Stability of Envelope Soliton

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In this decade, many works have been made to show that the stable pulse-like waves which are called solitons play an important role in the development of one-dimensional nonlinear wave phenomena. Several attempts have been proposed to describe nonlinear wave system as a superposition of solitons. The success of these attempts is dependent upon whether the soliton concerned is fully stable. It is known that the soliton is stable against a one-dimensional disturbance along the wave propagation. The two-dimensional stability has been studied by Kadomtsev and Petviashvili for a system which the Korteweg-de Vries equation applies. Zakharov has treated the two-dimensional stability of envelope solitons which display a remarkable property of permanence in a self-modulation process of nonlinear plane wave and has pointed out, by using a variational method, the possibility that envelope solitons can be unstable. In this paper we apply the reductive perturbation method to the two-dimensional stability problem and present a result more detailed than Zakharov's.

The complex envelope, \( \phi \), of a self-modulated plane wave propagating along the \( x \)-axis in a nonlinear dispersive medium is described by a nonlinear Schrödinger equation,

\[
i\phi_t + (\lambda'/2)\phi_{tt} + (\lambda/2k_0)\phi_\perp^2\phi + \gamma|\phi|^2\phi = 0,
\]

where the subscripts denote partial differentiations, \( k_0 \) the wavenumber of carrier wave, \( \lambda \) the group velocity, \( \lambda' \) its derivative with respect to \( k \), \( \gamma \) a constant giving the measure of nonlinearity, \( \xi = x - \lambda t \) and \( r_\perp = (0, y, z) \). By making use of the complex envelope \( \phi \), the modulated plane wave is expressed as \( \phi(\xi, r_\perp, t) \).

When \( \gamma' > 0 \), Eq. (1) has an envelope soliton solution vanishing at infinity:

\[
\phi = S_{\gamma} = A \sech \left( (\gamma'/\lambda')^{1/2}A(\xi - VT) \right) \times \exp \left[ i(\gamma' x - \omega_0 t) \right].
\]

Consider a two-dimensional perturbation such that \( A \) and \( V \) are slowly varying functions of \( r_\perp \) and \( t \). If the wavenumber and frequency of the perturbation are sufficiently small compared with those of the unperturbed soliton, \( (\gamma'/\lambda')^{1/2}A \) and \( \gamma A^2/2 \), the perturbed solution is expected to be

\[
\phi = S_{\gamma} - \delta \phi.
\]
little different from the solution (2). It is then anticipated that
\[ \phi(\xi, \rho, t) = A_0(A(\xi - \epsilon \xi_0), \rho, t) \times \exp(i\theta + i\xi \Omega/\lambda), \] (3)
\[ \theta = \gamma A^2/2 - \epsilon \dot{v}^2/(2\lambda'), \quad \xi_0 t = \nu, \] (4)
\[ A = A_0 + \epsilon a(\tau, \rho), \quad v = v(\tau, \rho), \] (5)
where \( \epsilon \) is a smallness parameter, \( \epsilon \ll 1 \), and \( \rho \) are the stretched variables to denote the slowness of \( t \)- and \( \rho \)-dependence of \( a \) and \( v \), \( \tau = \epsilon^{1/2} t \) and \( \rho = \epsilon^{1/2} \rho \). We solve Eq. (1) by substituting Eqs. (3) ~ (5), setting
\[ g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \cdots \] (6)
and solving the sequence of equations corresponding to the successive powers of \( \epsilon \).

In the lowest order, we have
\[ (\lambda'/\gamma) g_{0y} - 2 g_0 = 0, \] (7)
where \( \gamma = A(\xi - \epsilon \xi_0) \). Provided \( \gamma \lambda' > 0 \), this equation has a soliton solution vanishing at infinity, i.e.,
\[ g_0 = \text{sech}(\gamma/\lambda')^{1/\gamma}, \] (8)
which is equivalent to \( S_v \) with \( V = 0 \) (compare with Eq. (2)).

Dividing \( g_1 \) into the real and imaginary parts, \( g_1 = R + iI \), we get, in the order \( \epsilon^{1/2} \),
\[ (\lambda'/\gamma) R_{yy} - 6 g_0^2 R = (2/\gamma A^3) \{ \lambda^{-1} v \gamma g_0 + \epsilon^{1/2} (\lambda A^2/2k_0) P_\rho \dot{\xi} \dot{g}_0 \}, \] (9)
\[ (\lambda'/\gamma) I_{yy} - I + 2 \epsilon g_0^2 I = - (2/\gamma A^3) \{ a_e(\dot{g}_0 + \gamma \dot{g}_0) + \epsilon^{-1/2} (\lambda A^2/2k_0) P_\rho \dot{\xi} \dot{g}_0 \}. \] (10)

We now impose that \( R \) and \( I \) are bounded at \( \xi = \pm \infty \). Equation (9) is multiplied by \( g_{0y} \) and integrated by parts with \( \gamma \) over \((-\infty, \infty) \). Taking into account Eq. (10) and differentiating the equation so obtained with respect to \( t \), we have
\[ \nu_{yy} = (\gamma A^3/3k_0) P_\rho \dot{v} = 0. \] (11)

Applying the same procedure to Eq. (10) but replacing \( g_{0y} \) by \( g_0 \), we then obtain
\[ a_{ye} + (\lambda A^3/3k_0) P_\rho \dot{v} = 0. \] (12)

It is noted that Eqs. (11) and (12) are nothing but the condition for the boundedness of \( R \) and \( I \) at \( \xi = \pm \infty \).

Equations (11) and (12) indicate that for \( \gamma > 0 \) the soliton is stable with regard to a perturbation with \( v \neq 0 \) and \( a = 0 \) (this mode is called "flutter") and unstable against a perturbation with \( v = 0 \) and \( a \neq 0 \) ("granulation"). This case \( (\gamma > 0) \) corresponds to the case of positive dispersion, \( \lambda' > 0 \), since the envelope soliton solution is possible only for \( \gamma \lambda' > 0 \). Whilst, in a medium with negative dispersion \( \gamma < 0 \), the soliton exhibits a flutter-type of instability and is stable for granulations. It is concluded that the envelope soliton is, in general, not stable.