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INFLUENCE OF INSTANTANEOUS FERTILITY DECLINE TO REPLACEMENT LEVEL ON POPULATION GROWTH: AN ALTERNATIVE MODEL

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Abstract—If age-specific birth rates m_x of a stable population drop abruptly to m_x/R_0, where R_0 is the net reproduction rate, then, according to Keyfitz, the size of the ultimate stationary population relative to that at the beginning of the process is given by $I = be^0{}_0(R_0 - 1)/(r\mu R_0)$, where b and r are the birth rate and the rate of growth, respectively, of the stable population, $e^0{}_0$ the life expectancy at birth, and μ the average age at childbirth in the resulting stationary population. Noting that the decline in m_x need not necessarily be uniform, investigation has been carried out to examine the effect on I when fertility decline is more rapid at higher ages. In particular, the effect of the reduced age-specific rates such as $m_x e^{-rx}$ (which also produces a stationary population) has been analyzed, and simplifications of the results carried out separately for three different models of the net maternity function. It has also been shown that when m_x drops abruptly to some m^*_x, where the form of m^*_x need not be specified except for the restriction that the resulting population will be stationary, the value of the index can be approximately obtained from $I^* = be^0{}_0(1 - r\mu/2)$, where μ is the average age at childbearing of the initial stable population.

THE PROBLEM

In a very interesting article, Keyfitz (1971) has shown that "if age-specific birth rates drop immediately to the level of bare replacement, the ultimate stationary number of a population will be given by

$$I = \frac{be^0{}_0}{r\mu}\left(\frac{R_0 - 1}{R_0}\right) \qquad (1)$$

multiplied by the present number," where b and r are the intrinsic birth rate and intrinsic rate of growth, respectively, R_0 the net reproduction rate, $e^0{}_0$ the expectation of life at birth, and μ the average age of childbearing in the resulting stationary population. This result is obtained by assuming that the population is stable, and the regime of age-specific birth rates m_x transforms immediately to m_x/R_0, where m_x refers to a one-sex model only. Keyfitz has acknowledged that "this is not the only way an NRR can be constituted, and not even the most probable way" and that "the fall in the birth rate is likely to be more rapid for older women than for younger."

The purpose of this paper is to examine the modifications in (1), resulting from a pattern of more rapid fall in the birth rate for older women, by using a variable multiplier to m_x. The use of e^{-rx} immediately suggests itself, in which r is the intrinsic rate of growth corresponding to m_x and survivorship function l_x. In other words, r is the real root of the integral equation

$$\int_0^\infty e^{-rx} l_x m_x \, dx = 1. \qquad (2)$$

It may be pointed out that Keyfitz has shown with numerical examples and population projections that equation (1) fails when the initial population is not approximately stable. This should also be expected from the present model, which is based merely on an alternative pattern of fertility decline in a stable population. However, for approximately stable populations, Keyfitz found that the results were close to the model values, as they should be. To avoid unnecessary duplications, such projections will not be undertaken here, since the relative age compositions of the ultimate stationary populations will be identical regardless of the differences in the schedules of age-specific fertility rates, as long as the mortality rates remain the same. The ultimate populations will, of course, differ with respect to their sizes, the magnitudes of which can be ascertained mathematically, as is demonstrated in this paper.

The question regarding the actual pattern of instantaneous fertility decline is still a theoretical one, and empirical verification of a model describing such a phenomenon is not possible at the present moment. From that point of view, the exponential pattern of decline may be regarded as a possible alternative. It is interesting to note, however, that in several countries where fertility decline has been rapid over a reasonably short period of time, the exponential model does indeed provide a reasonable approximation of the pattern of this decline. For the United States (1953–1968), Canada (1953–1967), Hungary (1955–1968), Australia (1959–1968), and New Zealand (1951–1966), the linear correlations between age and the logarithm of the ratio of the five-year age-specific fertility rates (United Nations' *Demographic Yearbook, 1959* and *1969*) over the time intervals are found to lie between .90 and .99. Needless to say, these correlations are highly significant, thus providing a partial justification of the model.

It may be further pointed out that, although the exponential model is proposed and developed somewhat extensively in this paper, a generalized algebraic solution is also obtained (equation 30), in which no assumption is made about the pattern of fertility, except that of an instantaneous decline to a stationary level.

What follows next is an analysis of the impact of an exponential decline on population growth and a comparison of results with the model developed by Keyfitz. Particular cases, corresponding to varying patterns of fertility rates, are examined and results obtained for four countries.

Functional Expression for the Stationary Population

It is well known that the number of births $B(t)$ at a given time t, as the population approaches stability, is approximated. by Qe^{rt} where

$$Q = \frac{\int_0^\beta e^{-rt} G(t)\, dt}{\int_\alpha^\beta x e^{-rx} l_x m_x\, dx}, \qquad (3)$$

$G(t)$ being births to women who were present at the start of the process, and (α, β) the reproductive interval. If l_x remains unchanged, and m_x is transformed to

$$m^*_x = e^{-rx} m_x, \qquad (4)$$

the rate of growth intrinsic to l_x and m^*_x is zero, and the corresponding Q simplifies to

$$Q_0 = \frac{\int_0^\beta G(t)\, dt}{\int_\alpha^\beta x l_x m_x^*\, dx}, \qquad (5)$$

which becomes

$$Q_0 = \frac{\int_0^\beta G(t)\, dt}{\int_\alpha^\beta x e^{-rx} l_x m_x\, dx} \qquad (6)$$

because of (4). Interestingly enough, the denominator of (6) is the average age of childbearing, μ^*, not only in the resulting stationary population but also in the stable population generated by l_x and m_x.

The numerator of Q_0 depends on $G(t)$, which can be expressed as

$$G(t) = \int_{\alpha-t}^{\beta-t} P_x \frac{l_{x+t}}{l_x} m_{x+t}^* \, dx, \qquad (7)$$

where P_x is the population aged x at the beginning of the process. Following Keyfitz, if B is the number of births at the beginning of fertility decline, and P_x is stable, (7) simplifies to

$$G(t) = B \int_{\alpha-t}^{\beta-t} e^{-r(2x+t)} l_{x+t} m_{x+t} \, dx \qquad (8)$$

because of (4). The numerator of Q_0 can then be written as a double integral

$$\int_0^\beta G(t)\, dt = B \int_0^\infty \int_0^\infty e^{-r(2x+t)} \cdot l_{x+t} m_{x+t} \, dtdx. \qquad (9)$$

Note that, in (9), the limits of integration have been conveniently extended to $(0,\infty)$, which is possible because m_x is zero outside the range (α, β). The order of integration has also been interchanged, which is facilitated by the extension of the limits.

Writing $x + t = a$, so that $x \leq a \leq \infty$ and $dtdx = dadx$, the double integral in (9) can be written as

$$\int_0^\infty \int_x^\infty e^{-rx} e^{-ra} l_a m_a \, dadx. \qquad (10)$$

Changing the order of integration, (10) can be written as

$$\int_0^\infty \int_0^a e^{-rx} e^{-ra} l_a m_a \, dxda$$
$$= \frac{1}{r} \int_0^\infty (1 - e^{-ra}) e^{-ra} l_a m_a \, da$$
$$= \frac{1}{r} - \frac{1}{r} \int_0^\infty e^{-2ra} l_a m_a \, da \qquad (11)$$

because of (2). Substitution in (6) produces

$$Q_0 = \frac{B}{r\mu^*} \left(1 - \int_0^\infty e^{-2rx} l_x m_x \, dx\right). \qquad (12)$$

Since the size of the stationary population can be expressed as a product of the births and the life expectancy, this size relative to that at the beginning of the process, say P, can be expressed as

$$I^* = \frac{be^0_0}{r\mu^*} \left(1 - \int_0^\infty e^{-2rx} l_x m_x \, dx\right) \qquad (13)$$

which can be compared with (1). The major differences are that $1/R_0$ or $1/\int_0^\infty l_x m_x dx$ in the latter has been replaced by $\int_0^\infty e^{-2rx} l_x m_x dx$ in the former, and, further, that μ, the average age at childbearing in the stationary population with age-specific fertility rates m_x/R_0, is replaced by μ^*, the same in the stationary population with age-specific fertility rates $m_x e^{-rx}$. Interestingly enough, the latter is greater than or equal to the former, because

$$\int_0^\infty e^{-2rx} l_x m_x \, dx \int_0^\infty l_x m_x \, dx \geq \left(\int_0^\infty e^{-rx} l_x m_x \, dx\right)^2, \qquad (14)$$

according to the Cauchy-Schwarz inequality and the fact that the right-hand side equals 1.

Similarly, it can be shown that the average age of childbearing in the resulting stationary population with age-specific birth rate as $e^{-rx} m_x$ is smaller than or equal to that when the age-specific birth rate is given by $m(x)/R_0$. This is so, because the derivative with respect to u of

$$k(u) = \int_0^\infty x e^{-ux} l_x m_x \, dx \Big/ \int_0^\infty e^{-ux} l_x m_x \, dx \qquad (15)$$

is

$$-\frac{\int_0^\infty x^2 e^{-ux} l_x m_x \, dx \int_0^\infty e^{-ux} l_x m_x \, dx - \left(\int_0^\infty x e^{-ux} l_x m_x \, dx\right)^2}{\left(\int_0^\infty e^{-ux} l_x m_x \, dx\right)^2}, \qquad (16)$$

which is negative because of the Cauchy-Schwartz inequality. Thus, $k(u)$ is a decreasing function with respect to r, and, therefore, the value of (15) for $u = 0$ is greater than that for $u = r$, where $r > 0$.

Taking both of these inequality relationships into consideration while comparing the index values obtained by the two models, it becomes apparent that both the numerator and the denominator of (1) are larger than the corresponding quantities of (13). The net effect of this relationship on the index is not readily apparent and may have to be analyzed in individual cases.

Simplifications Based on Varying Assumptions About the Distribution of Net Maternity Function

It may be pointed out that the resolution of (13) in terms of first few moments is not possible without additional assumptions about the functional forms of the net maternity functions. In the past, different assumptions about its form have been made, and three are examined here.

Normality Assumption of Lotka (1939)

The net maternity function can be expressed as

$$(R_0/\sigma\sqrt{2\pi})\ \exp[-(x - \mu_1')^2/2\sigma^2], \quad (17)$$

where μ_1' is the average age of childbearing and σ^2 the variance. Now

$$\int_0^\infty e^{-2rx} l_x m_x\, dx$$

$$= \frac{R_0}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2rx} e^{-(x-\mu_1')^2/2\sigma^2}\, dx$$

$$= R_0 e^{-2r\mu_1' + 2\sigma^2 r^2}, \quad (18)$$

which follows from the well-known expression of the moment-generating function. Using the approximate equality relationship,

$$R_0 = e^{r\mu_1' - 1/2\sigma^2 r^2}, \quad (19)$$

(18) can be further simplified as

$$\int_0^\infty e^{-2rx} l_x m_x\, dx = \frac{e^{r^2\sigma^2}}{R_0}. \quad (20)$$

Substitution of (18) or (20) in (13) will produce an estimate of I^*.

Wicksell's (1931) Incomplete Gamma Function

According to Wicksell, $l_x m_x dx$ can be replaced by

$$\frac{Kc^k x^{k-1} e^{-cx}}{\Gamma(k)}\, dx, \qquad x \geq 0. \quad (21)$$

A modified version of the above was suggested by Mitra (1970) as

$$\frac{Kc^k (x - \alpha)^{k-1} e^{-cx}\, dx}{\Gamma(k)}, \qquad x \geq \alpha \quad (22)$$

where α is the lower limit of the childbearing period. It is easy to see that

$$\int_0^\infty e^{-2rx} l_x m_x\, dx = e^{-r\alpha}\left(\frac{r + c}{2r + c}\right)^k$$

For such a distribution,

$$R_0 = Ke^{-c\alpha},$$

$$c = \mu_1'/\sigma^2,$$

and

$$k = \mu_1'^2/\sigma^2,$$

where μ_1' is obtained by using α as the arbitrary origin, so that the average age of childbearing, μ, is equal to $\mu_1' + \alpha$.

Pearson's Type 1 Distribution

Mitra (1970) observed that, among several approximations, the best result is obtained when the net maternity function is expressed in the form of Pearson's type 1 distribution. Accordingly, using the limits α and β, one can write

$$l_x m_x\, dx = \frac{R_0 (x - \alpha)^{m_1 - 1} (\beta - x)^{m_2 - 1}}{(\beta - \alpha)^{m_1 + m_2 - 1} B(m_1, m_2)}\, dx,$$

$$\alpha \leq x \leq \beta, \quad (23)$$

where B stands for the Beta function conveniently defined as

$$B(m_1, m_2) = \int_0^1 x^{m_1 - 1} (1 - x)^{m_2 - 1}\, dx,$$

$$m_1, m_2 > 0. \quad (24)$$

Now

$$\int_0^\infty e^{-2rx} l_x m_x \, dx = \frac{R_0 e^{-2r\alpha}}{(\beta-\alpha)^{m_1+m_2-1} B(m_1, m_2)} \sum_{i=0}^{\infty} \frac{(-2r)^i}{i!} \int_\alpha^\beta (x-\alpha)^{m_1+i-1} (\beta - x)^{m_2-1} \, dx$$

$$= \frac{R_0 e^{-2r\alpha}}{B(m_1, m_2)} \sum_{i=0}^{\infty} \frac{(-2r)^i}{i!} (\beta-\alpha)^i B(m_1+i, m_2) \qquad (25)$$

$$= R_0 e^{-2r\alpha} \left[1 + \sum_{i=1}^{\infty} \frac{(-t)^i}{i!} \frac{m_1(m_1+1)\cdots(m_1+i-1)}{(m_1+m_2)(m_1+m_2+1)\cdots(m_1+m_2+i-1)} \right], \qquad (26)$$

where $t = 2r(\beta - \alpha)$. It can be shown that since $B(m_1, m_2) = B(m_2, m_1)$, another expression for the integral can be obtained as

$$R_0 e^{-2r\alpha} \cdot \left[1 + \sum \frac{(t)^i}{i!} \frac{\prod_{s=0}^{i-1} (m_2 + s)}{\prod_{s=0}^{i-1} (m_1 + m_2 + s)} \right]. \qquad (27)$$

However, for empirical evaluation of the results, the alternating series in (26) is preferable, because of its rapid convergence due to the fact that, generally, $m_1 < m_2$.

Results

Of the three models representing the net maternity function, the normal is the simplest, because, for a given value of R_0, only two more parameters must be known or estimated for purposes of graduation, in contrast with three for the type III and four for the type I distributions. However, for reasons of simplicity, the number of parameters in the latter distributions can be reduced to two by assuming, say, $\alpha = 15$ for the type III and, in addition, $\beta = 45$ for the type I distributions. The parameters necessary to estimate I* are thereby reduced to b, r, $e_0{}^0$, the average and the variance of the distribution of the net maternity function. While comparing the index values with those presented by Keyfitz (1971, Table 3), it was found that the latter two parameters were not presented by him since he did not need them. However, the average age of net maternity can be approximately obtained for small values of r from (16), when the same is known for the stationary population in which the age-specific birth rate is m_x/R_0. For $u = r$, the formula reduces to

$$\left. \frac{dk(u)}{du} \right|_{u=r} = -\sigma^2, \qquad (28)$$

where σ^2 is the variance of the distribution of the net maternity function. Thus,

$$\mu_* = \mu - r\sigma^2, \qquad (29)$$

where μ^* and μ are the average ages corresponding to $m_x{}^*$ and m_x/R_0, respectively. All these parameters are shown in Table 1, in which the index values obtained by Keyfitz (1971) are reproduced in the row corresponding to the Variable I [see equation (1)]. The I* values are obtained from the model developed in this paper and subjected to three different patterns assumed as the age-specific fertility rates.

Concluding Observations

The four sets of index values presented in Table 1 are quite similar, as they are expected to be. Of the four estimates, the size of the stationary population relative to that of the stable population at the beginning of the instantaneous fertility decline is lowest according to type I assumptions of net maternity function for countries experiencing a rapid rate of growth. For these two countries, namely, Chile and Colombia, differences in ultimate population sizes are of the order of 6 to 7 percent which is quite significant in terms of absolute numbers. However, fertility decline can rarely be expected to be instantaneous, and the ultimate size of the stationary population, if and whenever at-

TABLE 1.—Selected Characteristics of the Stable Population and Values[a] of I Under Different Approximations of the New Maternity Function

Parameters and Values of I	Chile 1965	Colombia 1965	Italy 1966	United States 1967
$1000b$	33.4	38.8	16.7	17.8
$1000r$	23.5	29.0	5.58	7.38
R_0	1.95	2.31	1.17	1.21
$\overset{0}{e}_0$	62.7	61.7	74.2	74.2
μ	29.1	29.6	28.6	26.3
μ^*	27.9	28.2	28.4	26.0
μ_2	49.4	50.9	37.4	34.9
I^b	1.49	1.59	1.13	1.18
I^c	1.52	1.61	1.16	1.21
I^d	1.49	1.57	1.08	1.23
I^e	1.41	1.48	1.13	1.19

a- I = the ratio of the size of the stationary population to that of the initial stable population (see text).

b- The values in this row are from Keyfitz (1972).

c- Normal approximation to net maternity function.

d- Type III approximation to net maternity function.

e- Type I approximation to net maternity function.

Source: Keyfitz, personal communication.

tained, will be larger than any of the estimates obtained from the index values. Such estimates, when fertility decline is spread over a number of years, can be made by making appropriate adjustments along the lines suggested by Keyfitz.

One final observation may be made at this point. If the initial population is stable, and the fertility decline to replacement level is based on a set of $m_x{}^*$ values different from m_x/R_0 or from $m_x e^{-rx}$, as assumed, respectively, by Keyfitz and by this investigator, the mathematical expression for the index value simplifies to

$$I^* = \frac{b\overset{0}{e}_0}{r\mu^*}\left(1 - \int_0^\infty e^{-rx} l_x m_x{}^* \, dx\right), \quad (30)$$

where

$$\int_0^\infty l_x m_x^* \, dx = 1.$$

This can easily be verified by following steps similar to those described earlier in developing equations (8)–(13). Expanding the exponential in (30) and then simplifying, the algebraic solution of (30) is obtained as

$$I^* = b^0 e_0 \left[1 - \frac{r}{2\mu^*} (\mu_2^* + \mu^{*2}) + \cdots \right],$$

The series will usually converge; however, knowledge of the moments beyond the second of the stationary population may be required for the solution. In practice, however, μ_2^* will be small compared to $(\mu^*)^2$, and, as a trial solution, one may use $be^0{}_0 (1 - r\mu^*/2)$, where the average age at childbearing of the stable population may be substituted for μ^*. For Chile, Colombia, Italy, and the United States, values of I^* thus obtained are 1.41, 1.42, 1.14, and 1.19, respectively, which, when compared with others, appear to be quite satisfactory.

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