The Koba-Nielsen-Olesen Scaling Function

R. C. MISRA
Department of Physics
University of Nigeria, Nsukka
East Central State, Nigeria

November 19, 1973

There has been a considerable interest shown in the scaling law of Koba, Nielsen and Olesen \(^1\) (KNO scaling) for asymptotic semi-inclusive topological cross sections,

\[
\langle n \rangle \sigma_n(s) \rightarrow \psi \left( \frac{n}{\langle n \rangle} \right),
\]

where \(\sigma_n(s)\) is the cross section for producing \(n\) charged particles at c.m. energy \(\sqrt{s}\), \(\sigma_{\text{tot}}(s)\) is the total inelastic cross section, \(\langle n \rangle\) is the average charged particle multiplicity and \(\psi(n/\langle n \rangle)\) is an universal energy independent function. Such a scaling law explains directly \(^2\) the linearity of the average charged particle multiplicity \(\langle n \rangle\) with its dispersion \(D=\langle n^2 \rangle-\langle n \rangle^2\), which was first observed by Wroblewski, \(^3\)

\[
D=A\langle n \rangle - B
\]

with \(A\) and \(B\) as positive constants. The stability of \(\langle n \rangle/D\) is also reported for a large high energy range upto the ISR energy, \(^4\) where \(\langle n \rangle\) is shown to vary \(^5\) as \(\ln s\).

The purpose of the present note is to show that the scaling function \(\psi(x)\) can be determined as

\[
\psi(x) \sim xe^{-(x/\langle n \rangle)x^2},
\]

from the first principles. Although various other complicated forms are reported by different authors, \(^6\), \(^7\) the scaling function obtained by us is in consistence with the KNO form \(^8\) obtained by Buras and Koba, which is given as

\[
\psi(x) \sim xe^{-(x/\langle n \rangle)x^2},
\]

The derivation of the relation given by Eq. (3) is as follows: The distribution for production of \(n\) charged particles in \(p-p\) inelastic high energy collision is given by the distribution function

\[
f(n) = \frac{1}{\langle n \rangle} \psi \left( \frac{n}{\langle n \rangle} \right),
\]

which must satisfy the normalization condition, multiplicity condition and the second multiplicity condition as given below by Eqs. (6), (7) and (8) respectively:

\[
\sum_{n=1}^{\infty} f(n) = 1,
\]

\[
\sum_{n=1}^{\infty} nf(n) = \langle n \rangle,
\]

\[
\sum_{n=1}^{\infty} n^2f(n) = \langle n^2 \rangle.
\]

Equations (6), (7) and (8) above can be cast in integral form, as follows, by considering the variable \(x=n/\langle n \rangle\) and by assuming that \(\langle n \rangle\) is very large for very high energy \(p-p\) collision:

\[
1 = \int_{\langle n \rangle}^{\infty} \frac{dn}{\langle n \rangle} \psi \left( \frac{n}{\langle n \rangle} \right)
\]

\[
\approx \int_0^{\infty} dx \psi(x),
\]

where

\[
x = \frac{n}{\langle n \rangle},
\]

\[
\int_0^{\infty} dx x \psi(x) = 1,
\]

\[
\int_0^{\infty} dx x^2 \psi(x) = \frac{\langle n^2 \rangle}{\langle n \rangle^2} = \frac{5}{4}.
\]

Therefore, the KNO scaling function must satisfy Eqs. (9), (10), (11), where \(\langle n^2 \rangle/\langle n \rangle^2=5/4\) is obtained by assuming the ex-
Experimental stable value $\langle n \rangle / D = 2$. With the substitution

$$\psi(x) = -\frac{d\phi(x)}{dx},$$  \hspace{1cm} (12)

Eqs. (9), (10) and (11) can be transformed into Eqs. (13), (14) and (15) respectively, as follows:

$$\phi(0) - \phi(\infty) = 1,$$  \hspace{1cm} (13)

$$\int_{0}^{\infty} \phi(x) dx = 1,$$  \hspace{1cm} (14)

$$\int_{0}^{\infty} dx x \phi(x) = \frac{5}{8}.$$  \hspace{1cm} (15)

From Eqs. (13) and (15), we generate a simple integral equation for $\phi(x)$, namely

$$\phi(x) = \phi(0) - \frac{8}{5} \int_{0}^{x} dx x \phi(x),$$  \hspace{1cm} (16)

with $x \phi(x) \to 0$ for $x \to \infty$)

whose solution is

$$\phi(x) = \phi(0) e^{-(4/5)x^2}.$$  \hspace{1cm} (17)

The normalization condition (14) then gives

$$\phi(0) = \frac{4}{\sqrt{5} \pi},$$  \hspace{1cm} (18)

which is of order unity. Thus, we have derived the form of the KNO scaling function as $\psi(x) \sim x e^{-(4/5)x^2}$. We must remark that $\phi(x)$ is more universal than $\psi(x)$, because of its gaussian character.

Our approximate KNO scaling function, which is derived from conditions (6), (7) and (8), together with the assumptions

(1) $\frac{\langle n \rangle}{D} = 2$ for very high energy, and

(2) $x \phi(x) \to 0$ for $x \to \infty$,

agrees very well with the phenomenological fit of Burns and Koba given by Eq. (4).

3) A. Wroblewski, Warsaw University preprint, IFD No. 72/2 (1972), and contribution to XVI Int. Conf. High Energy Physics, Sept. 1972, Batavia, U.S.A.
F. T. Dao, J. Lach and J. Whitmore, preprint, NAL-Pub-73/37-Exp, gives a compilation of $\langle n \rangle / D$ values for high energy $pA$ collisions, where $A=p, \bar{p}, \pi^+, K^+, \gamma$. Stability condition $\langle n \rangle / D = 2$ seems to be well maintained at momenta greater than $\sim 50$ (GeV/c).

5) For example, see M. Antinucci et al., Lett. Nuovo Cim. 6 (1973), 121.