ -1 \end{array}) + V^{TT},
\]

where the notations are those used in [II]. Furthermore the potential in the order of \( G^2 \rho^2 \) cannot be determined uniquely in quantized theory. These discrepancies should be considered seriously.

§ 5. **Hamiltonian for two-body system**

In this section we calculate the potential \( U^{TT} \) for two-body system, \( U^{TT}(1, 2) \), and give the Hamiltonian for this system up to post-post-Newtonian order. Using the relation (3·16), we can express the potential \( U^{TT}(1, 2) \) as

\[
U^{TT}(1, 2) = \frac{G}{2\pi} m_1 m_2 \int d^3 x \left( \frac{1}{r_1} \right) \left( \frac{1}{r_2} \right) \frac{\partial^2}{\partial z_1^i \partial z_2^j} \ln (r_1 + r_2 + r_{12}).
\]

The potential \( U^{TT}(1, 2) \) is explicitly given by

\[
U^{TT}(1, 2) = I_1 + I_2 - I_3,
\]

where

\[
I_1 = \frac{G}{2\pi} m_1^2 m_2^2 \int d^3 x \left( \frac{1}{r_1} \right) \left( \frac{1}{r_2} \right) \frac{\partial^2}{\partial z_1^i \partial z_2^j} \ln (r_1 + r_2 + r_{12})
\]

\[= \frac{G}{4\pi} m_1^2 m_2^2 \int d^3 x \frac{\lambda_{12}}{r_1^2 r_2^2 r_{12}^2}, \]

\[
I_2 = \frac{G}{2\pi} m_1^2 m_2^2 \int d^3 x \left( \frac{1}{r_1} \right) \left( \frac{1}{r_2} \right) \frac{\partial^2}{\partial z_1^i \partial z_2^j} \ln (r_1 + r_2 + r_{12})
\]

\[= \frac{G}{4\pi} m_1^2 m_2^2 \int d^3 x \left( \frac{\lambda_{12}}{r_1^2 r_2^2 r_{12}^2} \right) + \frac{\lambda_{12}^2}{r_1^2 r_2^2 r_{12}^2} - \frac{1}{2r_{12}} \left( \frac{1}{r_1^2} + \frac{1}{r_2^2} \right) \]
Higher Order Gravitational Potential for Many-Body System

Using the formulas given in Appendix 3, we can integrate each term in (5·3), (5·4) and (5·5). The results are

\[
I_1 = -2G^3 \frac{m_1^2 m_2^2}{r^3}, \quad I_2 = -\frac{3}{2} G^3 \frac{m_1^2 m_2^2}{r^3}, \quad I_3 = -\frac{5}{2} G^3 \frac{m_1^2 m_2^2}{r^3},
\]

where \( r = r_{13} \). Then the potential (5·1) is

\[
U^{TT}(1, 2) = -\frac{G^3 m_1^2 m_2^2}{r^3}.
\]

The static potential \( U(\ldots) \) includes the two-body potential

\[
2G^3 \frac{m_1^2 m_2^2}{r^3},
\]

which has the same form as \( U^{TT}(1, 2) \). Thus the transverse-traceless part of graviton manifestly contributes to the static potential.

For two-body system the Lagrangian up to post-post-Newtonian order is obtained from (I-3·7), (3·24) and (5·7). The Hamiltonian is derived from the Lagrangian. It is

\[
H(1, 2) = m_1 + m_2 + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{1}{8} \left\{ m_1 \left( \frac{p_1^2}{m_1^2} + \frac{p_2^2}{m_2^2} \right) \right\} + \frac{1}{16} \left\{ m_1 \left( \frac{p_1^2}{m_1^2} + \frac{p_2^2}{m_2^2} \right) \right\} = -G \frac{m_1 m_2 (m_1 + m_2)}{r^3} + G^2 \frac{m_1 m_2 (m_1^2 + m_2^2) + 7m_1 m_2}{r^3}
\]

\[
-\frac{G^3}{4} \frac{m_1 m_2 (m_1^2 + m_2^2 + 7m_1 m_2)}{r^3} + \frac{G^2}{4} \frac{m_1 m_2 (10 p_1^2 + 19 p_2^2)}{m_1^2 m_2}
\]

\[
+ \frac{G^2}{4} \frac{m_1 m_2 (19 p_1^2 + 10 p_2^2)}{m_2^2}
\]

\[
- \frac{G^2}{4} \frac{m_1 m_2 (m_1 + m_2)}{r^3} \left\{ 27 \frac{(p_1 \cdot p_2)}{m_1 m_2} + 6 \frac{(n \cdot p_1)}{m_1 m_2} \right\}
\]

\[
+ \frac{G}{8} \frac{m_1 m_2}{r} \left\{ -12 \frac{(p_1^2 + p_2^2)}{m_1^2 m_2} + 28 \frac{(p_1 \cdot p_2)}{m_1 m_2} + 4 \frac{(n \cdot p_1)}{m_1 m_2} \right\}
\]
§ 6. Discussion

We have obtained the Hamiltonian for many-body system up to post-post-Newtonian order. Here we would like to recall the process to get it. The first step is to solve Einstein's equation for the metric tensor $g_{ab}$. In this step there appear higher time-derivatives of the velocities of bodies. The equations of motion of the bodies are used, in order to express the metric tensor as the function of the coordinates and the velocities. The next step is to insert the expression of the metric tensor into the Lagrangian for many-body system. The equations of motion are used again to get the Lagrangian of the Fokker type, which is the function of the coordinates and the velocities alone. The authors think that it is not justified to use the equations of motion in Lagrangian. Thus it is necessary to calculate the Hamiltonian for many-body system in another formalism and to compare the results. In the subsequent paper\textsuperscript{7} we shall recalculate the Hamiltonian in canonical formalism.\textsuperscript{8} It will be shown that the results coincide.

In conclusion we want to mention the observability of our Hamiltonian. Let us consider, for example, the precession of Mercury. The observed value of the angle of the precession is about 43" per century. The contribution from the potential in post-post-Newtonian order is about $GM_0/Rc^2 (\approx 10^{-8})$ times the observed value, $M_0$ and $R$ being the mass of the sun and the distance between the sun and Mercury. Thus the readers may think that the present authors are calculating unobservable effects. However the potential has observable effects in many phenomena in very dense states.

There are many other phenomena which are affected by our potential even in low-density states. As an example, consider the energy level of a hydrogen atom on the surface of a star, which is surrounded by distant matter in the universe. For this phenomenon the distribution of matter (including the star) is not isotropic. Let $\rho(x)$ and $r$ be the density of distant matter at $x$ and the distance between the hydrogen atom and the position $x$. Using a reasonable value for the density $\rho(x)$, we get $(G/c^2) \int d^3x \rho(x)/r^2$. This numerical value shows that distant matter has effects on the energy level through three-, four- and many-body potentials.
Appendix I

Derivation of the expression (2.10).

In general a three-dimensional symmetric tensor $A_{ij}$ is uniquely decomposed into the form

$$A_{ij} = A_{ij}^{TT} + A_{ij}^T + A_{i,j} + A_{j,i},$$  \hspace{1cm} (A1.1)

where

$$A_{ij}^T = \frac{1}{2} (A^T \delta_{ij} - \frac{1}{d} A_{ij}^T),$$  \hspace{1cm} (A1.2)

$$A^T = A_{ii} - \frac{1}{d} A_{ij},$$  \hspace{1cm} (A1.3)

$$A_i = \frac{1}{d} \left( A_{ij,j} - \frac{1}{2d} A_{jk,ki} \right),$$  \hspace{1cm} (A1.4)

and $A_{ij}^{TT}$ and transverse-traceless part $A_{ij}^T$ satisfy the conditions

$$\partial_i A_{ij}^{TT} = 0,$$  \hspace{1cm} (A1.5)

$$\partial_i A_{ij}^T = 0.$$  \hspace{1cm} (A1.6)

The solution for $h_{ij}^0$ is given by the expression (I-2.39) with (I-A4.2). It contains the terms

$$f_{ij} = G \sum_a \sum_{b \neq a} m_a m_b \frac{\partial^2}{\partial z_a^i \partial z_b^j} \ln (r_a + r_b + r_{ab}) \quad \text{and} \quad G \sum_a \frac{m_a}{r_a} v_a^i v_a^j.$$

The transverse-traceless parts of these terms are

$$f_{ij}^{TT} = f_{ij} - \frac{1}{8} G \sum_a \sum_{b \neq a} m_a m_b \left[ \delta_{ij} \left( \frac{1}{r_a r_b} - \frac{2}{r_a r_b^2} + \frac{(n_a \cdot n_{ab})}{r_{ab}^2} \right) \right]$$

$$+ \partial_i \partial_j \left[ \ln (r_a + r_b + r_{ab}) - \frac{r_a^2}{r_{ab}} + \frac{r_{ab}^2}{2r_{ab}^2} \right] - 4 \frac{n_{ab}^i n_{ab}^j + n_{ab}^j n_{ab}^i}{r_{ab}^2},$$  \hspace{1cm} (A1.7)

$$G \sum_a m_a \left( \frac{1}{r_a} v_a^i v_a^j \right)^{TT}$$

$$= G \sum_a \frac{1}{r_a} v_a^i v_a^j - G \sum_a \frac{m_a}{r_a} \left[ \delta_{ij} \frac{v_a^2 + (n_a \cdot v_a)^2}{4 r_a} + \partial_i \partial_j \left( \frac{3}{8} r_a v_a^2 \right) + \frac{1}{16} r_a (v_a^2 - (n_a \cdot v_a)^2) + \frac{1}{2} \{ v_a^i \partial_j (n_a \cdot v_a) + v_a^j \partial_i (n_a \cdot v_a) \} \right].$$  \hspace{1cm} (A1.8)

The $f_{ij}^{TT}$ can also be expressed in the form

$$f_{ij}^{TT} = \frac{G^2}{2 (2 \pi)^5} \sum_a \sum_{b \neq a} m_a m_b \int d^3 k_a d^3 k_b e^{ik_a \cdot n_a + k_b \cdot n_{ab}} \times \frac{1}{k_i^2 k_j^2 k^2}$$

$$\times \left[ k_i^{i_a} k_j^{i_b} + k_i^{j_b} k_j^{i_a} - \left( (k_i \cdot k_a) (k_j \cdot k_b) \right) \left( \delta_{ij} - \frac{k_i^j k_j^i}{k^2} \right) \right]$$
where $k = -(k_1 + k_2)$.

Using the expressions (A1.7) and (A1.8) in (1-2.39) and choosing

$$j^{(4)} = \frac{G}{2} \sum_a \sum_{b \neq a} m_a m_b \ln |r_a + r_b + r_{ab}|,$$

for unknown functions $j^{(4)}$ and $\varepsilon_i^{(0)} \cdot \vec{K}^{(4)}$, we obtain the expression (2-10) for $h_i^{(4)}$ in the case $y = 0$.

**Appendix 2**

$L_{PN}^{PPN}$ and $L_{0}^{PPN}$ in the case $y = 0$

Inserting the expansion (2-3) into the right-hand sides of (1-2) and (1-3) and using the relation $h_i^{(4)} = \delta_{ij} h_i^{(2)}$, we obtain the Lagrangian densities $L_{M}^{PPN}$ and $L_{0}^{PPN}$ in the case $y = 0$. They are given by

$$L_{M}^{PPN} = \frac{1}{4} \sum_a m_a \delta(x - z_a) \left[ 2h_{60}^{(4)} + 4v_a h_{60}^{(4)} + 2v_i h_{ij}^{(4)} \right.$$ 

$$+ (h_{60}^{(4)} + v_a^{(2)}) (h_{60}^{(4)} + 2v_i h_{ij}^{(4)} + v_i v_j h_{ij}^{(4)})$$ 

$$+ \frac{1}{4} (h_{60}^{(4)})^2 + 3v_i h_{ij}^{(4)} + 3(v_a^{(2)})^2 + \frac{1}{4} (v_a^{(2)})^4 \right],$$

(2-1)

$$L_{0}^{PPN} = L_{0}(h^2) + L_{0}(h^4) + L_{0}(h^4),$$

(2-2)

where

$$L_{0}(h^2) = \frac{1}{32\pi G} \left[ -2h_{60}^{(4)} h_{60}^{(4)} + 2h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} - 2h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} + 4h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} + 2M h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} + 2h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} \right]$$

(2-3)

$$L_{0}(h^3) = \frac{1}{32\pi G} \left[ h_{ij}^{(4)} h_{60}^{(4)} h_{60}^{(4)} - \frac{1}{2} h_{ij}^{(4)} h_{60}^{(4)} h_{60}^{(4)} + \frac{3}{2} h_{ij}^{(4)} h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} + 2h_{ij}^{(4)} h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} \right]$$

(2-4)

$$L_{0}(h^4) = \frac{3}{64\pi G} h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)} h_{60}^{(4)}.$$  

(2-5)

The subscript PN in the term (No. 17) means that the equations of motion in
Higher Order Gravitational Potential for Many-Body System

post-Newtonian order should be applied for the acceleration \( x_{a,0} \) in \( h^{(6)}_{ij} \).

**Appendix 3**

**Integration formulas**

In this Appendix we write down integration formulas needed to calculate the Lagrangian \((1.1)\):

\[
G^3 \sum_i \sum_j \sum_k \sum_l m_i m_j m_k m_l \int d^3x \left( \frac{1}{r_i} \right) \left( \frac{1}{r_j} \right) \frac{1}{r_k r_l} = \frac{4\pi}{3} U(\mathcal{A}) \, . \tag{A3.1}
\]

\[
G^3 \sum_j \sum_k \sum_l m_j m_k m_l m_l \int d^3x \left( \frac{1}{r_j} \right) \left( \frac{1}{r_k} \right) \frac{1}{r_l r_k} = -2\pi U(\mathcal{A}) \, . \tag{A3.2}
\]

\[
\int d^3x \frac{n_a^i n_b^j}{r_a r_b r_c} = -4\pi \left\{ \frac{(n_{ab}^i + n_{ac}^i) (n_{ab}^j - n_{bc}^j)}{(r_{ab} + r_{bc} + r_{ca})^2} - \frac{\delta_{ij} - n_{ab}^i n_{ab}^f}{(r_{ab} + r_{bc} + r_{ca}) r_{ab}} \right\} \tag{A3.3}
\]

for \( b \neq a \) and \( c \neq a, b \).

\[
\int d^3x \frac{n_a^i n_b^j}{r_a r_b r_c} = 2\pi \frac{\delta_{ij} - 2n_{ab}^i n_{ab}^f}{r_{ab}} \, . \tag{A3.4}
\]

\[
\int d^3x \frac{n_a^i n_c^j}{r_a r_b r_c} = -\pi \frac{\delta_{ij} - n_{ab}^i n_{ab}^f}{r_{ab}} \, . \tag{A3.5}
\]

\[
\sum_a \sum_b \sum_c m_a m_b m_c \int d^3x \frac{(n_a \cdot n_b) (A \cdot B)}{r_a r_b r_c} = 2\pi \sum_a \sum_b \sum_c m_a m_b m_c \frac{(A \cdot B)}{r_a r_b r_c} \tag{A3.6}
\]

\[
+ 2\pi \sum_a \sum_b \sum_c m_a m_b m_c \frac{(A \cdot B)}{r_a r_b r_c} - 2\pi \sum_a \sum_b \sum_c m_a m_b m_c \frac{(A \cdot B)}{r_a r_b r_c} \, .
\]

where \( A \) and \( B \) denote the velocities of bodies.

\[
\int d^3x \left( \frac{1}{r_i} \right) \left( \frac{1}{r_j} \right) = 2\pi \frac{\delta_{ij} - n_{ab}^i n_{ab}^f}{r_{ab}} \, . \tag{A3.7}
\]

\[
\int d^3x \left( \frac{1}{r_i} \right) \left( \frac{1}{r_j} \right) = 2\pi n_{ab}^i \, . \tag{A3.8}
\]

\[
\int d^3x \left( \frac{(n_a \cdot v_a)^2}{r_a} \right) \left( \frac{1}{r_j} \right) = \frac{\pi}{r_{ab}} \left\{ \delta_{ij} \left\{ v_a^2 + (n_a \cdot v_a)^2 \right\} - n_{ab}^i n_{ab}^j \left\{ v_a^2 + 3(n_a \cdot v_a)^2 \right\} \right. \\
\left. + 2(n_{ab}^i v_a^j + n_{ab}^j v_a^i) (n_{ab} \cdot v_a) - 2v_a^i v_a^j \right\} \, . \tag{A3.9}
\]

\[
\sum_a \sum_b \sum_c m_a m_b m_c \int d^3x \frac{(n_a \cdot v_a)^2}{r_a r_b r_c} \left( \frac{1}{r_i} \right) \left( \frac{1}{r_j} \right) = 2\sum_a \sum_b \sum_c m_a m_b m_c \int d^3x \frac{(n_a \cdot v_a) (n_b \cdot v_a)}{r_a r_b r_c} \, .
\]
\[ + 2\pi \sum_a \sum_b \sum_c \sum_{\beta \gamma \alpha} m_a m_b m_c \frac{2(n_{ab} \cdot v_c)^2 - v_a^2}{r_{ab}^2}. \quad \text{(A3·10)} \]

\[ \sum_a \sum_b \sum_c m_a m_b m_c \int d^3 x \left( \frac{1}{r_a} \right) \left( \frac{1}{r_b} \right) \left( \frac{1}{r_c} \right) \left( \partial_i \frac{1}{r_a} \right) \partial_j \frac{1}{r_b} \left[ r_c \{ v_e^2 - (n_e \cdot v_e)^2 \} \right] \]

\[ - 2\pi \sum_a \sum_b \sum_c \sum_{\beta \gamma \alpha} m_a m_b m_c \frac{2v_e^2 - v_a^2}{r_{ab}^2} - 4\pi \sum_a \sum_b \sum_c \sum_{\beta \gamma \alpha} m_a m_b m_c \frac{(n_{ab} \cdot n_{ac}) v_c^2}{r_{ab}^2}. \quad \text{(A3·11)} \]

\[ \sum_a \sum_b \sum_c m_a m_b m_c \int d^3 x \left( \frac{1}{r_a} \right) \left( \frac{1}{r_b} \right) \left( \frac{1}{r_c} \right) \partial_j \frac{1}{r_b} \left[ r_c \{ v_e^2 + (n_e \cdot v_e)^2 \} \right] - 2(n_{ab} \cdot v_c)(n_{ac} \cdot v_e) \]

\[ - 4\pi \sum_a \sum_b \sum_c m_a m_b m_c \frac{1}{r_{ab}} \{ (n_{ab} \cdot n_{ac}) (v_e^2 + (n_e \cdot v_e)^2) - 2(n_{ab} \cdot v_c)(n_{ac} \cdot v_e) \}. \quad \text{(A3·12)} \]

\[ \sum_a \sum_b \sum_c m_a m_b m_c \int d^3 x \left( \frac{1}{r_a} \right) \left( \frac{1}{r_b} \right) \left( \frac{1}{r_c} \right) \partial_j (n_e \cdot v_c) v_e^i \]

\[ = - \sum_a \sum_b \sum_c m_a m_b m_c \int d^3 x \left( \frac{1}{r_a} \right) \left( \frac{1}{r_b} \right) \left( \frac{1}{r_c} \right) \left( \frac{1}{r_e} \right) \partial_j (n_e \cdot v_c) v_e^i \]

\[ + 2\pi \sum_a \sum_b \sum_c m_a m_b m_c \frac{v_e^2 - (n_{ab} \cdot v_b)^2}{r_{ab}^2} \]

\[ - 4\pi \sum_a \sum_b \sum_c m_a m_b m_c \frac{(n_{ab} \cdot v_c)(n_{ac} \cdot v_e)}{r_{ab}^2}. \quad \text{(A3·13)} \]

References

5) T. Ohta, H. Okamura, T. Kimura and K. Hiida, Prog. Theor. Phys. 50 (1973), 492 (referred to as [I]).