Microscopic Study of $^{16}$O-$^{16}$O Interaction by the Resonating Group Method

Akihiro TOHSAKI, Fumiya TANABE*,** and Ryozo TAMAGAKI**

Faculty of Textile Science and Technology, Shinshu University, Ueda
*Research Institute for Fundamental Physics, Kyoto University, Kyoto
**Department of Physics, Kyoto University, Kyoto

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A new method for constructing the kernels of the resonating group formulation in analytic way is presented in the $^{16}$O$+^{16}$O system, where the harmonic oscillator $(0s)(0p)^{12}$ configuration is assumed for the $^{16}$O ground state. The real phase shifts calculated from a simple effective two-nucleon potential are compared with those given by the phenomenological potential. The predicted resonance energies as well as the energy eigenvalues obtained in the bound-state approximation show the rotational feature similar to the energy surfaces given by the generator coordinate method. The behavior of the relative wave functions shows that the $^{16}$O-$^{16}$O interaction is characterized by the following three regions: In the innermost region (relative distance $r \leq 3.8$ fm) the inner oscillation is completely damped due to the dominance of the Pauli principle, in the outermost region ($r \geq 5$ fm) related to a few important surface partial waves the potential picture is valid and in the intermediate region ($3.8 \leq r \leq 5$ fm) the effects of nuclear forces and the Pauli principle equally play important roles.

§ 1. Introduction

The excitation function of $^{16}$O-$^{16}$O elastic scattering has shown the striking gross structures in the energy range of center-of-mass energy $E_{cm} = 20 \sim 30$ MeV, in the most pronounced manner among similar gross structures observed between $p$-shell nuclei. The optical model analysis has succeeded in reproducing the general trend of the gross structures by the use of the shallow real potential and improving the fit by the modifications such as angular-momentum-dependent imaginary part and real depth increasing with energy. These results indicate the validity of potential description for nucleus-nucleus interaction, but theoretical foundation is still open. As the energy range where the gross structures show up is considerably higher than the Coulomb barrier, it is not easy to obtain information on the inside of nucleus-nucleus interaction masked by absorption effect, while information about the grazing region is available through a few important surface partial waves. There are a lot of parts of nucleus-nucleus interaction whose properties cannot be determined phenomenologically. Therefore, it is important to theoretically investigate the properties of nucleus-nucleus interaction and to provide basis for the features empirically found.

The microscopic theory of nucleus-nucleus interaction, which takes full account

* Present address: Reactor Safety Evaluation Laboratory, Japan Atomic Energy Institute, Tokai-mura, Naka-gun, Ibarakiken.
of the effect of the Pauli principle, has been recently developed. Various attempts have been made to extend the applicability of the resonating group method (RGM) successful for scattering of s-shell nuclei into the heavier systems involving p-shell nuclei, especially of 4n-type. Although direct application of the generator coordinate method (GCM) to nucleus-nucleus scattering has been shown to be practically useful in the $\alpha$-$\alpha$ and $\alpha$-$^{16}\text{O}$ cases, there still remain mathematical problems concerning high frequency component entering into the GCM weight function. At present, therefore, the RGM seems to be the best way to describe the relative motion between nuclei. Recently systematic and practically possible ways have been developed to transform the kernels analytically in the generator coordinate space (we denote this parameter space by $R$-space) into those in the dynamical variable space (denoted as $r$-space). In this paper which deals with $^{16}\text{O}^{16}\text{O}$ interaction, a new method is presented to construct the kernels in $r$-space analytically. Thus we are free from the difficulty encountered in numerical transform of the kernels, which has been recently reported by Friedrich.

In a previous report, calculating the $^{16}\text{O}^{16}\text{O}$ relative energy under the angular momentum projection in $R$-space, we pointed out that three prominent peaks in the gross structures at $E_{\text{cm}}=20, 25$ and $30$ MeV can be regarded as corresponding to the quasi-molecular rotational levels and that the inner repulsive effect appears owing to the Pauli principle. This result corresponds to the "fast case" by Pruess and Greiner who adopted the two-center shell model. Yukawa proposed a method to describe a relative "wave function" (strictly speaking, a modified GCM weight function without redundancy) and obtained a local effective "potential" in $R$-space with diverging repulsive core. As is often emphasized in the $\alpha+\alpha$ system, the repulsive core is a model representation of the damped and energy-independent inner oscillation. Therefore, in order to assert the existence of an effective repulsive core in nucleus-nucleus interaction, it is necessary to confirm the damping of the relative wave function dominated by the Pauli principle in the region where two nuclei strongly overlap. For this purpose, we should treat the problem in $r$-space.

In this paper we first present a new method to derive analytically the RGM kernels by means of a transformation (which we call "double Fourier transform") from $R$-space to $r$-space. In performing this it is essential to have an analytic form of the GCM-kernels. One of the authors (A.T.) noticed that the analytic derivation of the GCM-kernels is possible even for the system involving many particles such as $^{16}\text{O}+^{16}\text{O}$ and found a way to derive both the GCM- and RGM-kernels analytically, where the coefficients characterizing each term of the kernels are stored in computer memory. For the $\alpha+^{16}\text{O}$ system, Horiuchi invented a new analytical method of calculating the RGM-kernels by treating the lighter particle as a cluster in the valence orbits outside the core of the heavier particle and Kamimura and Matsuse utilized the harmonic-oscillator expansion of the kernels. The method presented here has the following advantage for considerably
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large systems such as $^{16}\text{O} + ^{16}\text{O}$: It eliminates laborious manual work needed to write down all the expressions of the kernels and it enables us to get the kernels for the case where the harmonic-expansion method is hardly applicable.

Solving the RGM-equation we extract characteristic features of the relative wave functions. The calculated phase shifts (real) are compared with those obtained phenomenologically and the predicted resonance energies with the energy eigenvalues obtained in the bound-state approximation and the minima of the energy surfaces. A discussion is given of: At what extent the $^{16}\text{O} + ^{16}\text{O}$ interaction can be understood in logical extension from the $\alpha + \alpha$ and $\alpha + ^{16}\text{O}$ ones and what properties appear newly in the $^{16}\text{O} + ^{16}\text{O}$ system with an internal structure more abundant and complex than the $\alpha + \alpha$ and $\alpha + ^{16}\text{O}$.

Formulations and definitions are given in § 2, construction of kernels is explained in § 3, the results and discussion are given in § 4 and § 5 is devoted to concluding remarks.

§ 2. Formulation and definition

In this section, shown is a formulation to derive the kernels and the wave functions. We start with the Slater-determinantal wave function of 32 nucleons which represents the intrinsic states of two $^{16}\text{O}$-clusters separated by $R$ in the scheme of the GCM:

$$\Phi_R(1, \ldots 32) = \sqrt{32 \choose 16} \cdot \mathcal{A}_{12} \{ \phi_{R/2}(1, \ldots 16) \phi_{-R/2}(17, \ldots 32) \}$$

where $\phi_{\pm R/2}$ denote the normalized shell-model states of two $^{16}\text{O}$ centered about $\pm R/2$, $\phi_a$ the one-particle state in the product form. $\mathcal{A}$ denotes the total antisymmetrization and $\mathcal{A}_{12}$ is the antisymmetrizer between two clusters.

A variational wave function is given by the superposition of $\Phi_R$ with a generator coordinate $R$:

$$\Psi(1, \ldots 32) = \int \Phi_R(1, \ldots 32) f(R) dR.$$  (2)

The weight function $f(R)$ is determined by the Hill-Wheeler integral equation:

$$\int (H(R, R') - EN(R, R')) f(R') dR' = 0$$  (3)

with the normalization kernel $N(R, R') = \langle \Phi_R | \Phi_{R'} \rangle$ and the energy kernel $H(R, R') = \langle \Phi_R | H | \Phi_{R'} \rangle$. The Hamiltonian is given by

$$H = -\frac{\hbar^2}{2M} \sum_i p_i^2 - T_o + \sum_{i<j} V_{ij} + \text{Coulomb part},$$  (4)

where $T_o$ denotes the kinetic energy operator of the centre-of-mass, $V_{ij}$ the two-nucleon force and $M$ the nucleon mass. On the other hand, in the RGM the
wave function in the center-of-mass system is

$$\Phi^{(\text{RGM})}(1, \cdots, 32) = \frac{1}{2} \mathcal{A}_{12} \{ \phi^{(16\text{O})}_1 \phi^{(16\text{O})}_2 \} \chi(r)$$, (5)

where two $^{16}\text{O}$ states are assumed to be in the ground state with the internal energy $E_{^{16}\text{O}}^{(\text{g})}$ (no-polarization approximation) and the factor 1/2 is inserted for the system of two identical nuclei. The relative wave function $\chi(r)$ is the solution of the integro-differential equation in the RGM;\(^*)

$$\langle \phi^{(16\text{O})}_1 \phi^{(16\text{O})}_2 | E - H | \Phi^{(\text{RGM})} \rangle_r = \int \left(E \mathcal{K}^N(r, r') - K^H(r, r')\right) \chi(r') dr' = 0$$ (6)

with the normalization kernel

$$K^N(r, r') \equiv \delta(r - r') - K(r, r')$$ (6a)

and the energy kernel

$$K^H(r, r') \equiv (T_r + V_D(r) + 2E_{^{16}\text{O}}^{(\text{g})}) \delta(r - r') + K^C(r, r') + K^V(r, r')$$. (6b)

$T_r$ and $V_D(r)$ come from the non-exchange term of the relative kinetic energy and the relative potential part. Also the kernels are defined by

$$\left\{ \begin{array}{c} K^H(r, r') \\ K^N(r, r') \end{array} \right\} = \frac{1}{2} \left\langle \delta(r - S) \phi^{(16\text{O})}_1 \phi^{(16\text{O})}_2 \right| \left\{ \begin{array}{c} H \\ 1 \end{array} \right\} \mathcal{A}_{12} \left\langle \delta(r' - S) \phi^{(16\text{O})}_1 \phi^{(16\text{O})}_2 \right\rangle$$, (7)

where $S$ in the R.H.S. is the relative variable to be integrated.\(^7\)

The relative wave function $\chi(r)$ is related to $f(R)$ through the Gaussian transform\(^8\)

$$\chi(r) \propto \int \exp\{-4a(r-R)^2\} f(R) dR$$ (8)

because, by using the separability of the relative and center-of-mass part, we have

$$\Phi_R(1, \cdots, 32) = \sqrt{\frac{32}{16}} \left\{ \begin{array}{c} 16a \pi \end{array} \right\}^{1/2} \exp\{-16a \sum_{i=1}^{32} r_i / 32 a\}$$

$$\times \mathcal{A}_{12} \left\{ \exp\{-4a(r-R)^2\} \phi^{(16\text{O})}_1 \phi^{(16\text{O})}_2 \right\}$$ (9)

if we assume the harmonic oscillator (H.O.) $(0s)^4(0p)^{18}$ state for $\phi^{(16\text{O})}$; namely we take the one-particle state as

$$\phi_1(1) = (a/\pi)^{3/2} \exp\{-a/2 \cdot (r_1 - R/2)^2\} \times (\text{spin-isospin part}),$$

$$\ldots$$

$$\phi_{32}(32) = (a/\pi)^{3/2} (2a)^{1/2} (z_{32} + R_c/2) \exp\{-a/2 \cdot (r_{32} + R/2)^2\} \times (\text{c})$$. (10)

In order to obtain the RGM-kernels from the GCM-ones, we once need to trans-

\(^*\) The expression $<|O|>$ means the integration over all the variables involved and $<|O|>_r$ that over the variables except $r$. The terms involving $\delta(r-r')$ are called the direct term.
form the representation from $R$-space to the momentum space:

$$\begin{align*}
\{ \mathcal{H}^R(k, k') \} &= \frac{1}{(2\pi)^3} \int e^{-ik \cdot R + ik' \cdot R'} \langle \Phi_R | H \{ 1 \} | \Phi_{R'} \rangle dR dR' \\
\{ \mathcal{N}^R(k, k') \} &= \frac{1}{(2\pi)^3} \int \left\{ \frac{16a}{\pi} \right\}^{3/2} \left\{ \frac{\pi}{4a} \right\}^{3/2} \exp \left\{ -\frac{1}{16a} (k^2 + k'^2) \right\} \\
&\times \langle e^{ik \cdot \cdot \cdot} \phi^{(14O_1)} \phi^{(12O_2)} | H \{ 1 \} | \mathcal{N}^{1H} e^{ik \cdot \cdot \cdot} \phi^{(14O_1)} \phi^{(12O_2)} \rangle.
\end{align*}$$

(11)

The bracket in the R.H.S. is the Fourier transform of the RGM-kernels. We can get the RGM-kernels from $\mathcal{H}^R(k, k')$ and $\mathcal{N}^R(k, k')$ by the use of the following transform:

$$\begin{align*}
\frac{1}{2} \left\{ \frac{2a}{\pi} \right\}^{3/2} \int \exp \left[ ik \cdot r - ik' \cdot r' + \frac{1}{16a} (k^2 + k'^2) \right] \{ \mathcal{H}^R(k, k') \} dk dk' \\
= \frac{1}{2} \langle \delta (r - S) \phi^{(14O_1)} \phi^{(12O_2)} | H \{ 1 \} | \mathcal{N}^{1H} \{ \delta (r - S) \phi^{(14O_1)} \phi^{(12O_2)} \} \rangle.
\end{align*}$$

(12)

The "double Fourier transform" thus obtained leads to

$$\begin{align*}
\{ K^R(r, r') \} &= \frac{1}{2} \left\{ \frac{2a}{\pi} \right\}^{3/2} \int \exp \left[ -ik \cdot (R - r) + ik' \cdot (R' - r') + \frac{1}{16a} (k^2 + k'^2) \right] \\
&\times \{ H(R, R') \} dR dR' dk dk'.
\end{align*}$$

(13)

Using the product form of Eq. (1) as the wave function, we evaluate the RGM-kernels through the transformation of Eq. (13).

We will try to get the RGM-kernels after constructing the analytical form of the GCM-ones. For example, if the superposition of the Gaussian functions is assumed for the nucleon-nucleon interaction, both the normalization kernel and the energy one without the Coulomb part have in general the following form:

$$\begin{align*}
\{ H(R, R') \} &= \sum \frac{1}{\pi} C_i R^{\nu_1} R'^{\nu_2} (R \cdot R')^{\nu_3} \exp (-E_{\nu_1} R^2 - E_{\nu_2} R'^2 - E_{\nu_3} R \cdot R').
\end{align*}$$

(14)

Substituting Eq. (14) into Eq. (13), after straightforward calculations, we obtain the analytical form of the RGM-kernel:

$$\begin{align*}
\{ K^R(r, r') \} &= \frac{1}{2} \left\{ \frac{2a}{\pi} \right\}^{3/2} \sum \frac{1}{\pi} C_i (-1)^{\nu_1 + \nu_2 + \nu_3} \left( \frac{d}{dE_{\nu_1}} \frac{d}{dE_{\nu_2}} \frac{d}{dE_{\nu_3}} \right) \frac{1}{C_o^{3/2}} \\
&\times \exp (-E_{\nu_1} r^2 - E_{\nu_2} r'^2 - E_{\nu_3} r \cdot r') \\
= &\sum C_i r^{3\nu_1} r'^{3\nu_2} (r \cdot r')^{3\nu_3} \exp (-E_{\nu_1} r^2 - E_{\nu_2} r'^2 - E_{\nu_3} r \cdot r').
\end{align*}$$

(15)
The differentiation with respect to $E_{1t}$, $E_{2i}$ and $E_{3t}$ is carried out by the use of the recurrence formula. Then the sum is over the new sets $(C', n', l', m', E_{1t}, E_{2i}, E_{3t})$, where each term in Eq. (14) is separated into several terms with the same exponent in Eq. (15). When $C_0$ is zero, the expressions contain the $\delta$-function.

The relative wave function $\chi(r)$ contains the redundant solutions $\chi^{(i)}(r)$ forbidden by the Pauli principle. To say something definite in terms of the relative wave function, we should eliminate such redundancy. As the redundancy-free quantities we have the following three:

(a) $\tilde{\chi} = A\chi$ with $A = 1 - \sum \langle \chi^{(i)}| \chi^{(i)} \rangle$

(b) $\varphi = F\chi$ with $F = (1 - K)^{1/2}$

(c) $U = (1 - K)\chi$

where the operator $K$ denotes the overlapping kernel ($K(r, r')$ in Eq. (6a)) whose eigenfunction expansion is expressed as

$$K(r, r') = \sum_{j(\text{all})} \gamma_j \chi^{(j)}(r) \chi^{(j)}(r').$$

The forbidden states with $\gamma_i = 1$ are automatically eliminated in $\tilde{\chi}$, $\varphi$ and $U$ and these become orthogonal to $\chi^{(i)}$. As shown by Saito, Okai, Yasuno and one of the authors (R.T.), $\varphi$ obeying the usual normalization is suitable as the relative wave function. For the light clusters, these representations are essentially the same because of $|\gamma_j| (\text{non-redundant}) | < 1$. In the $^{16}$O+$^{16}$O system, however, a lot of almost forbidden states ($\gamma_i \approx 1$) are existent. In order to eliminate the effects arising from these almost forbidden states which prevent us from understanding physical situation, $U$ and $\varphi$ should be considered. We adopt $\varphi$ as the relative wave function hereafter. ($U$ is the reduced width amplitude.) From Eq. (6) symbolically rewritten as

$$\tilde{K}^H\chi = E(1 - \tilde{K})\chi,$$

we have the equation for $\varphi$:

$$F^{-1}\tilde{K}^H F^{-1}\varphi = E\varphi,$$

where

$$F = A + \sum_{j(\text{all})} (\sqrt{1 - \gamma_j - 1}) \chi^{(j)} \langle \chi^{(j)} \rangle$$

(19a)
and

$$F^{-1} = A + \sum_{\ell \geq n \geq 1} (1/\sqrt{1 - \ell - 1}) \chi^{(\ell)} \langle \chi^{(n)} \rangle.$$  (19b)

The relative wave function $\varphi$ is obtained by operating with $F$ on the solution of the RGM-equation, Eq. (6'), in this paper.

\section*{\S 3. Construction of kernels}

In this section we show a procedure for constructing the kernels under the no-polarization approximation in the elastic channel. For the case of the one-nucleon wave function given by Eq. (10), the normalization kernel in $\mathbf{R}$-space has the following form:

$$N(\mathbf{R}, \mathbf{R}) = | \det B_{ij} |,$$  (20)

where $B_{ij} = \langle \phi_{i\mathbf{R}} | \phi_{j\mathbf{R}} \rangle$, $\phi_{i\mathbf{R}}$ and $\phi_{j\mathbf{R}}$ denote the spatial parts of $\phi_a$ and others, which can be divided into eight parts (one $(0s)$ and three $(0p)$ states in each $^{16}$O) from the maximum orbital symmetry. As two-nucleon interaction we adopt a central potential

$$V(r_{12}) = u(r_{12}) (w + bP_z - hP_x - mP_y).$$  (21)

Then the energy kernel in $\mathbf{R}$-space is given by

$$H(\mathbf{R}, \mathbf{R}') = N(\mathbf{R}, \mathbf{R}') \left[ 4 \sum_{t, j} \langle \phi_{t\mathbf{R}} | t \phi_{i\mathbf{R}} \rangle B_{ji}^{-1} - 3/4 \cdot \hbar \omega \right] + \sum_{t, j, t'} \langle \phi_{t\mathbf{R}} | t' \phi_{i\mathbf{R}} \rangle \{ X_{a} B_{ji}^{-1} B_{ji}^{-1} + X_{b} B_{ji}^{-1} B_{ji}^{-1} \} + \text{Coulomb part}$$  (22)

with $X_a = 8w + 4b - 4h - 2m$ and $X_b = 8m + 4h - 4b - 2w$, where $B^{-1}$ is the inverse of the overlapping matrix $B$ given in Eq. (20) and $\hbar \omega$ is the H.O. quantum taken as $\hbar \omega = \hbar^2 a / M = 15.8$ MeV corresponding to $a = 0.38$ fm$^{-2}$. In Eq. (22) the first term in the bracket is the kinetic energy part, the second comes from the $T_a$ part and the third represents the interaction part, the sum of the direct and exchange matrix elements.

The $(8 \times 8)$ $B$ matrix is given by

$$B_{i,j} = \begin{pmatrix}
B_{11} & B_{1m} & B_{1s} & B_{1a} \\
B_{k1} & B_{km} & B_{ks} & B_{ka} \\
B_{51} & B_{5m} & B_{5s} & B_{5a} \\
B_{11} & B_{1m} & B_{1s} & B_{1a}
\end{pmatrix},$$  (23)

where $k$ and $m$ run from two to four and $l$ and $n$ from six to eight, corresponding to three components of the $(0p)$ state. Matrix elements are evaluated straightforwardly as
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$$\{ B_{11} \} = \langle (0s)_+ | (0s)_- \rangle = \exp \left\{ -a/16 \cdot (R - R')^2 \right\} = G_-,$$

$$\{ B_{13} \} = \langle (0s)_+ | (0s)_+ \rangle = \exp \left\{ -a/16 \cdot (R + R')^2 \right\} = G_+,$$

$$\begin{align*}
\{ B_{1m} \} &= \left\{ \langle (0p_+)_+ | (0p_+)_- \rangle \right\} - \left\{ \langle (0p_+)_+ | (0s)_- \rangle \right\} = - \left\{ \langle (0p_+)_+ | (0s)_+ \rangle \right\} - \left\{ \langle (0p_+)_- | (0s)_- \rangle \right\} \\
&= \left( a \right)^{1/2} \frac{1}{8} (R_+ - R_+) G_-, \\
\{ B_{1n} \} &= \left\{ \langle (0p_+)_- | (0p_+)_+ \rangle \right\} - \left\{ \langle (0p_+)_- | (0s)_+ \rangle \right\} - \left\{ \langle (0p_+)_- | (0s)_+ \rangle \right\} \\
&= \left( a \right)^{1/2} \frac{1}{8} (R_+ + R_+) G_+,
\end{align*}$$

where the suffix + (−) indicates the $^{16}$O (16O) in the plus (minus) side and $\kappa$ and $\lambda$ come from $x$, $y$ and $z$ components. The $\det B_{ij}$ is calculated as

$$\det B_{ij} = \begin{bmatrix}
1 & 0 \\
(a/2)^{1/2} R_+ \delta_{x\lambda} & \delta_{x\lambda}
\end{bmatrix} G_- \\
\begin{bmatrix}
1 & 0 \\
-(a/2)^{1/2} R_+ \delta_{x\lambda} & \delta_{x\lambda}
\end{bmatrix} G_+ = - (G_+^z - G_-^z)^2 A, \\
(25a)
$$

where

$$A = \left( G_+^z G_-^z \right)^2 + a^2/4 \cdot (R \cdot R')^2 G_+ G_-.$$

Then the inverse matrix elements are obtained as follows:

$$\begin{align*}
\{ B_{11}^{-1} = B_{13}^{-1} \} &= C_x [A - 2G_\pm^z (A_+ + A_-) + a/4 \cdot G_\pm^z (A_+ + A_-) (R^2 + R'^2) \mp 2A_\mp R \cdot R'] \\
&= -a/8 \cdot A (R \mp R')^2 - a^2/16 \cdot G_\pm^z A_\pm (R^2 \mp R'^2) (R'^2 \mp R \cdot R') / \det B_{ij}, \\
\{ B_{1m}^{-1} = B_{1n}^{-1} \} &= (a/8)^{1/2} C_x \{ 2G_\pm^z (A_+ + A_-) R_+ \mp A (R_+ \mp R_+) \} \\
&= -a/2 \cdot G_\pm^z A_\pm (R^2 \mp R \cdot R') R_+ / \det B_{ij}, \\
\{ B_{11}^{-1} = B_{13}^{-1} \} &= (a/8)^{1/2} C_x \{ 2G_\pm^z (A_+ + A_-) R'_+ \mp A (R'_+ \mp R_+) \} \\
&= -a/2 \cdot G_\pm^z A_\pm (R^2 \mp R \cdot R') R'_+ / \det B_{ij},
\end{align*}$$

where $C_x$ is the coefficient of the matrix element.
\[
\begin{align*}
\{ B_{km}^{-1} = B_{lm}^{-1} \} &= C_\pm (A \delta_{\lambda \lambda} \pm a/2 \cdot G_{\pm} \cdot A_{\pm} R_{\pm} R_{\pm}) / \det B_{ij}, \\
\{ B_{kn}^{-1} = B_{lm}^{-1} \} &= C_\pm = B_{\pm} = (1 \pm \alpha R \cdot R') G_{\pm}.
\end{align*}
\] (26)

where \( C_\pm = G_\pm (G_\pm - G_\pm) \), \( A_\pm = G_\pm - (1 \pm \alpha R \cdot R'/2) G_\pm \). Also we can derive the matrix elements \( \langle \phi_{iR} | \phi_{jR'} \rangle \) and \( \langle \phi_{iR} \phi_{jR'} | \mu | \phi_{iR} \phi_{iR'} \rangle \) by usual systematic calculations, if \( u(r_{12}) \) is a Gaussian potential given by \( u(r_{12}) = u_0 e^{-\beta r_{12}} \). (27)

Substituting the matrix elements thus obtained into Eq. (22) and taking the summation, we get the GCM-energy kernel of the type of Eq. (14). Also using

Table I. The sets \((n_i, l_i, m_i, C_i, E_{st}, E_{nt}, E_{nt})\) of the GCM-kernel for all the one-particle-pair exchange terms. Here \( C_i \) are represented by \( P_i \):

\[
C_i = \alpha^{l_i m_i n_i} P_i \quad \text{for the normalization kernel,}
\]

\[
C_i = \alpha^{l_i m_i n_i} P_i \quad \text{for the kinetic term of the energy kernel,}
\]

\[
C_i = 2 \left\{ \begin{array}{c} X_i \cr X_i \end{array} \right\} u_0 e^{\beta (a/2)} 1^{l_i m_i n_i} \sum_{\alpha \beta} \alpha^\beta P_{\alpha \beta}
\]

for the terms with \( X_i \) and \( X_i \) in the interaction part of the energy kernel,

(a) normalization kernel

\[
\begin{array}{c|ccc}
E_{st} & 1 & 2 & 3 \\
\hline
2a & 2a & \small{-3.5a} \\
\hline
\end{array}
\]

(b) kinetic term

\[
\begin{array}{c|ccc}
E_{nt} & 1 & 2 & 3 \\
\hline
2a & 2a & \small{-3.5a} \\
\hline
\end{array}
\]

| \( n_i, l_i, m_i \) | \( P_i \) | \( 16 \) | \( 1 \) |
|-----------------|-----------------|-----------------|
| 0 0 0           | -16             |                 |
| 0 0 2           | -1              |                 |

interaction term (internal energy part)

(c) \( X_i \)-dependent terms

\[
\begin{array}{c|ccc}
E_{st} & 1 & 2 & 3 \\
\hline
2a & 2a & \small{-3.5a} \\
\hline
\end{array}
\]

\[
\begin{array}{c|ccc}
\hline
n_i, l_i, m_i & k' & 2 & P_{i\alpha} \alpha^\beta \\
\hline
0 0 0           & 25.5           & -63            & -108.5  \\
0 0 1           & -2.5            & 1              & 1.5     \\
0 0 2           & -12.5           & -16            & -27.5   \\
\end{array}
\]

(d) \( X_i \)-dependent terms

\[
\begin{array}{c|ccc}
E_{nt} & 1 & 2 & 3 \\
\hline
2a & 2a & \small{-3.5a} \\
\hline
\end{array}
\]

\[
\begin{array}{c|ccc}
\hline
n_i, l_i, m_i & k' & 2 & P_{i\alpha} \alpha^\beta \\
\hline
0 0 0           & 52.5           & 105            & -108.5  \\
0 0 1           & -2.5            & 1              & 1.5     \\
0 0 2           & -12.5           & 26             & -27.5   \\
\end{array}
\]
(e) $X_x$-dependent terms

<table>
<thead>
<tr>
<th>$E_{st}$</th>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$2a$</td>
<td>$2a + a(1-\alpha)/8$</td>
<td>$-3.5a$</td>
</tr>
<tr>
<td>$n_t \ l_t \ m_t$</td>
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(f) $X_x$-dependent terms

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interaction term (relative energy part)

(g) $X_x$-dependent terms

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**Microscopic Study of ^16O^16O Interaction by RGM**

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Eqs. (20) and (25), we obtain the normalization kernel. These kernels can be rewritten with respect to the power of $G_\pm$. One half of the power of $G_+$ is related to the number of the particle-pair exchange as pointed out by Suzuki. As an example, to show the structures of the GCM-kernels, we extract all the one-particle-pair exchange terms and arrange them into the form given in Table I, where in the interaction part $\alpha$ defined by $\alpha = \alpha/(\alpha + 2\beta)$ is used. The choice of the set $(C, n, L, m, E_{1}, E_{2} \text{and } E_{3})$ and the summation of the terms belonging to this same set are carried out by the use of computer without writing down the full expressions. Also, when we make the transformation to the RGM-kernels according to the procedure given by Eq. (16), the choice of new sets $(C', n', L', m', E_{1}', E_{2}' \text{and } E_{3}')$ is carried out similarly. Thus we have obtained the analytic form of the RGM-kernels in computer memory.

The correctness of the RGM-kernels is confirmed by the following check with respect to the redundant eigenfunctions:

\begin{align}
\chi^{(i)}(r) &= \int K(r, r')\chi^{(i)}(r')dr', \\
(T_r + 2E^{(0)}_{\text{kinetic}})\chi^{(i)}(r) &= -\int K^T(r, r')\chi^{(i)}(r')dr', \\
(V_{\text{D}}(r) + 2E^{(0)}_{\text{potential}})\chi^{(i)}(r) &= -\int K^V(r, r')\chi^{(i)}(r')dr',
\end{align}

where the direct parts arising from the non-exchange term can be given by straightforward calculations:

\begin{align}
T_r &= -\hbar^2/16M \cdot V^2, \\
E^{(0)}_{\text{kinetic}} &= 69/4 \cdot \hbar \omega, \\
E^{(0)}_{\text{potential}} &= 1/8 \cdot \alpha^5 u_0 \{X_4 (15\alpha^2 + 18\alpha + 31) + X_5 (15\alpha^2 - 30\alpha + 31)\}
\end{align}

and

\begin{align}
V_{\text{D}}(r) &= 2u_0 X_4 (16\alpha/(15 + \alpha))^{3/2} \cdot \exp\{-8(1-\alpha)/(15 + \alpha) \cdot ar^2\} \\
&\quad \times \{16(73\alpha^2 + 78\alpha + 105)/(15 + \alpha)^3 + 1024(1-\alpha)^2(11\alpha + 5)/(15 + \alpha)^4 \cdot ar^2 + 16384(1-\alpha)^4/(15 + \alpha)^4 \cdot ar^4\}.
\end{align}

The check of each kernel appearing in Eqs. (28) is carried out in the accuracy of order $10^{-10}$. Also the bound-state calculation serves as a critical test of the total kernel.

Is it possible to construct the RGM-kernels from the numerical GCM-kernels by using Eq. (13)? The behavior of the RGM-kernels in the interactive region cannot be derived by this method because the treatment of high frequency components by means of a numerical computation is practically impossible. However, such a numerical method is expected to be useful as far as concerned with the outside region, if the RGM-wave function in the inside region can be shown to have a vanishingly small amplitude or to have a stable inner oscillation.
This is nothing but the task to be solved by microscopic treatment, the result of which has been first given by the present authors, as shown in the next section.

§ 4. Results and discussion

The calculations are carried out in the range of the center-of-mass energy $10 \sim 30$ MeV and the relative angular momenta $12 \sim 20$ because of our main interest in the gross structures. Also the bound state problem is solved only for the nuclear part and the energy-level positions with the Coulomb correction are estimated. The inter-nucleon potential used in this paper is the force of Volkov No. 2 type mostly with $m = 0.65$:

$$V(r_{12}) = u(r_{12}) \{(1-m) - mP_\pi P_r\},$$

where

$$u(r_{12}) = -60[\exp \{-(r_{12}/1.80)^3\} - \exp \{-(r_{12}/1.01)^3\}] \text{MeV}$$

with $r_{12}$ in fm-unit.

In correspondence to the expression (6), we write down the expression in the partial wave decomposition:

$$\int \{EK_{l}^H(r, r') - K_{l}^H(r, r')\} \chi_{l}(r') dr' = 0,$$

where

$$\{K_{l}^H(r, r')\} = rr' \int Y_{lm}^{*}(r, r') \{K_{l}^H(r, r')\} Y_{lm}(r, r') d\Omega_r d\Omega_{r'},$$

and $\chi_{l}(r)$ is defined as

$$\chi(r) = \sum_{l} \chi_{l}(r)/r \cdot Y_{lm}(\Omega_r).$$

The effective local potential for $\varphi$ adopted as a suitable relative wave function without redundancy can be found through

$$V_{l}^{\text{eff}}(r) = E_{\text{cm}} + \left(\hbar^2/16M\right) \left[d^2/dr^2 - l(l+1)/r^2\right] \varphi_{l}(r)/\varphi_{l}(r),$$

where $E_{\text{cm}} = E - 2E_{10}^{(0)}$ and $\varphi_{l}(r)$ is the $l$-th partial wave of $\varphi$. The validity and limitation of $V_{l}^{\text{eff}}(r)$ thus derived should be considered in close correspondence to the behavior of $\varphi_{l}$.

(1) Energy levels and phase shifts

We have shown in a previous work that the energy surfaces given by the diagonal parts of the GCM-energy kernel, $E_{l}(R)$,

$$E_{l}(R) = H_{l}(R, R) - H_{l}(\infty, \infty)$$

with

$$H_{l}(R, R) = \int H(R, R) Y_{10}(\Omega_R) d\Omega_R \left/ \int N(R, R) Y_{10}(\Omega_R) d\Omega_R\right.,$$

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Fig. 1. Energy surfaces including the Coulomb energy. The solid lines show the level position evaluated from the energy eigenvalues and the dashed lines $E_i(\text{res})$. The experimental excitation function at $90^\circ$ is illustrated in the right-hand part.

Fig. 2. Real phase shifts $\delta_1$. The dashed lines are the values calculated from the phenomenological real potential of Ref. 1). These results indicate that the RGM is expected to provide a realistic description of the real part of the optical potential if a suitable two-body interaction is adopted. Also the energy eigenvalues shifted by the Coulomb direct term, $V_D^{\text{Coul}}(r=4.2 \text{ fm}) = 22.46 \text{ MeV}$ and the resonance energies $E_i(\text{res})$ defined by $\delta_i(E_i) = 90^\circ$ show the good correspondence with the energy minima $E_i(\min)$ with the rotational series as illustrated by lines in Fig. 1. Such correspondence has been found by means of the GCM calculations in the bound state approximation.

The Coulomb energy kernel in the RGM is approximated by $\sqrt{1 - K_i(r, r')} \times V_D^{\text{Coul}}(r) \sqrt{1 - K_i(r, r')}$ since the precision has been confirmed for the $\alpha^{16}\text{O}$ system by Kamimura and Matsuse. On the other hand the numerical values of the Coulomb part in the GCM-kernel are exactly calculated with the two-body Coulomb force.

(2) Wave functions

In Fig. 3(a), the relative wave functions $\varphi_i$ are shown for some typical values of $l$. The behavior of $\varphi_i$ suggests that it is appropriate to consider the $^{16}\text{O}^{16}\text{O}$ interaction in the three regions in correspondence to the characteristic
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features of $\varphi_i$. The outermost region (region I), $r \geq 5$ fm, can be described by an ordinary potential picture. The extremely damped inner oscillation takes place due to the dominance of the Pauli principle in the innermost region (region III), $r \leq 3.8$ fm. The number of inner-oscillation nodes $n$ is in accordance with $2n + l = 24$. $\varphi_i$ show in the intermediate region (region II), $5$ fm $\geq r \geq 3.8$ fm, a characteristic feature which may be understood as considerably weak attraction. This point, however, needs further detailed investigation with changing force parameters and comparing the calculated results with the data.

In order to investigate the behavior of region III, we consider the role of the normalization kernel, in other words that of $F$ in (16b). The GCM-kernel is given by

$$N(R, R') = \sum_{k=0}^{4} (-1)^k \left( \frac{4}{k} \right) \sum_{j=0}^{k} (-1)^j \left( \frac{16 - 2k}{j} \right) \left( \frac{a}{2} R \cdot R' \right)^{2k} G_x^{2(j+k)} G_x^{2(16-k-j)}.$$

(34)

The RGM-kernel is obtained by means of the transformation of Eq. (15) without the $\delta$-function $\rho_{\nu}$.

$$K(\nu, r') = -\frac{1}{2} \sum_{k=0}^{4} (-1)^k \left( \frac{4}{k} \right) (2k)! \sum_{j=0}^{k} \exp \left[ -2a \left( D_{k,j}^{+}/D_{k,j} \right) \cdot (r + r')^2 + D_{k,j}^{+}/D_{k,j} \cdot (r - r')^2 \right]$$

$$\times \sum_{s=0}^{2k} (-1)^s \left( \frac{2k}{s} \right) \left( D_{k,j}^{+} \right)^{-s} \left( D_{k,j} \right)^{-2k} L_{s/2}^{\nu} \left[ -16a (r + r')^2 / D_{k,j} \right]$$

$$\times L_{s-1/2}^{\nu} \left[ -16a (r - r')^2 / D_{k,j} \right]$$

$$= \sum_{N} \gamma_N^2 N(\nu) \chi_N(r'),$$

(35)

where $D_{k,j}^{\pm} = 4 \pm (8 - k - j) / 2$ and $L_{s/2}^{\nu}$ is the Laguerre polynomial. The eigenvalues $\gamma_N$ of $K$ expanded by the H.O. wave functions $\chi_N(r)$ are evaluated straightforwardly as

$$\gamma_N = \begin{cases} 1, & N = 2n + l \leq 22, \\ 1 - 1/8^{N} \sum_{k=0}^{4} (-1)^k \left( \frac{4}{k} \right) \left( \frac{N!}{(N-2k)!} \right) \sum_{j=0}^{k} (-1)^j \left( \frac{16 - 2k}{j} \right) (8 - k - j)^{N-2k}, & N > 22. \end{cases}$$

(36)
where \( N \) is the principal quantum number and \( \tilde{\gamma}_N \) is independent of \( l \) as proved by Horiuchi.\(^7\) The numerical values of \( \tilde{\gamma}_N \) are shown in Fig. 4. The redundant solutions are in the range of relative quanta \( N \leq 22 \), several almost redundant solutions continue and \( \tilde{\gamma}_N \) of non-redundant solutions decrease very slowly. In the \( \alpha + \alpha \) system \( \tilde{\gamma}_N \) of non-redundant solutions are so small that there is no important difference among \( \varphi_{l} \), \( U_l \) and \( \tilde{\varphi}_{l} \). Also the existence of a lot of non-redundant solutions with \( \tilde{\gamma}_N \) continuously distributed around 1/2 is a notable situation in the \( {^{16}}O + {^{16}}O \) system distinguished from the \( \alpha + \alpha \) and \( \alpha + {^{16}}O \) ones. In order to illustrate the effects of such properties for \( \varphi_{l} \) and \( U_l \), a Gaussian function is assumed for simplicity as the relative wave function with redundancy:

\[
\chi_{l=0}(r) \propto e^{-b(r^2+r_0^2)} J_{l=0}(2br^2)
\]

with \( r_0 = 4.0 \) fm, where \( J_l(\rho) = (-i)^l \widetilde{J}_l(i\rho) \), the modified spherical Bessel function and \( b \) is an arbitrary constant to make the peak of \( \chi_{l=0} \) at \( r = 4 \) fm. Figure 4 shows the redundancy-free wave functions \( \varphi_{l=0} \), \( U_{l=0} \) and \( \tilde{\varphi}_{l=0} \) derived from \( \chi_{l=0} \) of Eq. (37). \( \tilde{\varphi}_{l=0} \) has twelve nodes \( ((24-l)/2) \), in the inside region, corre-
sponding to the number of redundant solutions and these amplitudes are still quite large. However, $\phi_i$ and $U_i$ have the following different features from $\bar{\chi}_i$: 
(i) The inner oscillation is completely damped reflecting the important role of the Pauli principle through the almost forbidden states. This property physically indicates the existence of a hard-core-like repulsion in the innermost region. (ii) Many non-redundant solutions shift the outer wave to larger separation by $\sim 0.5$ fm. These characteristic features cannot be reproduced simply by increasing the number of the orthogonality conditions $\langle \chi_N | \varphi \rangle = 0$ with $\gamma_N \approx 1$, $N \geq 24$ in addition to $\langle \chi_N | \varphi \rangle = 0$ for $N \leq 22$. As reported earlier, these features are also confirmed by the wave functions $\varphi_i$ obtained through the Gaussian transformation (8) from the GCM-weight function in the bound-state approximation.

In region II, the direct potential $V_D$ becomes deeply attractive and the Pauli principle plays an important role through the exchange kernels. The wave function $\varphi_i(r)$ in this region has no node at low energies and shows the oscillatory behavior as the energy goes higher. The energy eigenvalues which are calculated only for the nuclear part and shifted by the Coulomb direct term are shown in Fig. 5. There exists no bound state below the threshold energy into two $^{16}$O-nuclei for $m > 0.65$. These facts indicate that the exchange kernels play the role to cancel the effect of the deeply attractive $V_D$ and make the total interaction in this region fairly weak.

(3) Effective potential

Effective local potentials $V^\text{eff}_1(r)$ calculated from $\varphi_i(r)$ through Eq. (32) are shown in Fig. 3(b). The $V^\text{eff}_1(r)$ in the outer regions are much weaker than $V_D$. The energy and angular momentum dependence of $V^\text{eff}_1(r)$ is not so strong. Therefore, the potential picture is valid in region I and the interaction may be substituted by a rather simple potential.

The modified GCM-weight function without redundancy used by Yukawa is defined by

$$g(R) \propto \int N(R, R') f(R') dR'. \tag{38}$$

The effective local "potential" for $g(R)$ is treated as

$$V^\text{GCM}_1(R) = \sum_{i(\ell \pi \Omega), j(\ell \pi \Omega), k, \text{trall}} \langle \phi_i R \phi_j R | V | \bar{A} \{ \phi_k R \phi_l R \rangle, \tag{39}$$

if $H^\text{rel}_0(R, R')/N(R, R') \sim V(R) \delta(R - R')$ is assumed where $H^\text{rel}_0(R, R')$ is a part of $H(R, R')$, the term of relative potential kernel. The angular momentum projection of $V^\text{GCM}_1(R)$, $V^\text{GCM}_1(R)$ is shown in Fig. 3(b). The $V^\text{GCM}_1(R)$ has a diverging core in region III where the inner oscillation is strongly damped and shows the features similar to $V^\text{eff}_1(r)$ in the outer regions.
§ 5. Concluding remarks

We point out that the $^{16}\text{O}^{36}\text{O}$ interaction is composed of three characteristic regions:

(1) grazing region (region I), \( r \geq 5 \text{ fm} \),

(2) surface region (region II), \( 5 \geq r \geq 3.8 \text{ fm} \),

(3) core region (region III), \( r \leq 3.8 \text{ fm} \).

The features of region I suggest that on the basis of the reasonable energy-dependent behavior of the theoretical \( \delta \), the $^{16}\text{O}^{36}\text{O}$ interaction is described by the attractive real potential, although resonance energies are rather sensitive to two-nucleon force parameters. The strongly damped inner oscillation in region III means the role of the Pauli principle preventing two nuclei from complete melting (the existence of a structural core in the fast process), which includes the effect not only of the forbidden states but also of the almost forbidden states in the $^{16}\text{O}^{36}\text{O}$ system.

The interaction in region II seems to be reasonably described by the direct potential plus exchange kernels of the RGM, whose net effect is weakly attractive. For determination of the interaction in this region, it is necessary to investigate the dependence on two-nucleon forces in fuller details and to compare the calculated cross sections with the experimental data by making use of the RGM-interaction obtained in this paper, introducing a phenomenological imaginary potential.

Although the starting point is the same GCM as other microscopic methods, our method to construct the RGM-kernels has an advantage in the treatment of the heavier system than s-shell nuclei because it enables us to eliminate the ambiguity of numerical calculations in \( R \)-space and to get the analytical form of the RGM-kernels stored in computer memory. This method is expected to provide a possible way of investigating the interaction between the other \( p \)-shell nuclei and even some multi-cluster problems of light nuclei by the RGM.

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Note added in proof: Ando, Ikeda and Suzuki have recently investigated the role of the Pauli principle for the interaction between the closed shell nuclei, $\alpha$-$\alpha$, $\alpha$-$^{16}$O and $^{16}$O-$^{16}$O by taking the correspondence with the number of particle-exchange (to be published in Prog. Theor. Phys. 54 (1975), No. 1). It has been shown there that the effect of the least number of particle-exchange has the longest range and works repulsively, in accordance with the results for $\alpha$-$^{16}$O in ref. 9). In the $^{16}$O-$^{16}$O case, although the exchange effects of several particles contribute to an effective potential with the alternative sign, the interaction in the intermediate region is fairly shallower than the direct potential. This result agrees with the conclusion of the present paper.