Oceanic induction and shifting the spectrum

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Summary. The recently produced method of analytic continuation in geomagnetic induction is shown for linear problems to be equivalent to the method of shifting the spectrum developed in this context nearly four years earlier. The changes are ones of nomenclature. The periods of decay are shown to be real and negative; this is necessary underpinning for both methods.

1. Introduction

The methods of functional analysis are often couched in such simple terms that they deceive the reader, whose previous experience has often been that complicated problems have complicated solutions. The method of shifting the spectrum is typical in this respect. When all the paraphernalia of proof are concealed the method appears to be so simple that its comprehensive nature and the generality of its application may be overlooked. The method was introduced into geomagnetic induction by Hutson, Kendall & Malin (1972, 1973) (hereafter HKM) whose first use of it was in three dimensions using tesserae on the sphere. In 1972 at the IAGA Workshop on Electromagnetic Induction, extensions of the theory were made by Hewson-Browne et al. (1973) (hereafter H-BHKM) to take account of formulations in terms of the stream function. Successive over-relaxation was introduced. Hewson-Browne (1973) reported at the same Workshop that he had successfully treated the first asymmetrical configuration in three dimensions.

Since then Hobbs & Brignall (1976) have put forward a method of analytic continuation. We here show that for linear problems this is in fact the method of shifting the spectrum, with appropriate changes in nomenclature.

2. Basic equations

If $f$ is an unknown quantity such as the current stream function $\psi$ in a thin ocean, the surface current density $J$, or the horizontal electric field components on the surface, an equation for $f$ may be obtained in the linear operator form

$$(\lambda + L)f = g.$$  

(1)
Here $g$ is a known right-hand side obtained from the inducing electromagnetic field, $\lambda$ is a constant composed of several physical constants such as frequency, depth and conductivity, and $L$ is a linear operator with a spectrum consisting of a set $\{\sigma\}$ such that for any element $\sigma$ of the spectrum there exists $s$ such that $Ls = \sigma s$. These correspond to the normal modes of decay of the system. If $L$ is real, if equation (1) is cast in the right form, and if $\sigma_M$ is arithmetically the largest element of $\{\sigma\}$, we have $\sigma_M < \sigma < 0$. Then $\lambda$ is purely imaginary. Hobbs & Brignall (1976) refer to eigenfrequencies: these would be proportional to $\sigma^{-1}$, but would lie on the imaginary axis in an Argand diagram. This means that they have chosen $L$ to be a purely imaginary operator, and $\lambda$ to be real. One may have to invert an elliptic differential operator to obtain $L$.

HKM observe that, as is well known, if $|\lambda| > |\sigma_M|$, the iterative scheme
\[
\lambda f_{n+1} = g - Lf_n
\]
converges from any $f_0$ to the unique solution of (1). This is equivalent to Price’s first method (Price 1949). Moreover, given the circumstances of these problems, the case $|\lambda| < |\sigma_M|$ often occurs; a constant $\beta$ can then be found such that the scheme
\[
(\lambda - \beta)f_{n+1} = g - (\beta + L)f_n
\]
converges to the solution $f$ of (1) for any $f_0$. It is convenient to choose $f_{-1} = 0$ so that $f_0 = (\lambda - \beta)^{-1}g$. Then the solution of (1) is
\[
f = \left\{ 1 + \sum_{n=1}^{\infty} \left( \frac{\beta + L}{\beta - \lambda} \right)^n \right\} f_0,
\]
provided that the series converges. This may be easily verified by substituting from (4) into (1) using the value $f_0 = (\lambda - \beta)^{-1}g$.

Hobbs & Brignall (1976) propose an analytic continuation in which the frequency is mapped onto the $p$ plane by the transform $p = \omega/(\gamma - \omega)$. As our $\lambda$ in (1) is proportional to $\omega^{-1}$, for (1) $p$ becomes $p = (\gamma \lambda - 1)^{-1}$, giving
\[
\lambda = \frac{1}{\gamma} \left( p^{-1} + 1 \right).
\]
Substituting into (1) and multiplying through by $\gamma p$ we obtain
\[(1 + p + \gamma pL)f = \gamma pg,
\]
leading to the iterative scheme
\[f_{n+1} = p \{ \gamma g - (1 + \gamma L)f_n \} .\]
Taking $f_0 = p\gamma g$ for the same reasons as before, the solution of (1) is
\[
f = \left\{ 1 + \sum_{n=1}^{\infty} (-p)^n(1 + \gamma L)^n \right\} f_0.
\]
Equations (4) and (6) should be carefully compared. They are identical if we take
\[
\frac{\beta}{\beta - \gamma} = \gamma - p \quad \text{and} \quad \beta = \frac{1}{\gamma}.
\]
As equation (7) is consistent with equation (5), and $f_0 = p\gamma g = (\lambda - \beta)^{-1}g$, we see that the analytic continuation procedure is equivalent to the method of shifting the spectrum for a linear operator $L$. 


This is confirmed further when we come to examine closely the benefits, for they also accrue to the credit of the method of shifting the spectrum. They are as follows:

2.1 CONVERGENCE

HKM guarantee convergence of the series (4) for a real negative spectrum and all values of purely imaginary \( \lambda \) (no matter how small) provided that \( \infty > \beta > -\sigma_M/2 > 0 \). For real \( L \) using (7) this is equivalent to

\[
0 < \gamma < -2/\sigma_M. \tag{8}
\]

Remembering that Hobbs & Brignall (1976) chose a purely imaginary \( L \) (i.e. replacing \( L \) by \( iL \)) we see that the HKM convergence criterion (8) using a real acceleration parameter \( \beta \) is the same as the criterion quoted by Hobbs & Brignall (1976), namely, \( 0 < \alpha < 2\Omega_1 \), with \( \gamma = \alpha \) (their notation).

2.2 EFFICIENT USE OF STORAGE

If large amounts of storage are available, we may store \( f_0, Lf_0, L^2f_0, \ldots \) and use the expansion of (4), namely,

\[
f = \left\{ 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} (\beta - \lambda)^{-n} \binom{n}{m} \beta^{n-m} L^m \right\} f_0. \tag{9}
\]

This is, however, not necessary for broad meshes of five degrees for which HKM obtained satisfactory convergence. Since Hobbs & Brignall (1976) chose a series for the stream function which had alternately purely imaginary and purely real terms, real and imaginary values alternate in their series. If we follow HKM in using a real operator \( L \) and a real value for the acceleration parameter \( \beta \) equation (9) becomes better. There seems to be no reason for restricting \( \beta \) to be real. On a one-degree mesh over the globe, each term of the series would require about 130 K in floating point storage; a two-degree mesh would require only 32 K in floating point storage for each of its terms.

3 The nature of the spectrum

Taking the operator \( L \) to be real it can be so chosen that its spectrum \( \{\sigma\} \) is real and negative, for each \( \sigma \) is proportional to one of the normal periods of decay \( T \) of the system. Consider a finite system of arbitrary conductivity \( \eta \) extending over all space. Neglecting the displacement currents, Maxwell’s equation gives

\[
\mu_0 B = \tau \text{curl} (\eta^{-1} B). \tag{10}
\]

At infinity we assume \( B = 0(1/r^3) \). Taking the scalar product of (10) with \( B \) and integrating by parts using the divergence theorem gives

\[
\mu_0 \iiint B^2 \, dx \, dy \, dz = \tau \iiint B \cdot \text{curl} (\eta^{-1} B) \, dx \, dy \, dz = \tau \iiint \eta^{-1} (\text{curl} B)^2 \, dx \, dy \, dz,
\]

where the integral is over all space. It follows that any non-zero \( \tau \) is real and positive, giving \( \sigma < 0 \).
References


