Parafermi Field Theory and Elementary Particles*}

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A model of elementary particles based on parafermi field theory is discussed. Urbaryons (that constitute hadrons) and leptons are assumed to be described by parafermi fields of \( p = \text{odd} \) and \( p = 2 \), respectively (\( p \): order), and the theory is subject to the minimal gauge group \( O(p) \times SO(2) \). It is emphasized that the existence of the gauge group is a natural consequence of the locality condition of the theory.

§ 1.

In a recent paper\(^1\) we have discussed the relation between the theory of a parafermi field of order \( p \) (PF-theory) and that of an ordinary fermi field with a hidden variable which takes on \( p \) different values (F-theory), with the conclusion that the relation depends on what kind of operators are adopted as observables. For the sake of convenience let us first summarize the main points of our previous results together with some supplementary remarks.

Let \( \hat{\phi}(x, t) \) be a parafermi field of order \( p \) (where and in what follows \( \hat{\phi}(x, t) \) stands for either \( \phi(x, t) \) or \( \phi^*(x, t) \)) and \( F(V, t) \) is an arbitrary observable defined in a spatial region \( V \) at time \( t \), that is, a hermitian functional of \( \{\phi(x, t)\} \). We require that any observables \( F(V, t), F'(V', t), \ldots \) shall satisfy the locality condition of the form

\[
[F(V, t), F'(V', t)] = 0 \quad \text{for } V \cap V' = \emptyset.
\]

The most general forms of the \( F \)'s that satisfy (1) are given as follows:

(i) In the case \( p = \text{odd} \), \( F(V, t) \) must be a functional of \( [\hat{\phi}(x, t), \hat{\phi}(y, t)] \) \( (x, y \in V) \). In this case, \( F(V, t) \) is found to satisfy further the locality condition of a stronger form

\[
[F(V, t), \hat{\phi}(x, t)] = 0 \quad \text{for } x \in V.
\]

(ii) In the case \( p = \text{even} \), \( F(V, t) \) must be a functional of \( [\hat{\phi}(x, t), \hat{\phi}(y, t)] \) and

\[
\{\hat{\phi}(x_1, t), \hat{\phi}(x_2, t), \ldots, \hat{\phi}(x_p, t)\}.
\]

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where the operators \( \tilde{\phi}^{(\alpha)}(x, t) \) (\( \alpha = 1, 2, \ldots, p \)) are the Green-component fields of \( \hat{\phi}(x, t) \):

\[
\hat{\phi}(x, t) = \sum_{\alpha=1}^{p} \tilde{\phi}^{(\alpha)}(x, t)
\]

and satisfy the anomalous commutation relations:

\[
\{ \psi^{(\alpha)}(x, t), \psi^{(\beta)}(y, t) \} = \delta^\alpha(x - y),
\]

\[
\{ \phi^{(\alpha)}(x, t), \phi^{(\beta)}(y, t) \} = 0
\]

and

\[
[\tilde{\phi}^{(\alpha)}(x, t), \tilde{\phi}^{(\beta)}(y, t)] = 0. \quad (\alpha \neq \beta)
\]

Such \( F(V, t) \)'s for the case (ii), although satisfying the relation (1), are known in general to violate the cluster property of the theory. The difficulty can be removed, however, by assuming that observables \( F \) take different forms \( F^{(\pm)} \) in the even and odd sectors of the para-Hilbert space, such that \( F^{(\pm)} = F([\tilde{\phi}(x, t), \hat{\phi}(y, t)], \tilde{\phi}(x, t), \phi^{(\alpha)}(y, t)] = \pm \delta_\alpha^\delta \delta^\alpha(x - y), \{ \phi^{(\alpha)}(x, t), \phi^{(\alpha)}(y, t) \} = 0. \]

By performing a Klein transformation to the \( \tilde{\phi}^{(\alpha)}(x, t) \)'s we can go over to the corresponding \( F \)-theory which is described by a set of fermi fields \( \phi^{(\alpha)}(x, t) \) (\( \alpha = 1, 2, \ldots, p \)), where the \( \phi^{(\alpha)} \)'s are the Klein transforms of the \( \tilde{\phi}^{(\alpha)} \)'s and satisfy the normal commutation relations:

\[
\{ \phi^{(\alpha)}(x, t), \phi^{(\beta)}(y, t) \} = \delta_\alpha^\delta \delta^\alpha(x - y), \{ \phi^{(\alpha)}(x, t), \phi^{(\alpha)}(y, t) \} = 0.
\]

Observables in the \( F \)-theory are then obtained by making the following replacement in the expressions for the corresponding observables in the \( PF \)-theory:

\[
[\hat{\phi}(x, t), \hat{\phi}(y, t)] \leftrightarrow \sum_{\alpha=1}^{p} [\phi^{(\alpha)}(x, t), \phi^{(\alpha)}(y, t)],
\]

\[
\pm : \{ \hat{\phi}(x_1, t), \hat{\phi}(x_2, t), \ldots, \hat{\phi}(x_p, t) \} : \quad \text{for } \{ \text{even} \} \text{ sector}
\]

\[
\leftrightarrow p! \sum_{\alpha_1, \alpha_2, \ldots, \alpha_p} \varepsilon_{\alpha_1, \alpha_2, \ldots, \alpha_p} \phi^{(\alpha_1)}(x_1, t) \phi^{(\alpha_2)}(x_2, t) \cdots \phi^{(\alpha_p)}(x_p, t),
\]

where \( \varepsilon_{\alpha_1, \alpha_2, \ldots, \alpha_p} \) is a totally antisymmetric tensor such that \( \varepsilon_{1,2,\ldots,p} = 1. \)

An important point here is that the right-hand sides of (4) and (5) are invariant under the transformations \( g \in O(p) \) and \( g \in SO(p) \), respectively:

\[
\phi^{(\alpha)}(x, t) \rightarrow \sum_{\alpha=1}^{p} g_{\alpha \beta} \phi^{(\beta)}(x, t).
\]

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* This can also be expressed as a polynomial of degree \( p \) in \( \hat{\phi} \) when use is made of the para-commutation relations for the operator \( \hat{\phi} \).

** It is to be remarked that the above correspondence relation denoted by \( \leftrightarrow \) does not imply equality. In fact, the relations (4) and (5) hold true effectively when one applies the Klein transformation to state vectors as well.
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That is to say, observables in the F-theory, corresponding to those in the PF-theory which are subject to the locality condition (1), remain invariant under the above group of transformations, which we shall call the minimal gauge group. Thus, in the cases \( p=\text{odd} \) and \( p=\text{even} \), the minimal gauge groups are \( O(p) \) and \( SO(p) \), respectively.

In addition to the condition (1), which is a basic requirement for observables, we may impose on them some further restrictions which are independent of statistics. In so doing we obtain a larger gauge group which contains the minimal gauge group as a subgroup. It is easy to see, for example, that if observables are restricted to functionals of \( [\psi, \phi'] \) only, the gauge group becomes \( U(p) \), and that if in the case \( p=\text{even} \), observables are restricted to functionals of \( [\psi, \phi'] ; \{\psi(x_1, t), \psi(x_2, t), \cdots, \psi(x_p, t)\} \) and the hermitian conjugate of the latter, then the gauge group becomes \( SU(p) \). Since all observables remain invariant under the gauge transformations, it is absolutely impossible to observe individual values of the variable \( \alpha \). In this sense \( \alpha \) may be interpreted as a hidden variable.

The relation between the PF- and F-theories is established completely when one further specifies the relation between the corresponding Hilbert spaces. For this purpose we have previously introduced the following three kinds of equivalence relations. When for a given state of the total system of the PF-field (enclosed in the entire universe) one can always find a corresponding state for the total system of the F-field which gives the same results for all possible observations, and the converse is also true (not necessarily true), then the PF-theory is said to have the relation of strong (weak) equivalence with the F-theory. On the other hand, when the above type of correspondence exists in either direction with respect to states of any subsystem of the PF- and F-fields (enclosed in a spatially localized region), the two theories are said to be in the relation of local equivalence.

In this respect the following theorem can be proved:\(^1\) (i) For the case of the gauge group \( U(p) \) the relation of strong and local equivalence holds true. (ii) For the case of the gauge group \( O(p), SO(p) \) or \( SU(p) \) (the latter two cases being possible only for \( p=\text{even} \)) the relation of weak equivalence holds true, and when the subsystem in question, which is susceptible of our observations, is sufficiently small compared with the total system (this condition is obviously satisfied in the realistic situation) the relation of local equivalence holds true.\(^1\)

\(^1\) We are assuming here that the vacuum state \( |0\rangle \) for the fields \( \hat{\phi}^{(0)} \) is invariant under gauge transformations, so that for the special elements \( K_\alpha K_\beta (\alpha \neq \beta) \) of the gauge group concerned there holds a relation \( K_\alpha K_\beta |0\rangle = |0\rangle \) (Ref. 3), where \( K_\alpha \) is the Klein operator that relates \( \phi^{(0)} \) to \( \phi^{(0)} \). Thus, if the gauge symmetry is spontaneously broken in the F-theory for the case of a non-Abelian gauge group, there does not exist any equivalent PF-theory. In other words the non-Abelian gauge symmetry that originates from the PF-theory can never be broken spontaneously.
The relation between the PF- and F-theories being established, we can then prove the cluster property of the former on the basis of the same property of the latter.

§ 2.

The foregoing arguments have been confined to the case of a single parafermi field of order $p$. It is expected, however, that the arguments can further be extended without any difficulties to the case of coexisting parafields. In fact, some authors have already developed to a certain extent the parafield theory for such a case, which, when elaborated further in connection with gauge groups, will enable us to discuss problems such as the relation between parafields and ordinary fermi or bose fields with hidden variables, the possible types of gauge groups, etc. In the following, however, we shall not be concerned with the general theory of such cases, but confine ourselves to the case that seems to be physically the most realistic. Here, from the general theory we quote only the following theorem: Between any two fields that obey parastatistics of different orders one must assume, for equal-time quantities, either (bilinear) commutation or anticommutation relations, the choice of which must be so determined as to be compatible with the locality condition.

Thus, in the following we shall consider a system consisting of (i) urbaryons with spin 1/2 that constitute hadrons, (ii) leptons and (iii) certain kinds of bosons that mediate the interactions (such as the photon, intermediate bosons, etc.).

It is a well-established fact that baryons and leptons obey ordinary fermi statistics. In order for any baryon to be a composite system of an odd number of urbaryons, it is necessary that the minimal gauge group, which the urbaryon field is subject to, contains at least one one-dimensional irreducible representation among those expressed by tensors of odd rank; for in such a representation the degree of freedom corresponding to the hidden variable $\alpha$ is, so to speak, frozen, so that the composite systems can behave as ordinary fermions. That is to say, baryons which obey fermi statistics must belong to such irreducible representations of the minimal gauge group. It is not difficult to see that the minimal gauge groups which meet this requirement are $O(p)$ with $p=\text{odd}$ and $SO(2)$. In the former (latter) case those composite systems which obey fermi statistics consist of $(p+2r)$ particles ($(1+2r)$ particles), where $r$ is a non-negative integer to be determined by the dynamics of the model. As for leptons we can argue in a similar manner. If we assume that leptons are not composite systems, minimal gauge groups are restricted to $O(1)$ and $SO(2)$. In a recent paper we have studied in detail the lepton model based on the minimal gauge group $SO(2)$.

In view of this we shall assume in the following that urbaryons (or hadrons) and leptons are subject to the minimal gauge groups $O(p)$ with $p=\text{odd}$.
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and $SO(2)$, respectively. That is to say, the urbaryon field, to be denoted by $u$ (spin, unitary-spin and other indices being omitted), is a parafermi field of odd order $p$, whereas the charged and neutral leptons, to be denoted by $\chi_i$ and $\chi_2$, respectively, are parafermi fields of order 2. From the theorem quoted above we then find that $u$ and $\chi_i(i=1,2)$ at equal times must be either commutable or anticommutable.

If we further assume that the boson fields $b$, which mediates the interactions of $u$ and $\chi$, are commutable at equal times with $u$ and $\chi$, then the locality condition (1) will restrict observables $F$ of the system to the following form:

$$F = F([\hat{u}, \hat{u}], [\chi_i, \chi_j], : \{\chi_i, \chi_j\} : , b),$$

where $: \{\chi_i, \chi_j\} :$ in the present case can be written as

$$: \{\chi_i(x,t), \chi_j(y,t)\} : = \{\chi_i(x,t), \chi_j(y,t)\} - 2\delta_{ij}\delta^k(x-y),$$

and its hermitian conjugate.

We see therefore that any $F$ is given as a functional of the bilinear forms or currents of the urbaryon and lepton fields.* The minimal gauge group, that leaves all observables (7) invariant, is now found to be a direct product $O(p)_h \times SO(2)_l$ of the respective minimal gauge groups of hadrons and leptons, that is, $O(p)_h$ and $SO(2)_l$, where the suffices $h$ and $l$ stand for hadrons and leptons, respectively. Thus, the superselection rules associated with the respective gauge groups, $O(p)_h$ and $SO(2)_l$, hold true as well in the present case. If we introduce a further restriction that the currents which constitute observables $F$ are to conserve the respective particle numbers, then the gauge group will be enlarged to $U(p)_h \times SO(2)_l \times U(1)_n$, where $U(1)_n$ is the gauge group associated with the lepton number. It is known,[6] on the other hand, that for the leptons with the gauge group $SO(2)_l$, $(N_\mu - N_e)$ is a superselection quantum number, where the muon number $N_\mu$ is the total number of $\mu^-$ and $\nu_\mu$, and the electron number $N_e$ is the total number of $e^-$ and $\nu_e$. We find therefore that in the case of the enlarged gauge group the urbaryon number, $N_\mu$ and $N_e$ are all superselection quantum numbers.

So far, the parameter $p$ for the hadron gauge group has not been specified except that $p = \text{odd}$. In fact, the distinguishable 3-triplet model by Nambu and Han[6] corresponds to the case $p = \text{1}$, and the paraquark model by Greenberg[6] corresponds to the case $p = \text{3}$. As is clear from the preceding discussion Gell-Mann's colored quark model[6] is in the relation of strong and local equivalence

*$^*$ As seen from (7), the locality condition alone does not specify whether $\hat{u}$ and $\hat{\chi}$ at equal times are commutable or anticommutable, and this difference in commutativity is not susceptible of observations. We may say therefore that $\hat{u}$ and $\hat{\chi}$ form different families in the sense of Umezawa et al. (Ref. 5).
with the paraquark model with the gauge group $U(3)_s$, so that the Green index $\alpha=1, 2, 3$ may be identified with the color=$R, W, B$. As already discussed in our previous paper, it is not possible, on the basis of the strange nature of parastatistics alone, to guarantee that only color-singlet states, that is, those composite systems which obey ordinary bose or fermi statistics are realized in nature. In order to make this possible it is therefore necessary to introduce, in addition to the kinematical restriction imposed by statistics, some new mechanism, perhaps of a dynamical nature, the formulation of which, however, is completely unknown to us at present.

In this note we have been assuming that the fields $b$ which have integer spins and which mediate the interactions obey ordinary bose statistics. Evidently we cannot a priori exclude the possibility that they may even obey parabose statistics, and this remains a problem to be investigated further.

Lastly we should emphasize the following specific feature of the present theory. The gauge groups which leave observables in the F-theory invariant originate from the locality condition in the PF-theory: The invariance of the theory has not been imposed as an ad hoc assumption. In the usual theories, on the other hand, any symmetry transformations related to the inner structure of elementary particles (especially, non-Abelian transformations), with the exception of inhomogeneous Lorentz transformations, are only fictitious transformations that can never be realized in the laboratory. We believe that if physical theories are to be strictly invariant under a certain group of transformations, such a property should be founded on a realistic physical basis. In this sense our approach based on the locality condition in parafermi field theory seems to indicate a new possible direction in the pursuit of the problem of symmetry.

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*) Gell-Mann's "quark statistics" (Ref. 8), which requires further that only color-singlet states be realized in nature, may therefore be regarded as corresponding to "parafermi statistics of order 3 plus the mechanism mentioned above". As far as this mechanism remains unknown, it is not clear whether and how the quark statistics can be founded on a theoretical basis. In this connection the following point should be made clear: The above-mentioned equivalence relation between the two types of quark model does not necessarily imply that corresponding to any given theory for a system consisting of more than one color-multiplets, there always exists an equivalent theory of parafields (cf., the following footnote).

**) We can prove that as far as free and interacting fields are assumed to obey one and the same kind of statistics, it is not possible to introduce colored gauge fields of the Yang-Mills type which obey parabose or any other unusual statistics. The color-octet gluon (Ref. 9), on the other hand, may be identified, in our model, with a composite system of a colored-quark and an anti-colored-quark. In a previous paper (the second paper of Ref. 1) we have defined the asymptotic field corresponding to a composite system of parafermi particles as the asymptotic limit of the product of the parafermi fields that constitute the composite system. This procedure, however, is not generally legitimate, except for the case in which composite systems obey bose or fermi statistics. A correct and simple method to deal with composite systems in parafermi field theory consists in utilizing the equivalence relation; We have only to go over to the F-theory and to solve the problem there. The same method applies to the case of parabose field theory.
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