

## AGE COMPOSITION IN THE ABSENCE OF MORTALITY AND IN OTHER ODD CIRCUMSTANCES

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*Abstract*—The age composition of populations experiencing no mortality and various kinds of fertility is derived. It is then shown that, if successive mortality schedules can be expressed as the sum of a component that varies with age and a component that varies with time, only the former has any effect on the age structure of the population. One implication is that a population in which mortality varies with time but not with age has the same composition as a population with no mortality at all.

There is some value in exploring the implications of contrary-to-fact conditions, if they represent interesting limiting cases of real variations, or if the implications are especially simple and thus help to clarify theoretical reasoning. Justifications of this sort are required for the analysis, presented below, of the age composition of closed populations in which there is no mortality, and of the age distribution that results when mortality rates are the sum of a component that varies with age and a component that varies with time.

If there were no mortality—if every person once born lived forever—the future development of a closed population would be easy to calculate. The population alive at the moment mortality was abolished would be moved, intact, to an age one year greater with the passage of each calendar year, so that the population at ages from 100 to 200 a century later would have the same size and internal age composition (with a uniform 100-year displacement in age) as the initial population zero to 100. To calculate births it would be necessary, as in any projection, to apply fertility rates to the population of

women of childbearing age, but once the projected number of births for a given year was determined, the size of the cohort in question would be fixed for all future years. In the absence of death, the age composition of successive populations could thus be readily calculated from the initial distribution of persons by age and the distribution through time of future births. If  $\omega$  is the highest age attained in the initial population,  $\omega + t$  will be the highest age  $t$  years later, under a newly started regime of no mortality.

### *A. Age composition with constant fertility and no mortality*

If the initiation of an epoch with no mortality were accompanied by the institution of a schedule of fertility by age that did not change through time, the stream of births would follow an evolution well known in formal demography: an evolution from the initial number of births through a period of irregular oscillations to an exponential sequence with a fixed exponent. The exponent of this sequence is Lotka's (1939) *intrinsic rate of increase*, the one real root of the equation

$$\int_0^\beta e^{-ra} m(a) da = 1.0,$$

where  $r$  is the ultimate annual rate of increase in the number of births,  $\beta$  is the highest age of childbearing, and  $m(a)$  is the proportion of women at age  $a$  giving birth to a female child. Because of the absence of mortality, the term usually present expressing the proportion of women surviving to age  $a$  is omitted. The long-run rate of increase of births is positive, zero, or negative according to whether the gross reproduction rate

$$\left( \int_0^\beta m(a) da \right)$$

is greater than, equal to, or less than 1.0. (In the no-mortality population, the net and gross reproduction rates are obviously the same.)

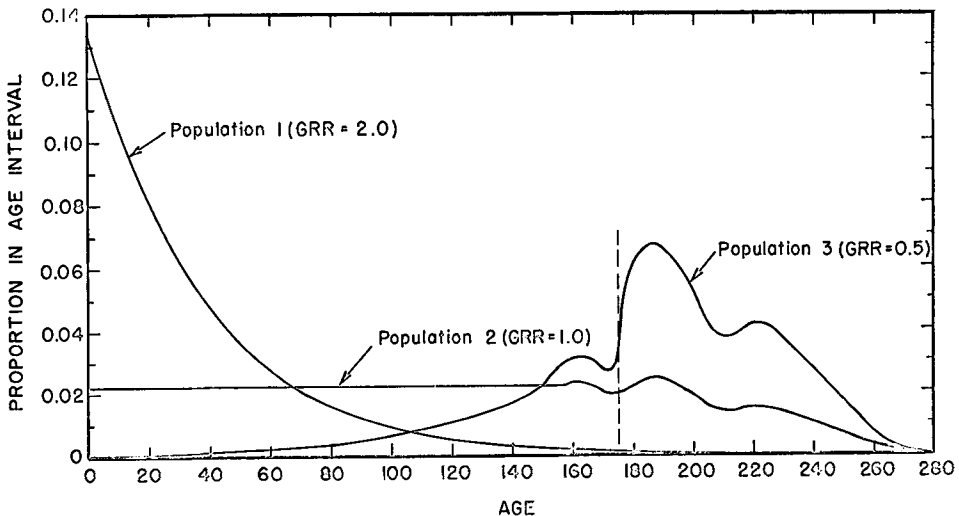
Examples of age distributions of the United States that would exist 175 years after the institution of regimes of no mortality and constant fertility are shown in Figure 1. In each instance the age distribution up to an age of about 140 years follows an exponential (positive exponential, negative exponential or a horizontal line, the zero exponential),

and from 140 to 175 an irregular sequence resulting from the births during the period when the sequence converged to the exponential. From age 175 to  $175 + \omega$  the shape of the initial age distribution is exactly reproduced, displaced by 175 years.

With the passage of time, the number of births approaches infinity if  $r$  is positive, zero if  $r$  is negative, and remains fixed if  $r$  is zero. After  $t$  years ( $t$  a large number), the age composition when  $r$  is positive becomes a negative exponential that is truncated by an irregular descent to zero (at  $a + t + \omega$ ); when  $r$  is zero becomes an ever-extending flat ribbon (like the output of a continuous strip mill) terminating in the same descent to zero at  $a + t + \omega$ ; and when  $r$  is negative becomes a moving island, aging one year annually. It is an "island" because, in the long run, with negative  $r$ , there is no one at the youngest ages, and then there is an exponentially rising section of the age distribution, an irregular section, and finally the same descent to zero as in the other cases.

The total population at a given date can be approximated by determining  $Q_0$ , the coefficient of the real exponential in

FIGURE 1.—Age Distribution of the Projected Female Population of the United States, Projected from 1960 for 175 Years with No Mortality



Lotka's expansion of a birth sequence as the sum of an exponential term and a sequence of damped oscillatory components

$$\left( B(t) = \sum_{i=0}^{\infty} Q_i e^{r_i t}, \text{ only } r_0 \text{ is real} \right).$$

The damped oscillatory components ( $r_i$  complex) are ignored on the assumption that the positive and negative parts cancel.

$$Q_0 = \left( \int_0^{\beta} B_1(t) e^{-rt} dt \right) / \bar{A},$$

where  $B_1(t)$  is the births that occur at time  $t$  to the women in the initial population and  $\bar{A}$  is the mean age of childbearing in the stable population (Lotka, 1939). Unless the initial age distribution is highly irregular,  $Q_0$  is of about the same magnitude as the number of births recorded in the year of initiation of no mortality and constant fertility. The total population at time  $t$  is the sum of the initial population plus all births that occur after the initiation of zero mortality.

If  $r$  is zero, annual births become constant, and the total number after the initiation of the regime of no mortality is  $t \cdot Q_0$ . If  $r$  is not zero, total births are

$$Q_0 \int_0^t e^{rx} dx = (Q_0/r)(e^{rt} - 1).$$

Thus,

$$P(t) = P(0) + (Q_0/r)(e^{rt} - 1) \quad (r \neq 0),$$

$$P(t) = P(0) + t \cdot Q_0 \quad (r = 0).$$

When  $r$  is positive, the total population increases indefinitely, and its rate of growth approaches the constant value  $r$ ; when  $r$  is zero, the growth in total population becomes arithmetic, with a constant annual increment and hence a rate of growth that slowly approaches zero, although there is no finite upper limit of population size; when  $r$  is negative, the size of population approaches

an upper limit  $(P(0) + B(0)/-r)$ , annual births and the rate of increase of the population approach zero.

A hypothetical condition of no mortality is thus inconsistent with the concept of a stable population, since a stable population achieves a fixed age composition and a rate of increase for the whole population (not just births) equal to  $r$ . However, when the value of  $r$  is large (two percent per year or more), the absence of mortality has a surprisingly small effect on the age composition that develops in the long run; and the conditions of a stable population are closely approximated (by a projected population) within less than a century. With a positive value of  $r$ , the principal difference between the stable population and the projected one is that a stable population has a purely exponential age distribution with no finite upper age, while the projected population duplicates the initial age distribution from age  $t$  to  $t + \omega$ , where it terminates. However, with high fertility the proportion at high ages is small even with no mortality. (The proportion above age  $x$  is  $e^{-rx}$ ; with a GRR of 4.0, less than one percent of the population would be over 100 even if no one died.) Thus the effect of terminating the age distribution of the projected population with the initial population (displaced in age) instead of with a continuing exponential is trivial if  $r$  is large. After 175 years, as is evident in Figure 1, the age distribution with no mortality and a GRR of 2.00 is not visibly different from an exponential. (The fraction over 175 is 0.899 percent in the projected population, compared to 0.892 percent in an exponential age distribution when the rate of increase of births is the same.)

It was pointed out some years ago that the substitution of zero mortality for a low-mortality schedule of death rates would have only a slight effect on the long-run growth of a population (provided the gross reproduction rate is

greater than 1.0); such a substitution would cause an increase in the order of two per thousand in the long-run rate of growth (Coale, 1959). The effect is minor because the future sequence of births is affected only by mortality prior to the last childbearing ages. When the number of births is increasing each generation, the long-run average rate of growth of the population is equal to the average rate of growth of the number of births. An illuminating comparison is between a population having no mortality prior to  $\beta$  (the highest age of childbearing) followed by total extermination at  $\beta$ , and a population experiencing no mortality at all, both populations having a GRR greater than 1.0. If the two populations have the same constant fertility, they will experience the same sequence of births. One will have no one above age  $\beta$ ; the other will have a fraction above this age that approaches  $e^{-r\beta}$  as time passes. Thus in the long run the population with no mortality will surpass that in which everyone dies at  $\beta$  by a fixed multiple,  $1.0/(1.0 - e^{-r\beta})$ ; but eventually the rate of increase of the two populations will be the same. In the short run, the no-mortality population increases more rapidly to attain its permanently greater size. It should be noted that the effect on long-run growth of substituting zero mortality for a low set of death rates is slight only if GRR is greater than 1.0. If GRR equals 1.0, the absence of mortality implies a positive constant arithmetic increase, compared to a slight negative rate of increase with a low-mortality schedule. If GRR is less than 1.0, the no-mortality population approaches a fixed maximum (with a growth rate of zero); while with a GRR below 1.0 and low-mortality rates (but a finite duration of life), the growth rate is negative, and the population ultimately vanishes.

*B. Age composition when mortality rates do not vary with age*

If variations over time in age-specific mortality rates consisted of the same increases or decreases of the rates at all ages, the age composition of the population would be exactly the same as if mortality were constant. Suppose, in more specific terms, that the mortality rate at age  $a$  and time  $t$ ,  $\mu(a, t)$ , could be expressed as the sum of an age-dependent mortality rate  $\mu_1(a)$  and a time-dependent mortality rate  $\mu_2(t)$ . For a given initial population and sequence of age-specific fertility schedules, the age distribution is independent of  $\mu_2(t)$  and in particular is the same as if  $\mu_2(t) = 0$ . The proportion surviving from age  $a$  to  $a + n$  during the period  $t$  to  $t + n$  is

$$\exp \left[ - \int_a^{a+n} \mu_1(a) da - \int_t^{t+n} \mu_2(t) dt \right]$$

= (survival according to  $\mu_1(a)$ ) · (factor that does not vary with age). The survival of every cohort alive at time  $t$  is multiplied by a factor that depends on  $\mu_2(t)$ , but not on  $a$  — every cohort is affected to the same extent. The size of a cohort born at  $t'$  (between  $t$  and  $t + n$ ) is reduced by a factor

$$\exp \left[ - \int_t^{t'} \mu_2(t) dt \right]$$

because of the additional mortality of the mothers before the birth of the children and by a factor of

$$\exp \left[ - \int_t^{t+n} \mu_2(t) dt \right]$$

because of the additional mortality of the children after birth. The combined effect on the size of the cohorts born during the interval is the same as for cohorts alive at time  $t$ . We have thus shown that the population at every age at time  $t + n$  is multiplied by

$$\exp \left[ - \int_t^{t+n} \mu_2(t) dt \right]$$

as a consequence of the sequence of changes in mortality  $\mu_2(t)$ , and hence

that the proportionate age distribution is unaffected. This result is made intuitively obvious by a less cumbersome argument: If an increase in mortality randomly kills additional persons without regard to age—one can picture a firing squad selecting a representative sample—the age composition of the survivors is the same as if mortality had not changed.

We now turn to the age composition of populations in which mortality rates are the same at every age, but variable over time—mortality rates are  $\mu(t)$  at all ages. This situation is a special case of the proposition established in the preceding paragraph. If mortality is not age-selective, the age composition of the population is the same as would prevail with the actual sequence of fertility, and *no mortality at all*. If mortality is not age-selective, age composition is wholly determined by the sequence of fertility schedules to which the population is subject.

To calculate the age composition of a population in which mortality was not age-selective, it would suffice to make the kind of population projection described in the opening paragraphs of this article—a projection in which the initial population is advanced in age without a change in age structure and in which fertility schedules alone are used to determine the stream of births.

*C. Quasi-stable populations: Age composition when fertility varies with age (but not with time) and mortality schedules are the sum of a schedule that varies with age (but not with time) and a schedule that varies with time (but not with age).*

If there is an age-specific fertility schedule that does not vary with time and a mortality schedule of the form  $\mu(a, t) = \mu_1(a) + \mu_2(t)$ , the age distribution would be that of the stable population defined by the given fertility

schedule and the mortality schedule  $\mu_1(a)$ . However, the growth rate of the population at any moment  $t$  would be  $r - \mu_2(t)$ , where  $r$  is the growth rate of the stable population with  $\mu_2(t) = 0$ . The population arising from fixed fertility and changing mortality of this form might be considered the *true* quasi-stable population: the population that has a stable age distribution even though mortality is changing. The true quasi-stable population is not found in human experience because changes in mortality are always in fact age-selective.

If mortality rates themselves (rather than just variations in mortality rates) were not age-selective, and the age schedule of fertility were constant through time, the age composition of the population would follow the form described in section A (above) for populations with fixed fertility schedules and no mortality. That is to say the age composition would match that in a population in which the birth sequence approaches  $Q_0 e^{rt}$ , where  $r$  is the root of

$$\int_0^a e^{-ra} m(a) da$$

and mortality is zero. As a result of positive mortality rates, rather than zero, the size of the population would be reduced from  $t$  to  $t + n$  by a factor

$$\exp \left[ - \int_t^{t+n} \mu_2(t) dt \right].$$

If the fertility schedule were constant with age as well as time (while mortality varied with time, but not with age), the birth rate would equal the fixed annual probability  $b$  of bearing a female child, and the age composition would be  $be^{-ba}$ , the same age distribution as would occur with zero mortality. There may in fact be unicellular organisms whose mortality and reproduction are independent of age. If there

are such and if, in addition, reproduction does not vary with time, the age composition is invariant and independent of mortality. However, if fertility and mortality are independent of age, age may be an insignificant attribute—organisms of different age may be functionally equivalent.

## REFERENCES

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