Superfluidity of Liquid He$^3$

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Superfluid properties of the A phase of the liquid He$^3$ is discussed by assuming the superfluid state to be the equal spin pairing of the triplet $P$ pairing. The Ginzburg-Landau equations are given valid to the third order terms in the order parameters and to the second order terms in the spatial derivatives. They are solved for the uniform flow and vortex states. Most of discussion is devoted to the bulk systems. The superfluid density has a tensor character and brings about correlations of the flow direction or the vortex axis to the symmetry axis of the pair wave function. Inclusion of the paramagnon effect does not change the main results stated here.

§ 1. Introduction

The recently found anomalous phase of the liquid He$^3$ in the mili Kelvin region$^{1-5}$ is widely convinced to be the pairing state, or the so-called BCS state, which has long been expected.$^7$ In contrast to the early predictions, the observed magnetic properties in the high temperature phase $A$, such as the absence of the reduction in the static susceptibility from the normal value and the finite frequency shift in the nuclear magnetic resonance, suggest that the pairs are in the triplet spin state instead of the singlet $D$. In the $P$ pairing the most stable state is the state proposed by Balian and Werthamer (B-W).$^8$ Since in this state the magnetic susceptibility is also reduced, the $P$ pairing state seemed to be ruled out. Recently, it has been pointed out that the spin fluctuation effect in the pairing interaction (or the paramagnon effect) favours the triplet pairing to the singlet one.$^9$–$^{12}$ Moreover, taking account of the paramagnon effect, Anderson and Brinkman (A-B)$^{11}$ have proposed a state with the $P$ pairing which has the lower free energy than that of the B-W state near $T_c$ and is consistent with the many of the observed properties of the $A$ phase. From the detailed experiment on the phase diagram in the lower pressure regions of the melting curve under magnetic field,$^9$ both the higher temperature $A$ phase and the lower temperature $B$ phase must be due to the pairings with the same orbital angular momentum. If these are in the $^3P$ state, the $A$ phase is most likely the A-B state, and the $B$ phase the B-W state.$^{11,12}$ The $A \rightarrow B$ transition takes place by the change in the relative magnitude of the condensation energy (which favours B-W state) and the energy corresponding to the paramagnon effect. Although we cannot yet rule out the triplet $F$ pairing,$^{13}$ we will study
the superfluid properties assuming the triplet $P$ pairing. Assuming that the temperature is close to the transition temperature, we use the Ginzburg-Landau (G-L) equations which are the gap equations expanded up to third order with respect to the order parameters and up to second order in spatial variations.

There are in general nine (complex) order parameters for the $P$ pairing state, and they are coupled in the differential terms as well as in the nonlinear terms. For a homogeneous system the paramagnon effect favours the A-B state, which belongs to the so-called equal spin pairing (ESP) state, that is the state which, through a proper choice of the spin quantization axis, can be considered as the pairing state between the particles with the parallel spin orientations. In the ESP case, the order parameters for the pairs with spin 1 and $-1$ can be treated separately if we neglect the contributions from the paramagnon effect and the dipole interaction. We will study the properties of the A phase using the G-L equations for 1 or $-1$ spin pairs (they have the same form). It is sufficient to take the paramagnon effect into account when we determine the relative form of the order parameters for the 1 and $-1$ spin pairs. The dipole interaction, which plays an important role in the magnetic resonance, is finally included to decide the relative orientation between the spin and the orbital quantization axes.

In § 2 we give the G-L equations and the relating formulae. The equations are solved for the uniform flow state in § 3. The tensor character of the superfluid density is discussed. The state with one vortex line is studied in § 4, the main result of which have been presented in a preliminary report by the present author and Tsuneto. In the last section we briefly remark on the paramagnon effect. The studies along the similar line by several authors are compared with the present work also in that section.

§ 2. General formalisms

Our starting point is the Ginzburg-Landau equations. Since there are plural number of order parameters for the anisotropic pairing, the equations are a set of coupled equations. We follow the same procedure as we used to discuss the superfluid properties of the $^3P_2$ pairing. We adopt the following representations for the order parameters:

$$\mathbf{A}_\alpha(R, \hat{n}) = \sum \mathbf{F}_\mu(R) S^{\alpha}_{\mu}(R) Y^\mu_{\nu}(\hat{n}) = \sum \mathbf{\Phi}_\mu(R) S^{\alpha}_{\mu}(\sqrt{3/4\pi} n)$$

where $\mu$ represents the $z$ component of the pair spin, $m$ runs 1, 0 and $-1$, $i$ represents the Cartesian component and the $S^{\alpha}$'s are the spin matrix for the pairs,

$$S^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We will use the same reduced unit for the order parameters and the coordinate
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defined in Eq. (47) of Ref. 17),
\[ \Psi_{\mu m} = \Delta \phi_{\mu m}, \quad \Phi_{\mu} = \Delta \phi_{\mu}, \quad R = \xi r, \eqno(3) \]
where \( \Delta^2 = 160 \pi (\pi T)^4 \ln \left( T_c/T \right) /21 \zeta (3) \) and \( \xi^2 = 7 \zeta (3) \rho_f /240 (\pi T m)^3 \ln \left( T_c/T \right) \).
If we assume that the pairs are in the ESP state, the order parameters with \( \mu = 0 \) are identically zero, and without taking account of the paramagnon effect (see § 5) the order parameters with \( \mu = 1 \) and \( -1 \) are independent of each other.\(^{18}\) For this reason we drop the index \( \mu \) until necessary. By the use of the Gor'kov method\(^{13,16}\) we arrive at the following G-L equations in the weak coupling limit:
\[ \sum_{m} (\bar{I} - \bar{D} - \bar{N})_{mm} \phi_m = 0, \eqno(4) \]
where \( \bar{I} \) is the unit matrix,
\[ \bar{D} = -3 \begin{pmatrix} \partial_x^2 + 2 \partial_x \partial_y, & -\sqrt{2} \partial_x \partial_y, & -\partial_y^2 \\ -\sqrt{2} \partial_x \partial_y, & 3 \partial_x^2 + \partial_x \partial_y, & \sqrt{2} \partial_x \partial_y \\ -\partial_y^2, & \sqrt{2} \partial_x \partial_y, & \partial_x^2 + 2 \partial_x \partial_y \end{pmatrix} \]
and
\[ \bar{N} = \begin{pmatrix} 2 \lambda + 2 \phi_+ \phi_-, & \phi_+^* \phi_+ & 0 \\ -\phi_+^* \phi_-, & 2 \lambda + \phi_+^* \phi_0, & -\phi_0^* \phi_+ \\ 0, & -\phi_0^* \phi_0, & 2 \lambda + 2 \phi_0^* \phi_+ \end{pmatrix}, \eqno(5) \]
with \( \lambda = \sum_m \phi_m^* \phi_m \) and \( \partial_x = \partial_x \pm i \partial_y \). If we allow the order parameters with \( \mu = 0 \) to be nonvanishing, the order parameters with \( \mu \) and \( \mu \pm 1 \) are coupled due to the nonlinear terms. Even in this case, since the differential terms do not generally cause the couplings between order parameters with different \( \mu \), the form of the differential matrix (5) is the same for \( \mu = \pm 1 \) and 0. Since Eqs. (4) are at the same time the minimization condition of the free energy \( F = \int f(R) d^3 R \) with respect to the \( \phi_m \)'s, we can readily write down the form of the integrand \( f \) as follows:
\[ f(\xi r) = \frac{m \rho^f \ln (T_c/T)}{16 \pi^4} \Delta^2 \{ -\lambda + \sum_{m} \phi_m^* D_{mm} \phi_m \\ + \left( \lambda^2 + \frac{1}{2} |\phi_0^* \phi_0 - 2 \phi_+ \phi_-|^2 \right) \}. \eqno(7) \]
The kinetic energy term in the above can be rewritten in an explicitly scalar form\(^{18}\)
\[ \frac{3 m \rho^f \ln (T_c/T)}{16 \pi^4} \Delta^2 [3 (\text{div} \phi) \cdot (\text{div} \phi) + (\text{rot} \phi) \cdot (\text{rot} \phi)] . \eqno(8) \]
We can determine the coefficient of each term by the straightforward calculations from Eq. (7). They can also be determined through the comparison with Eq. (37) of Ref. 17) if we recall that \( \phi_{ii} \) here corresponds to \( \phi_2 \) of Ref. 17).

The current density is obtained from the thermal average of the current.
operator \( - (i/2) \sum_a [\psi_a \nabla \psi_a - (\nabla \psi_a) \psi_a] \). The same result is obtained if we introduce a fictitious vector potential by replacing \( \partial_t \) with \( \partial_t - iA_t \) in Eq. (5). In order to obtain the correct form for the current density, we must symmetrize the product \( (\partial_t - iA_t) (\partial_t - iA_j) \) with respect to \( i \) and \( j \). The current density is obtained by the following functional derivative:

\[
j^t(\xi r) = -(2m\xi) \frac{\partial F}{\partial A_t(\xi)} \bigg|_{\xi=0}.
\]

Taking care of the above symmetrization, we find from Eq. (8) the following expression:

\[
j^t = \frac{1}{2m\xi} \text{Re} \left[ \sum_{i,r} \phi_i^* \mathcal{D}^{(i,r)}_{ij} (-i\partial_\nu) \phi_j \right],
\] (9)

where

\[
\mathcal{D}^{(i,r)}_{ij} = \frac{2m\rho_j^r}{\xi} \ln \left( \frac{T_c}{T} \right) ( \delta_{ir} \delta_{jj'} + \delta_{ij} \delta_{jr'} + \delta_{ij'} \delta_{jr} ).
\] (10)

Substituting \( \phi_j(\xi) = \phi_j \exp(2im\xi \nu \cdot r) \) into Eq. (9), we can obtain the superfluid density \( \rho_s \) in the relation

\[
\rho_s = \sum_{i,r} \phi_i^* \mathcal{D}^{(i,r)}_{ij} (-i\partial_\nu) \phi_j.
\]

Before proceeding our discussion, we must notice that the form of the order parameters depends on the orientation of the quantization axes. Let the coordinate system \( Ox'yz' \) is obtained from the system \( Oxyz \) by the rotation with the Euler angles \( (\alpha, \beta, \gamma) \) as shown in Fig. 1. Using the rotation matrix

\[
(\langle 1m'|R|1m \rangle^*)
\]

\[
\begin{pmatrix}
\frac{1}{2} e^{i\alpha} (1 + \cos \beta) e^{-i\epsilon r}, & -\frac{1}{\sqrt{2}} e^{i\alpha} \sin \beta, & \frac{1}{2} e^{i\alpha} (1 - \cos \beta) e^{-i\epsilon r} \\
\frac{1}{\sqrt{2}} \sin \beta e^{i\epsilon r}, & \cos \beta, & -\frac{1}{\sqrt{2}} \sin \beta e^{-i\epsilon r} \\
\frac{1}{2} e^{-i\alpha} (1 - \cos \beta) e^{i\epsilon r}, & \frac{1}{\sqrt{2}} e^{-i\alpha} \sin \beta, & \frac{1}{2} e^{-i\alpha} (1 + \cos \beta) e^{i\epsilon r}
\end{pmatrix},
\] (11)

the representation of the orbital wave function of the pairs \( \phi_m^s \)'s with respect to \( Oxyz \) are related to the representation \( \phi_m^{s'} \)'s with respect to \( Ox'y'z' \) by \( \phi_m^s = \sum \phi_m^{s'} \langle 1m'|R|1m \rangle^* \). By the rotation of the spin quantization system the order parameters are transformed by the same matrix as Eq. (11) with \( \mu \) instead of \( m \).

Without spatial variations, the G-L equations are easily solved. The solutions are classified into the following two cases:

a) \( 2\lambda = 1, \ (\phi_0^s - 2\phi_1^s) = 0 \).

b) \( 3\lambda = 1, \ |\phi_0^s - 2\phi_1^s| = \lambda \).

Using Eq. (11), it is easily seen that, by proper rotations of the orbital quantization axes relative to the spin quantization axes, these states can be represented...
by one parameter as the only nonvanishing component, such as \( \phi_1 \) (or \( \phi_{-1} \)) for the class a) and \( \phi_0 \) for the class b). The \( z \) axis of the orbital quantization system chosen in this way is the symmetry axis of the pair wave function, and is denoted by \( \hat{l}_\mu \) in the following. Since the class a) is the so-called A-M state\(^{19} \) and belongs to the lower free energy state than the class b), we shall call the conditions for the class a) the A-M condition. Within the A-M condition, \( \hat{l}_1 \) and \( \hat{l}_{-1} \) can be directed independently. The paramagnon effect then requires the directions of \( \hat{l}_1 \) and \( \hat{l}_{-1} \) to coincide with each other. This is the A-B state. In this case we can select the orbital quantization system so as the order parameters for 1 and \(-1\) spin pairs to be related by \( \psi_{11} = \psi_{-11} = (1/\sqrt{2})e^{ix} \).

The relative phase \( 2\chi \) can be made zero by a rotation of the spin quantization system around its \( z \) axis by an angle \(-\chi\). There remains an arbitrariness in the relative orientations of the spin and the orbital quantization system. The dipole energy depends on these orientations as

\[
E_d = \frac{\gamma_0^2}{160g^2} \beta (2 - 6 \sin^2 \beta \cdot \sin^2 \gamma),
\]

(12)

where \( g \) is the coupling constant of the attractive \( l=1 \) component, \( \gamma_0 \) is the gyromagnetic ratio and \((\alpha, \beta, \gamma)\) are the Euler angles necessary to bring the orbital quantization system coincide with the spin system (Fig. 1). The dipole energy is minimized if \( \hat{l} \) is along the \( y \) axis of the spin system.

If the \( z \) axis of the initial coordinate system coincide with the vector \( \hat{l} \) and the state is represented by one order parameter \( \phi_1 = 1/\sqrt{2} \), the representation in the rotated system is given from Eq. (11) as

\[
\{\phi'_1, \phi'_0, \phi'_{-1}\} = \phi_1 e^{i\alpha} \left\{ \frac{1}{2} (1 + \cos \beta) e^{ix}, \frac{1}{\sqrt{2}} \sin \beta, \frac{1}{2} (1 - \cos \beta) e^{-ix} \right\}. \quad (13)
\]

For the A-B state, the dipole energy is minimized if the order parameters for \( \mu = \pm 1 \) have the following form with respect to the spin quantization system: 

\[
\{i\phi^1, \phi^0, \phi^{-1}\} = \{1/2, \pm i/\sqrt{2}, -1/2\}.
\]

For the sake of the following discussion, we give another expression for the current density in terms of the order parameters \( \psi_m \)'s:
\[
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\]

\[j^t = \frac{f^t}{6\pi^2} \ln \left( \frac{T_c}{T} \right) \Re \left[ \sum_{m\nu} \psi_m^* \hat{J}_{m\nu} \psi_m \right], \tag{14}
\]

where

\[
\hat{J}_{m\nu} = -\frac{1}{2} \left( \begin{array}{ccc}
4i\partial_x & -\sqrt{2}i\partial_z & -2i\partial_-

-\sqrt{2}i\partial_z & 2i\partial_x & \sqrt{2}i\partial_z

-2i\partial_+ & \sqrt{2}i\partial_z & 4i\partial_z
\end{array} \right),
\]

\[
\hat{J}^v = -\frac{1}{2} \left( \begin{array}{ccc}
4i\partial_y & -\sqrt{2}i\partial_z & -2i\partial_

\sqrt{2}i\partial_z & 2i\partial_y & \sqrt{2}i\partial_z

2\partial_+ & -\sqrt{2}i\partial_z & 4i\partial_y
\end{array} \right)
\]

and

\[
\hat{J}^s = -\frac{1}{2} \left( \begin{array}{ccc}
2i\partial_x & -\sqrt{2}i\partial_+ & 0

-\sqrt{2}i\partial_+ & 6i\partial_z & \sqrt{2}i\partial_-

0 & \sqrt{2}i\partial_+ & 2i\partial_z
\end{array} \right).
\]

§ 3. Uniform flow state

As the simplest case with spatial variations, we find solutions of the G-L equations (4) corresponding to states with a uniform flow. The uniform flow state is represented by the order parameters \(\psi_m(r) = e^{i\kappa \cdot r} \phi_m\), where the \(\phi_m\)'s in the right-hand side are constant in space. Equations (4) then reduce to the following equations for the constant \(\phi_m\)'s:

\[
(2\lambda - 1 + 2|\phi_{+1}|^2 + 3k^2)\phi_{+1} - \phi_{-1} \phi_{+}^* = 0,
\]

\[
(2\lambda - 1 + |\phi_0|^2 + 9k^2)\phi_0 - 2\phi_0^* \phi_0 \phi_{-1} = 0,
\]

where the \(z\)-axis is selected along the total momentum of the pair \(k\). If we express the order parameters by \(\phi_m = \rho_m \exp(i\chi_m)\), where the \(\rho_m\)'s and the \(\chi_m\)'s are real, the relation \((\chi_{-1} - \chi_0) + (\chi_{-1} - \chi_0) = 0\) follows directly from Eqs. (15).

The remaining equations for the \(\rho_m\)'s are solved easily. The solutions are classified into the following four cases:

a-1) \(2\lambda = 1 - 3k^2, \rho_0 = 0, \rho_{-1} = 0, f = -(1 - 6k^2)f_0\) and \(j^t = j_b/2\).

a-2) \(2\lambda = 1 - 6k^2, \rho_0 = 1/8, \rho_{-1} = (1 - 12k^2)/4, f = -(1 - 12k^2)f_0\) and \(j^t = j_b\).

b-1) \(3\lambda = 1 - 9k^2, \rho_0 = 0, f = -(2/3) (1 - 18k^2)f_0\) and \(j^t = j_b\).

b-2) \(3\lambda = 1 - 3k^2 = 6|\rho_0\rho_{-1}|, \rho_0 = 0, f = -(2/3) (1 - 6k^2)f_0\) and \(j^t = j_b/3\),

where \(f_0 = m_f \ln \left( \frac{T_c}{T} \right) / 64\pi^2 k^2\) and \(j_b = p_f \ln \left( \frac{T_c}{T} \right) / 6\pi^2 k\). The free energy density and the current density are calculated by substituting the corresponding form of the order parameters into Eqs. (7), (13) and retaining the second order terms in \(k\). The superfluid density along the flow is given by \(2m^2 j_b/k\). Taking account of the following remarks, we obtain the same superfluid density as...
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In the limit of \( k \to 0 \), the classes a–1) and a–2) approach the A-M state a) in the preceding section and the class b–1) and b–2) approach the higher free energy state b). The direction of the symmetry axis \( \hat{l} \) is parallel to the pair momentum in the class a–1) and b–1), and perpendicular to \( k \) in the classes a–2) and b–2). In all the above solutions the direction of the flow coincides with the direction of \( \mathbf{k} \). We may conclude that, if there is uniform flow, the stationary state of the pairs has the symmetry axis \( \hat{l} \) parallel to or perpendicular to the flow direction.

Now we would have a question whether or not states with \( \hat{l} \) neither parallel nor perpendicular to the momentum \( \mathbf{k} \) is possible. Suppose the direction of \( \hat{l} \) and the flow direction are controlled by external conditions differently. For example, if the system is exposed to a magnetic field, spin is quantized along the field and \( \hat{l} \) is forced to lie on the plane perpendicular to it to minimize the dipole energy. The flow may be directed along the capillary, which can be set at our will. We may naturally think that the direction of \( \hat{l} \) is determined by the minimization condition of the total free energy and can be in an intermediate direction between parallel and perpendicular to \( \mathbf{k} \). Before considering this problem we must make the following remarks:

From the expression (14) for the current density, or from the tensor character of the superfluid density, we might expect nonvanishing components of the current density perpendicular to \( \mathbf{k} \) (i.e., nonvanishing \( j^x \) and \( j^y \) as we have selected the \( z \) axis along \( \mathbf{k} \)) for a general form of the order parameters. According to the fact that \( \partial_x \) appear linearly in Eq. (5), however, the states with nonzero \( j^x \) and/or \( j^y \) are shown to be unstable against appearances of small \( x \) and \( y \) components in the momentum \( \mathbf{k} \). In fact the change of the free energy density caused by the appearance of small \( k_x \) and \( k_y \) is given by the following, in the lowest order with respect to \( k_x \) and \( k_y \):

\[
\delta f = \frac{1}{2m^2} (k_x j^x + k_y j^y).
\]

(16)

Since this is linear in \( k_x \) and \( k_y \), only the states with vanishing \( j^x \) and \( j^y \) correspond to the stable ones. More precisely speaking, the rotation of \( \mathbf{k} \) from \((0, 0, k)\) to \((k_x, k_y, \sqrt{k^2 - k_x^2 - k_y^2})\) causes the variation of the free energy density (16) in the lowest order with respect to \( k_x \) and \( k_y \). If there are nonvanishing \( j^x \) and \( j^y \), the free energy is lowered by the rotation of \( \mathbf{k} \), or, for a fixed \( k_x \) by the rotation of the state inversely. The coincidence of the direction of current with \( \mathbf{k} \) in the stable states is physically reasonable. Among the states which approach the A-M state in the limit of \( k \to 0 \) (we call them the states in the A-M class in the following) only the previous solutions a–1) and a–2) correspond to the state with vanishing transverse current. The former corresponds to the minimum and the latter to the maximum of the free energy against the
fluctuations $k_x$ and $k_y$.

Thus, even if we start with the representation for the state with a uniform flow along the $z$ axis by the direction of $\hat{l}$ neither parallel nor perpendicular to $k$, we cannot stabilize this state without taking account of the variations in the $x$-$y$ plane, or the appearance of $k_x$ and $k_y$, which contradict the starting assumption. Consequently, if the direction of the flow is neither parallel nor perpendicular to the external magnetic field, the following two cases are possible: 1) $\hat{l} \parallel k$ at the expense of the dipole energy. 2) $\hat{l}$ is not parallel to the flow, and the flow pattern may be changed. Because of the smallness of the dipole energy, the case 1) is more probable than the case 2). These features have observable effects on the frequency shift in the n.m.r. just as we have reported previously in connection with the vortex state.

Although in the presence of spatial variations the A-M condition can be satisfied only approximately, with the deviations in second order with respect to the spatial variations, we can always find the states belonging to the A-M "class". We may picture the state in the A-M class by the spatially varying amplitude $\phi_1$ and $(\alpha, \beta, \gamma)$ in the representation of Eq. (13). In closing this section we derive the condition of the vanishing transverse current with respect to $k$ for this representation. Allowing the variations of $(\alpha, \beta, \gamma)$ along the $z$ axis, the transverse components of the current density is given from Eqs. (14) and (18) as follows:

$$j^x = \frac{\rho_s}{4m^2} \left[ (2 \cos \beta \cdot \partial_z \alpha + \partial_z \gamma) \sin \beta \cdot \cos \gamma - \cos \beta \cdot \sin \gamma \cdot \partial_z \beta \right],$$

$$j^y = \frac{\rho_s}{4m^2} \left[ -\cos \beta \cdot \cos \gamma \cdot \partial_z \beta - (2 \cos \beta \cdot \partial_z \alpha + \partial_z \gamma) \sin \beta \cdot \sin \gamma \right],$$

(17)

with $\rho_s = 2m \rho_s^2 \ln (T_c/T) / 12 \pi^3$. Equating $j^x$ and $j^y$ to zero, we obtain the conditions $\cos \beta \cdot \partial_z \beta = 0$ and $(2 \cos \beta \cdot \partial_z \alpha + \partial_z \gamma) \sin \beta = 0$. The case a-1) corresponds to $\sin \beta = 0$, $\partial_z \beta = 0$ and $\alpha + \cos \beta \cdot \gamma = k_z$. The case a-2) corresponds to $\cos \beta = 0$, $\alpha = k_z \pm \cos \beta \cdot \gamma = k_z$. Moreover, states with $\beta = \text{const}$, $\alpha = k_0 z$ and $\gamma = -2 \cos \beta \cdot k_z$ have the vanishing transverse current. The last states should be solutions of the G-L equations with the form $\phi_m(z) = e^{i k_m z} \psi_m$, where $k_1 - k_0 = k_0 - k_{-1}$. After the substitution of this form into Eq. (4) the resultant equations for the constant parameters $\psi_m$'s can be solved. For solutions belonging to the A-M class directions of $\hat{l}$ are determined as a function of $k_0$ and $k_1$, $\cos \beta = 2k_0 k_1 / (2k_0^2 + k_1^2)$, which is different from that obtained from the condition of vanishing transverse current. It seems peculiar that the solutions of the G-L equations are not always the stable ones. This paradox corresponds to the limitation on the form of the order parameters within some kinds. By assuming the special form for the order parameters, fluctuations against which the free energy is to be minimized are confined within a special class.
4. Vortex state

In this section we consider the ESP state with one vortex line. The vortex state is an axially symmetric state described by the order parameters of the form \( \psi_m(r, z, \theta) = \exp(iL_m\theta)\phi_m(r, z) \), where the symmetry axis is taken as the axis of the cylindrical coordinates. The relative magnitude of the integers \( L_m \)'s is determined by the condition that the free energy density equation. (7) is independent of \( \theta \), and is given by the following:

\[
L_m + m = J. \quad (J: \text{integer}) \tag{18}
\]

This equation is derived by us previously for the \( ^1\!P_2 \) pairing. Then the G-L equations reduce to the equations for the functions \( \phi_m(r, z) \)'s, which have the same form as Eq. (4) if we use the differential matrix of the following form:

\[
D' = \begin{pmatrix}
\partial_r^2 + 2\{\partial_r + \frac{\partial_z}{r} - \frac{(J-1)^2}{r^2}\}, & -\sqrt{2}\partial_r\left(\partial_r + \frac{J}{r}\right), & -\left\{\partial_r + \frac{2J+1}{r}\right\} - \frac{J^2-1}{r^2}\left\{\partial_r + \frac{J}{r}\right\} \\
-\sqrt{2}\partial_z\left(\partial_r - \frac{J-1}{r}\right), & 3\partial_z^2 + \frac{\partial_r^2 + \partial_z^2 - \frac{J^2}{r^2}}{r^2}, & \sqrt{2}\partial_z\left(\partial_r + \frac{J+1}{r}\right) \\
-\left\{\partial_r - \frac{2J-1}{r}\right\} - \frac{J^2-1}{r^2}\left\{\partial_r - \frac{J}{r}\right\}, & \sqrt{2}\partial_z\left(\partial_r - \frac{J}{r}\right), & \partial_z^2 + 2\left\{\partial_r^2 + \frac{\partial_z^2}{r} - \frac{(J+1)^2}{r^2}\right\}
\end{pmatrix}
\tag{19}
\]

The quantization rule equation (18) can be interpreted as follows: The statement that the system has an axial symmetry means that, for a fixed value of \( r \) and \( z \), the state of the pairs is independent of \( \theta \). This does not mean, however, that the order parameters \{\( \psi_m(r, z, \theta) \)\} and \{\( \psi_m(r, z, \theta') \)\} are identical apart from a phase factor, because they describe the states in terms of the fixed reference coordinate system. To state the equivalence of the order parameters, we should describe the state in terms of the local reference system which is obtained by rotating the fixed one by \( \theta \) as in Fig. 2. In this representation the local order parameters with different \( \theta \) coincide with each other apart from a phase factor, which is \( \exp(iJ\theta) \) from the one valuedness of the order parameters. Using Eq. (11) the representation of the pair states in the fixed reference system is then accompanied by an additional phase factor \( \exp(-im\theta) \) for each \( \phi_m \). Thus, as a consequence of the complete axial symmetry

**Fig. 2.** The fixed coordinate system \((x_0y_0z_0)\) is taken as the local reference system of \( \theta = 0 \), the \( x \) axis of which is arbitrarily chosen in the radial direction. The local reference systems for \( \theta \) and \( \theta' \) are denoted by \((x'y'z')\) and \((x'z'y')\), respectively. \( AA' \) is the axis of the axial symmetry.
the different component makes different contribution to the vortex flow.

If there are no variations along the $z$ direction, the order parameters with odd $m$ and those with even $m$ are decoupled in the differential terms. From this fact and from the form of the nonlinear couplings, the G-L equations allow such solutions that either the odd or the even $m$ components of the order parameters are identically zero. For example, in the case where $\psi_{\pm 1} = 0$ and $\psi_0 \neq 0$, the G-L equations reduce to the equation used in the discussion for the vortex state of He$^4$. In the asymptotic regions of $r \to \infty$, the solution of this equation approaches the state b) of § 2, which is not realized without appreciable loss of the condensation energy. Since it is difficult to solve even numerically the equations for the order parameters with odd $m$, which are the coupled equations for 1 and $-1$ component, and it is even more difficult for general cases, we study the main features of the vortex state by solving the G-L equations asymptotically for $r \to \infty$.

Now we consider the asymptotic G-L equations corresponding to the vortex state, assuming no spatial variations along the $z$ axis. For large $r$ we may expand the order parameters in terms of $1/r$ as $\psi_m = \sum_n c_{m,n} (1/r)^n$. Since we are interested only in the possible form of the lowest order coefficients $\{c_{m,0}\}$, we can use the following asymptotic G-L equation: Retaining terms up to $1/r^2$, we can neglect the terms with derivatives with respect to $r$ and obtain

$$\{ \delta_1 + 6 (J \pm 1)^2 / r^3 \} \psi_{\pm 1} - \delta_2 \psi_{\pm 1}^* + \{ 3 (J^2 - 1) / r^3 \} \psi_{\pm 1} = 0,$$

$$\{ \delta_1 + 3J^2 / r^3 \} \psi_0 + \delta_2 \psi_0^* = 0,$$

where $\delta_1 = 2\lambda - 1$ and $\delta_2 = \psi_0^2 - 2\psi_1 \psi_{-1}$. The relative phase of the $\psi_m$'s is determined at first, for example by the requirement that $\delta_2 \psi_0^*$ is a real quantity from the second equation of (20). Using the same notation for the phase as in § 3, they are given by

$$(x_1 - x_0) + (x_{-1} - x_0) = p\pi,$$

$$(x_1 - x_0) - (x_{-1} - x_0) = q\pi,$$

where $p$ and $q$ are integers. Without loss of generality, we can take $p = 0$. Then $\psi_{n^*} \psi_{-n} = (-)^q \rho_{n^*n}$.

The solutions which approach the A-M state in the limit $r \to \infty$ are of our interest. They are obtained if we assume $\delta_1$ and $\delta_2 = O(1/r)$, and can be classified into the following three cases:

a-1) $c_{1,0} \neq 0$, and $c_{0,0} = c_{-1,0} = 0$ (or $c_{-1,0} \neq 0$ and $c_{0,0} = c_{1,0} = 0$).

a-2) $q = $ even integer, and $c_{m,0} \neq 0$ for all $m$.

a-3) $q = $ odd integer, and $c_{m,0} \neq 0$ for all $m$.

Using the description of the pairing state in terms of $\ell$, pairs of a-1) in regions remote from the vortex core are represented by $\ell$ parallel to the vortex line. The class a-2) is possible only when $J = 0$, in which case the contributions of
the order parameters $\psi_1$ and $\psi_{-1}$ to the angular momentum of the system have the opposite sign and the same magnitude, and they cancel out. The pairs in the remote regions are represented by $\hat{l}$ directed along the radial axis. In the expression (13), they are represented by $\alpha = 0$, $\beta = \pi/2$ and $\gamma = \theta$. The class $\alpha = 3$ is possible except for $J = 1$, because if $J = 1$ the A-M condition cannot be satisfied even asymptotically. For $J = 0$, $\hat{l}$ of pairs in the remote regions from the vortex core is directed along the vortex flow, i.e., $\alpha = 0$, $\beta = \pi/2$ and $\gamma = \pi/2 + \theta$. For $J \geq 2$ direction of $\hat{l}$ is obtained by rotating above $\hat{l}$ for $J = 0$ around the radial axis. This state is represented by $\alpha = 0$, $\gamma = \pi/2 + \theta$ and $\beta = \pi/2$. The angle $\beta$ is determined by the comparison of $\rho_1/\rho_{-1} = (2J^2 + 4J + 1)/(2J^2 - 4J + 1)$ with Eq. (13).

So far as we are concerned with a state with no variations along the $z$ axis, the $z$ component of the current density must vanish identically by the stability condition, just as we noted in connection with Eq. (16). Let us examine the above solutions by this requirement. Since the A-M state is realized asymptotically, we may use the representation by Eq. (13). The axial symmetric state is described by $\alpha = \epsilon J\theta$, $\beta = \beta_0$ and $\gamma = -\theta + \gamma_0$, where $\beta_0$ and $\gamma_0$ depend on $r$. From the preceding discussion about the asymptotic forms, $(d/dr)\beta_0$ and $(d/dr)\gamma_0 = O(1/r^2)$ and can be eliminated in the present approximation. From Eq. (14) the $z$ component of the current density in this case is

$$j_z = \frac{\rho_{1\ell}}{4m_R^2} \sin \beta \left[ (\partial_x T + 2 \cos \beta \cdot \partial_x \alpha) \cos \gamma - (\partial_y T + 2 \cos \beta \cdot \partial_y \alpha) \sin \gamma \right]. \quad (22)$$

For the class $\alpha = 1$ and 2) this component of the current density vanishes identically. For the class $\alpha = 3$ this component does not vanish. Thus the states in $\alpha = 3$ are unstable against the variation of the phases along the $z$ axis. Since there correspond no angular momentum of the system to $\alpha = 2$), the case $\alpha = 1$) is the only possible vortex state in our situation. Thus, when vortex lines are created to lower the rotation energy, the symmetry axis of the pair wave function far from the vortex cores will be directed along the rotation axis.

In the class $\alpha = 1$ the creation energy and the angular momentum per unit length associated to one vortex line in a cylinder of radius $R$ are given as

$$\varepsilon_v = \frac{3m_R}{8\pi^2} \ln \left( \frac{T_c}{T} \right) \alpha^2 \ln (R/R_0) (J - 1)^2, \quad (23)$$

$$L_v = \frac{\rho_R^2}{6\pi} \ln \left( \frac{T_c}{T} \right) (R^2 - R_0^2) (J - 1), \quad (24)$$

respectively, where $R_0$ is the core radius. This form coincides with the usual formulae for the superfluid He$^4$ if we use the transverse superfluid density. The vortex quantum number $(J - 1)$ arises because, for the vortex state characterized by $J$, the order parameter $\psi_1$ has the angular dependence $\exp \left( i(J - 1) \theta \right)$ whose contribution to the angular momentum of the system is $(J - 1)$ a pair.
Up to now, there correspond two states to the lowest vortex quantum number; one is the state with \(\ell\) parallel to the \(z\) axis and \(J=2\) and the other is one with \(\ell\) antiparallel to the \(z\) axis and \(J=0\). Since \(\phi_1\) and \(\phi_{-1}\) are coupled in the G-L equations, the order parameters \(\phi_{-1}\) for the former case and \(\phi_1\) for the latter case cannot be taken identically zero. This coupling effect is important for the solutions close to the vortex core. There remains a problem whether or not it is possible to extend the above asymptotic solutions toward \(r=0\) without divergence. If possible, we expect the above two vortex states with the lowest vortex quantum number to have different energy according to different core structures.

§5. Discussion

1) Up to now we have taken the paramagnon effect into account only to select the A-B type solutions, in other words, to have \(\ell_1\) and \(\ell_{-1}\) coincide with each other, and considered each spin component of the order parameters separately. The paramagnon effect, however, contributes to the fourth order terms of the free energy expansion with respect to the order parameters. According to the formula of Brinkman and Anderson, the following term is to be added to the \(\mu\) sum of Eq. (7) for the ESP state:

\[
\frac{mp_f \ln (T_c/T)}{16\pi^3} \int \left[ -\delta \left| \sum_m \phi_{1m} \phi_{-1m} \right|^2 \right],
\]

(25)

where \(\delta\) is a parameter given in Ref. 11). This term contains the couplings between \(\phi_{1m}\)'s and \(\phi_{-1m}\)'s, such that it is minimized if \(\{\phi_{1m}\}\) and \(\{\phi_{-1m}\}\) coincide with each other apart from an arbitrary phase factor. Besides this coupling, we can include the additional effect by changing the coefficient of \(\lambda^2\) from 1 to \((1-\delta/2)\). After this modification most of the preceding arguments remain valid.

2) In all the previous work on problems involving spatial variations of the order parameters the nonlinear terms of the G-L equations were not included. Their effect was taken into account by restricting the form of the order parameters within the A-M state. Then the state is represented by Eq. (13) with \((\alpha, \beta, \gamma)\) varying in space. Since the deviation from the A-M state is in second order with respect to the spatial derivatives of the order parameters, this representation is useful for the intuitive description of the state when the spatial variations of the order parameters are small within a distance \(\sim \xi\). In the case where the spatial variations are important, however, we must take the nonlinear terms into account and treat \(\phi_m\)'s in a general form. For example, to obtain the vortex solutions we should impose no assumptions such as the A-M condition other than the axial symmetry upon the order parameters. We must also include the nonlinear terms in order to discuss fluctuations of the order.
parameters around the stable state. For this purpose the representation (13) is misleading because the deviations from the A-M state cannot be included.

3) In describing the uniform flow state in § 3, we have seen that the A-M state the direction of whose symmetry axis \( \hat{l} \) is assumed to be in some specific orientation is not generally stable against the transverse fluctuations of \( k \). To obtain stable states we have had to require the transverse current with respect to \( k \) to vanish. In general we have to take additional conditions into account when we restrict the direction of \( \hat{l} \), because in doing so we restrict the variations of the order parameters within some class against which the free energy is to be minimized. The representation (13) is not satisfactory to find stable states, because it contains no correlations between the direction of \( \hat{l} \) and that of the flow.

In § 4 we have seen that not all the solutions of the asymptotic G-L equations correspond to the stable state. This difficulty may correspond either to our asymptotic treatment or to the assumed form of the order parameters.

4) Assuming slow variations in space, we can expect that there are always states which approach the A-M state in the homogeneous limit. For the states in this class, which we have called the A-M class in § 3, the A-M condition is satisfied approximately. In this case the state of the pairs can be denoted by the Euler angles \( (\alpha, \beta, \gamma) \) at each point in space. This is just the same as the specification of the orientation of a rigid body. Corresponding to this, we must notice that the representation by the vector \( \hat{l} \) and the angle \( \alpha \) is not satisfying, because we cannot specify the state by these quantities without specifying the reference system \( Ox'y'z' \) in Fig. 1.

The current density is related to \( \text{grad} \alpha \) and \( \text{rot} \hat{l} \) in Refs. 15) and 16). The explicit form is given for the special case of the \( \hat{l}/z \) axis. Since \( \alpha \) is not a scalar and \( \text{grad} \alpha \) is not a vector, the expression for the current density must contain more complicated terms in general, for example terms relating to \( \text{grad} \hat{l} \). Such a relation is less meaningful than Eq. (14).

5) To obtain the correct current density, we have had to symmetrize the product \( (\partial_i-iA_i)(\partial_j-iA_j) \). From this fact the vector potential is suggested to enter into the G-L equations as the same symmetrized product in the place of the corresponding differential operator in Eq. (5).

6) We have been discussing the bulk properties of the system. Because of the anisotropy of the pairing state, boundary effects are expected to be important. Recently Ambegaoker et al.15) have discussed it from the microscopic point of view, and have found that toward the wall \( \psi_0 \) is to approach zero linearly, whereas \( \psi_{\pm 1} \) remain constant. For the A-M state, this means that \( \hat{l} \) is directed perpendicular to the wall. To satisfy this condition on all the boundary, \( \hat{l} \) must change in space. It is also expected that the vortex state is affected by the shape of the container. In a cylindrical vessel with a small radius, it is difficult to form a vortex line. There the state a-2) of § 4 are the only
possible state, with which no net current is accompanied.

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