ward and outward as in Fig. 5, forming a barrier about the eye. The flow situation is similar to that in the emptying tank shown in Fig. 4, with the one difference that now the converging current turns upward rather than downward through the outlet.

The strong inflow near the ground leads to a large positive momentum gradient \( \frac{\partial (p \nu)}{\partial z} \). (This is opposite to the case of the tank vortex.) A clockwise vorticity will be generated in the converging current in the Northern Hemisphere. On the (approximate) basis of constant density, equation (27) shows that the lower, slower-moving layers will suffer the greater deflection. In this case, the flow turns upward. Helmholtz's theorem requires that clockwise vorticity about the upward vertical be accumulated at the wall of the eye. Looking down onto the hurricane as one does on a map or radar screen, one sees the counterclockwise rotation in the Northern Hemisphere.

Concluding Remarks

The formulation for (relative) steady secondary flow development in a stratified fluid in a rotating frame of reference shows that a streamwise component of vorticity will be progressively generated first from a binormal energy gradient, then from a body force-density gradient interaction represented by a streamwise component of the vector \( \mathbf{g} \times \text{grad } \rho \), and finally by two terms explicitly dependent upon the angular velocity of the rotating frame of reference; see equations (16), (17), and (18). Of the terms involving the angular velocity of the frame of reference, the first does not involve a momentum gradient perpendicular to the flow direction but is only operative if the streamlines constitute a family of three-dimensional curves; it vanishes identically if the streamlines are plane curves. The second term, generally operative, involves momentum gradients in directions perpendicular to that of the streamline.

The direct effect of viscosity on the secondary flow generation process has been discounted; however its indirect effect is acknowledged insofar as it represents a prime cause for the establishment of momentum gradients in general. The direct effect of viscosity may be incorporated in the present formulation as in reference [11]. It represents a dissipation and allows an equilibrium to be established against the vorticity generating processes leading to a steady secondary flow or swirl.

It is to be emphasized that the aim of the paper was to present a general theory in a form suitable for future application and to include, in addition, two physical examples which it was felt would be of some interest. Thus the basic formulation is deliberately given in a greater degree of generality than is required for the examples considered.

This paper is dedicated to M. J. S.

References


DISCUSSION

A. G. Hansen

Professor Marris is to be commended for adding to his already excellent contributions on the generation of secondary flow. In keeping with his previous publications, one again sees how a relatively simple type of analysis affords understanding of reasonably complex phenomena. It is perhaps the chief contribution of this analysis and the earlier ones as well to afford qualitative insight into flow behavior rather than precise quantitative predictions of such behavior.

Two questions occurred while reading the paper on which I hope the author might comment. In the discussion of the slowly emptying tank, the "operative momentum gradient" is assumed to be negative. The tank-bottom boundary layer is disregarded. One suspects that the gradient in this region exceeds that of the gradient referred to by the author. On the other hand, the boundary layer region is apt to be quite small. Is there some relatively simple guide lines which might be suggested for determining the "operative momentum gradient?"

Second, the problem of secondary flow in turbomachines is a problem of long standing interest. In fact some of the earliest papers on secondary flow dealt with the generation of secondary flows in airfoil cascades. In the type of analysis presented here, the secondary flow in rotating blade rows might well be investigated. Comments of the author on this possibility and also his assessment of the importance of the rotational effect would be of interest.

It is hoped that future experimental work will further demonstrate the generality and applicability of Professor Marris' analysis. Again, he is to be complimented for his contribution.

Author's Closure

The writer thanks Professor Hansen for his comments. The question raised in the second paragraph of his discussion is incisive and deserves particular consideration.

One might perhaps suggest that in the last stages of tank-drainage when the water is shallow and the circulation takes place in the boundary layer region, the positive (boundary layer) momentum gradient could cause a slow generation of flow-wise...
vorticity in the opposite sense and that as this vorticity is swept downward through the outlet it could cause an annulment and eventual reversal of the existing vorticity. The effect has been observed by Sibulkin [3] and also possibly by Shapiro. In this respect we note that equation (26) for the vertical variation in curvature of horizontal streamlines above a horizontal boundary rotating about a vertical axis, has the full integral

$$\int_{x}^{\infty} \kappa_{y} \delta s^{2} = 2\omega \sin \lambda (\eta - v)$$  \hspace{1cm} (28)$$

implying that the change in centripetal acceleration balances the change in Coriolis acceleration.

If at a certain level \(z_{0}\) the velocity is \(v_{0}\) and the streamline curvature in \(\kappa_{y}\), then the streamline curvature will be zero [streamline rectilinear] at a lower level \(z_{1}\) where the velocity is \(v_{1} < v_{0}\) where

$$v_{1} - v_{0} = \frac{2\omega \sin \lambda}{\kappa_{y} \delta s^{2}}$$  \hspace{1cm} (29)$$

At this point the curvature and rotation effects annul one another.

However these results and particularly the schematic drawing of Fig. 4 may be misleading in terms of the full physics of the tank vortex. Except in the last stages of drainage when the water is very shallow in the tank, the discharging streamlines will not be as nearly horizontal as implied by Fig. 4. For more steeply vertical streamlines, flow-wise and therefore vertical vorticity will be generated by equation (24) from the spatial flow-wise acceleration — as the (deep) water in the tank flows down toward the outlet.

Using Bjørgums expressions for \(\nabla v\) in intrinsic coordinates (reference 1, p. 34), one finds for constant \(\omega\) and \(\nabla v = 0\),

$$t \cdot \text{curl} (\omega \times v) = \text{curl} (\omega \times v) \cdot t$$

$$= -\omega \cdot \nabla v \cdot t = -\omega \cdot \nabla v$$  \hspace{1cm} (30)$$

Using (30), the result (17) for the steady generation of flow-wise vorticity in a stratified fluid may be written in the alternative form exhibiting the role of the velocity gradient.

$$\frac{d}{ds} \left[ \frac{\Omega_{1}}{v} \right] = \frac{k}{v} \frac{d}{d\eta} \log (\rho^{v})$$

$$- t \cdot (g - 2\omega \times v) \times \nabla \rho + \frac{2\omega \cdot \nabla v}{v^{2}}$$  \hspace{1cm} (31)$$

In particular if the density is uniform

$$\frac{d}{ds} \left[ \frac{\Omega_{1}}{v} \right] = \frac{k}{v} \frac{d}{d\eta} \log (\rho^{v}) + \frac{2\omega \cdot \nabla v}{v^{2}}$$  \hspace{1cm} (32)$$

$$= \frac{2\kappa}{v} \frac{d\omega}{v} + \frac{2\omega \cdot \nabla v}{v^{2}}$$  \hspace{1cm} (33)$$

$$= \frac{2\omega}{v} \Omega_{n} + \frac{2\omega \cdot \nabla v}{v^{2}}$$  \hspace{1cm} (34)$$

It appears that the downwardly directed \(\nabla v\) of approximate magnitude \(\frac{\Omega_{n}}{\eta}\) would interact with the upwardly directed \(\omega\) to create a negative \(\frac{\partial}{\partial \eta} \left[ \frac{\Omega_{1}}{v} \right]\), i.e., a generation of counter-clockwise circulation looking down on the tank in the Northern Hemisphere.

The present analysis, centered about the steady generation of a flow-wise vorticity component (swirl), can only give part of the picture of the tank-vortex. The vorticity generation in this problem would appear to be a time dependent process whose analysis would require solution of the full vorticity equation (rather than merely the equation pertaining to the flow-wise vorticity component) and the continuity equation. It is hoped to present a more complete study of this problem in the future.

One formal result which may be of theoretical interest does emerge from the analysis.

The result embodied in equations (32) to (34) may be regarded as a generalization of a result due to Proudman. Proudman considered an incompressible homogeneous inviscid fluid in rigid rotation about the \(z\)-axis (vertical) and showed that for a small relative flow (in which the nonlinear terms in the acceleration could be neglected), the relative motion was two-dimensional in the \((x, y)\) plane with no variation of velocity in the direction of the axis of rotation \(z\). In the light of (34) we may assert that for any steady relative motion, if the vorticity is directed wholly along the binormal [i.e., perpendicular to the plane of the curvature] over a region of the flow, then there will be no velocity gradient parallel to the axis of rotation. This result is derived without approximating the acceleration expression and will be true though the relative flow be large (as long as it be steady).

Equation (31) shows how this effect is modified by the density gradient in inhomogeneous flow. The gravitational acceleration is modified by the Coriolis acceleration is modified by the Coriolis acceleration is modified by the Coriolis acceleration is modified by the Coriolis acceleration.

Equation (32) or its equivalent for stratified flows would seem to be well adapted to the problem of swirl generation in rotating blade rows. One simplification emerges in this complicated problem, the fact that the angular velocity \(\omega\) of the frame of reference is large. Secondary vorticity generation from the rotation must greatly exceed any effect from the streamline curvature. The governing equation would be of simple form

$$\frac{\partial}{\partial \eta} \left[ \frac{\Omega_{1}}{v} \right] = \frac{2\omega \cdot \nabla v}{v^{2}}$$  \hspace{1cm} (35)$$

The cause of a steady progressive generation of swirl is thus attributed to a velocity gradient [e.g. from end plates] in a direction parallel to the axis of rotation. If this gradient remains fixed independently of the swirl then there will be a progressive and continual generation of swirl as opposed to the reversal effect resulting from a rotation of the velocity gradient by the secondary flow as in the dish pan model considered in the paper. For a system with upper and lower end plates with planes perpendicular to the axis of rotation of the vane, one would expect that the positive and negative velocity gradients from the maintained boundary layers would interact with \(\omega\) to cause a pair of secondary flow loops of opposite rotations similar to those created in a stationary curved channel (by the first term on the right of equations (32)-(34)).

The writer again expresses his appreciation of Dr. Hansen's comments which have so greatly assisted in the physical interpretations of the analytical results.

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