Bag Theory with the Dirichlet Boundary Conditions and Spontaneous Symmetry Breakdown

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The Dirichlet boundary conditions of bag theory are derived for models which have chiral symmetry or gauge symmetry. In order to confine the fields inside the bag, preserving the above symmetries, the masses of the fields are generated by spontaneous symmetry breakdown, and taken to infinity outside the bag. By this method, a chiral invariant model is successfully constructed, and the Dirichlet boundary conditions are derived for a gauge invariant bag model. It is shown that, in this model, a bag is colour singlet as in the bag models (with the Neumann boundary conditions) of the MIT group, but one-quark states can exist, whose colour charges are completely shielded by the Higgs scalars and the Yang-Mills fields. This means that the shielding mechanism does not necessarily lead to quark confinement. In the Appendix, it is shown that bag models with the Neumann boundary conditions have continuous mass spectra.

§ 1. Introduction

Recently Chodos, Jaffe, Johnson, Thorn and Weiskopf have proposed a new extended model of hadrons (MIT Bag), in which a hadron consists of fields confined to a finite region of space called a “bag”. This model is obtained by assuming that the bag possesses a constant, positive energy $B$ per unit volume, and imposing boundary conditions to confine the hadronic fields inside the bag. In Ref. 1), they have derived two kinds of boundary conditions for scalar fields, namely the Neumann boundary conditions and the Dirichlet boundary conditions, but we may consider that the Dirichlet boundary conditions are preferable in the following two points: 1) As Creutz has shown, the MIT bag theory with the Dirichlet boundary conditions can be realized as a limit of local field theory, and this strongly suggests a connection of the theory with other field theoretical bag theories, and 2) in a certain class of bag models, the Neumann boundary conditions make the mass spectrum of bag states continuous when quantized. (See the Appendix.)

In this paper, we consider bag models which have chiral symmetry or gauge symmetry. In order to preserve the symmetry in the procedure of confining the fields inside the bag, we generate masses of the fields by spontaneous symmetry breakdown, and take these masses to infinity outside the bag.

In § 2, we consider the $\sigma$-model and construct a chiral invariant bag model. In Ref. 1), boundary conditions for massless fermions have been derived by giving the fermion fields infinite masses outside the bag, but this procedure is not chiral.
invariant, and in the resulting theory, the axial charge is not conserved though
the divergence of the axial current vanishes inside the bag. Here we generate the
fermion mass by spontaneous symmetry breakdown, and obtain a model in which
the axial charge as well as the electric charge and the energy-momentum are
conserved. In this model, massless pions remain outside the bag, and consequently,
pions are not bag states but usual Goldstone bosons.

In §3, we consider gauge invariant bag models, and derive the Dirichlet
boundary conditions for gauge fields by generating their mass through Higgs mecha­
nism and taking the limit of infinite mass. The coloured quark-gluon model has
been investigated in Ref. 1), but their boundary conditions for vector gluons are
the Neumann boundary conditions which make the mass spectrum of the bag states
continuous when quantized (see the Appendix). They have shown using these
Neumann boundary conditions that only colour-singlet bag states exist, and this
ecludes states with non-zero triality. On the other hand, a bag is also colour­
singlet in our model. However there can be non-vanishing colour charges inside
the bag, and these charges are completely shielded by the charges of the Higgs
scalars which accumulate on the surface of the bag. It can be shown that these
charges on the bag surface do not carry energy-momentum, and further that, in the
unitary gauge, these charges and the charges inside the bag are conserved independ­
ently. This shows that the charges on the bag surface do not correspond to an
independent freedom of the bag. In our model, a bag can be visualized as an
extended object surrounded by a superconducting medium, and we show that the
magnetic flux is confined inside the bag. The shielding mechanism does not neces­
sarily lead to quark confinement, and the problem of quark confinement remains
unsolved.

Section 4 is devoted to discussion of our results. In the Appendix, we prove
that the Neumann boundary conditions make the mass spectrum of the bag states
continuous in a certain class of models. We show explicitly that bag models with
the Neumann boundary conditions which involve massless scalar fields or Yang­
Mills fields have continuous mass spectra.

§2. Chiral invariant bag models

In this section, we consider the σ-model, and construct a chiral invariant bag
model in which the fermion fields are confined inside the bag. In order to confine
the fermion fields, we generate their masses by spontaneous symmetry breakdown,
and take them to infinity outside the bag.

The fermion fields acquire a mass $M=gv$ when spontaneous symmetry break­
down takes place ($g$ is the coupling constant, and $v$ the vacuum expectation value
of $\sigma$), and there are two choices to make $M$ infinity (i.e., $g\to\infty$ or $v\to\infty$). How­
ever, since the field $\pi$ remains outside the bag even after taking the limit, $v$ is a
physical quantity (the pion decay constant). Consequently, we shall take $g$ to
infinitesimal outside the bag. Let us start from the following action:

\[ W = \int d^4x \left[ \frac{i}{2} \left( \bar{\psi} i \gamma \cdot \partial \psi + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} (\partial_\mu \pi)^2 - \frac{\mu^2}{2} (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2 \right) - \theta(V) \{ B + g_\sigma \bar{\psi} (\sigma + i\pi_\tau) \psi \} - (1 - \theta(V)) \{ g_\pi \bar{\psi} (\sigma + i\pi_\tau) \psi \} \right]. \]  

(1)

Here \( \theta(V) = 1 \) inside the bag, and \( \theta(V) = 0 \) outside the bag, and \( \mu^2 < 0 \). This action can be considered as a limit of (unrenormalizable) field theory. We demand the continuity of \( \psi, \sigma, \pi, \partial \sigma, \text{ and } \partial \pi. \)

The equation of motion of the fields and boundary conditions can be obtained from the variational principle. By varying the surface of the bag, we obtain the following on the surface:

\[ g_\sigma \bar{\psi} (\sigma + i\pi_\tau) \psi = g_\pi \bar{\psi} (\sigma + i\pi_\tau) \psi + B. \]  

(2)

Now, let us consider the limit \( g \to \infty \). We assume that, in this limit, there are solutions of the following nature, and restrict ourselves to these solutions.

(i) The hypersurface swept out by the bag surface in space-time is well defined and smooth in this limit, that is, the curvature of the hypersurface is finite and the normal to the hypersurface \( n_\mu \) is spacelike.\(^*\)

(ii) All the fields and their derivatives along the hypersurface are finite (and continuous, for Bose fields) in this limit; let us consider the neighborhood of order \( 1/M \) \( (M = g \nu = \sqrt{-\mu^2/\lambda}) \) of an arbitrary point on the hypersurface, and rescale the coordinates by \( M \) and put \( Mx = X \). In terms of the coordinates \( X \), the bag hypersurface can be approximated by a hyperplane (assumption (ii)), and the fields are approximately constant along the surface direction. We define the normal \( n_\mu \) to be the inward normal and \( n_\tau n_\tau = -1 \), and introduce the coordinate in the direction of outward normal \( r \). Then \( n_\mu \partial_\mu = \partial / \partial r \) (our metric is \( (+, -, -, -) \)). The equations of motion outside the bag become (in the leading order) Eqs. (3), (4) and (5) where \( R = Mr. \)

\[ \left( \frac{\partial}{\partial R} + in_\tau r^* \cdot \frac{\sigma + i\pi_\tau}{\nu} \right) \psi = 0, \]  

(3)

\[ \frac{\partial^2 \pi}{\partial R^2} = \frac{i}{M \nu} \bar{\psi} \psi, \]  

(4)

\[ \frac{\partial^2 \sigma}{\partial R^2} = \frac{1}{M \nu} \bar{\psi} \psi. \]  

(5)

From Eqs. (4) and (5), we see that \( \pi \) and \( \sigma \) can be regarded as constants when we solve Eq. (3). The solution is as follows:

\(^*\) One can generalize the action (1) to the case where the constants \( \mu^2 \) and \( \lambda \) take different values outside and inside the bag. We have also investigated a model in which there are no \( \sigma \) and \( \pi \) fields inside the bag. Since these models can be treated in a similar manner, we shall not describe them here.

\(^*\) This condition is rather too restrictive. The bag surface may have edge-like singularities.
\[ \psi = e^{A \Phi} \]  \( A = -i n_\mu \gamma^\mu \sigma + i n_\mu \gamma^\mu \)

Since \( A^2 = (\sigma^2 + \pi^2)/v^2 \), the matrix \( A \) has eigenvalues \( \pm \sqrt{\sigma^2 + v^2}/v \). In order that \( \psi \) vanish outside the bag when \( M \to \infty \), \( \Phi \) must satisfy the condition \( A \Phi = -\alpha \Phi \) \((\alpha = \sqrt{\sigma^2 + \pi^2}/v)\). From this, we obtain the following boundary condition:

\[ \frac{1}{v} (\sigma + i n_\mu \gamma^\mu) \psi = i \alpha n_\mu \gamma^\mu \psi \]  

\[ (7) \]

on the surface.

Let us multiply both sides of Eq. (7) by \( \bar{\psi} \) from the left. Then the left-hand side becomes real, and the right-hand side pure imaginary. Hence it follows that

\[ \bar{\psi} (\sigma + i n_\mu \gamma^\mu) \psi = 0 \]  

\[ (8) \]

and

\[ \bar{\phi} n_\mu \gamma^\mu \psi = 0. \]  

\[ (9) \]

Inserting the solution \( \psi = e^{-a R} \Phi \) in Eqs. (5) and (6), and integrating once, we obtain

\[ \frac{\partial}{\partial r} \pi = \frac{i}{-2\alpha v} [e^{-2aMr} - 1] \Phi + C \]  

\[ (10) \]

and

\[ \frac{\partial}{\partial r} \sigma = \frac{1}{-2\alpha v} [e^{-2aMr} - 1] \Phi + D, \]  

\[ (11) \]

where \( C, D \) are integration constants. Equations (10) and (11) show that, in the limit \( M \to \infty \), the normal derivatives of \( \pi \) and \( \sigma \) become discontinuous on the surface. Denoting the fields inside the bag by subscript \( i \), and outside the bag by subscript \( e \), we obtain the following equations from Eqs. (10) and (11).

\[ (n \cdot \partial) \pi_i - (n \cdot \partial) \pi_e = \frac{i \bar{\psi} \gamma^\mu \phi}{2\alpha v}. \]  

\[ (12) \]

\[ (n \cdot \partial) \sigma_i - (n \cdot \partial) \sigma_e = \frac{\bar{\phi} \phi}{2\alpha v}. \]  

\[ (13) \]

We can derive another boundary condition from Eq. (2). First, let us consider the energy-momentum tensor outside the bag.

\[ T^\mu_\nu = \frac{i}{2} \bar{\psi} \gamma^\nu \gamma^\rho \psi_e + \partial^\mu \sigma \partial^\nu \phi_e + \partial^\mu \pi_e \partial^\nu \pi_e \]

\[ -g^\mu_\nu \left\{ \frac{1}{2} (\partial_e \sigma_e)^2 + \frac{1}{2} (\partial_e \pi_e)^2 - \mu^2 (\sigma_e^2 + \pi_e^2) - \frac{1}{4} (\sigma_e^2 + \pi_e^2)^2 \right\}. \]

\[ (14) \]

From Eq. (6) and the condition \( A \Phi = -\alpha \Phi \), we can put \( \psi_e = e^{-aMr} \Phi(x) \). When
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Consider an element \(dS\) of the hypersurface swept out by the bag surface. We translate \(dS\) in the direction of outward normal by an infinitesimal quantity \(d\), and denote this element by \(dS'\). Then \(dV\) is the infinitesimal volume element between \(dS\) and \(dS'\). In the figure, the dimension of space is 2.

\[ M \text{ goes to infinity, } \partial' \psi_e \text{ and } \partial' \bar{\psi}_e \text{ become infinite on the surface, but it can be easily verified that these infinities cancel each other in } T''_{M \to \infty}. \] Thus, \( T''_{M \to \infty} \) is finite.

Let us consider the integral \( \int_A \partial_\mu T''_{M \to \infty} d\Sigma x = 0 \), where \( \Delta V \) is an infinitesimal volume attached to the bag hypersurface as shown in Fig. 1. Transforming this integral to a surface integral, and first letting \( M \) to infinity and then \( d \) to zero, we obtain the following:

\[
\lim_{M \to \infty} \lim_{d \to 0} n_\mu T'' = \lim_{M \to \infty} \lim_{d \to 0} n_\mu T''.
\] (15)

Since \( \lim_{M \to \infty} (i/2) \bar{\psi}_e n_\mu \partial' \psi_e \) vanishes, the above equation can be written as

\[
\lim_{M \to \infty} \lim_{d \to 0} \frac{i}{2} \bar{\psi}_e n_\mu \partial' \psi_e = \left[ (n_\mu \partial' \sigma_e + (n_\mu \partial' \pi_e) \partial' \nu_e - n_e' \frac{1}{2} (\partial_\mu \sigma_e)^2 + \frac{1}{2} (\partial_\mu \pi_e)^2 \right]

- \text{[the same term with } \sigma_i \text{ and } \pi_i].
\] (16)

From Eq. (16), we obtain the following equation by decomposing the derivatives into the normal part and the surface part (that is, \( \partial_e = -n_\mu (n \cdot \partial) + (\partial_e + n_\mu (n \cdot \partial)) \)), and using the boundary conditions (7), (12) and (13).

\[
\lim_{M \to \infty} \lim_{d \to 0} \frac{1}{\alpha v} (n \cdot \partial) \{ \bar{\Phi} (\sigma + i \pi \gamma) \Phi \} = \frac{1}{(2 \alpha v)^2} \{ (\bar{\psi}_e \psi_e)^2 - (\bar{\psi}_e \gamma \psi_e)^2 \}.
\] (17)

After some manipulations, it can be verified from Eq. (17) that

\[
\lim_{M \to \infty} \lim_{d \to 0} g \bar{\psi}_e (\sigma + i \pi \gamma) \psi_e = \lim_{M \to \infty} \lim_{d \to 0} \frac{i}{2} \bar{\psi}_e \gamma \partial' \psi_e

= -\frac{1}{2 \alpha v} \bar{\psi}_e (n \cdot \partial) (\sigma_i + i \pi_i \gamma) \psi_i + \frac{i}{2} \bar{\psi}_e (\partial_\mu + n_\mu n \cdot \partial) \psi_i

- \frac{1}{2 (2 \alpha v)^2} \{ (\bar{\psi}_e \psi_e)^2 - (\bar{\psi}_e \gamma \psi_e)^2 \}.
\] (18)
This equation, together with Eq. (2), gives the boundary condition

$$\frac{1}{\alpha V} (n \cdot \partial) \{ \bar{\psi} \psi + i n \gamma_5 \bar{\psi} \psi \} = - \frac{1}{(2\alpha V)^2} \{ (\bar{\psi} \psi) - (\bar{\psi} \gamma_5 \psi) \} - 2B.$$  \hspace{1cm} (19)

The boundary conditions (7), (12), (13) and (19) guarantee the conservation of the electric charge, the axial charge, and the energy-momentum. Here we show the conservation of the axial charge. The axial current, inside the bag, is

$$A^a_r = \frac{i}{2} \bar{\psi} \gamma_r \gamma_5 \psi - \epsilon_0 \partial^a \sigma_t + \epsilon_0 \partial^a \pi_t,$$

and, outside the bag,

$$A^a_r = - \epsilon_0 \partial^a \sigma_t + \epsilon_0 \partial^a \pi_t.$$

Since the divergences of $A^a_r$ and $A^a_s$ vanish, the axial charge is conserved if $n_\alpha A^a_r = n_\alpha A^a_s$ on the surface. This follows immediately from Eqs. (12) and (13). Thus, we have succeeded to construct a bag theory based on the $\sigma$-model, preserving the chiral symmetry. The essential feature of this model is that pions are not bag states, but the usual Goldstone bosons. When the pion fields are introduced as canonical fields in the original action, this feature is a general consequence. Then, what happens when the symmetry is dynamically broken? In this case, there seems to be a very interesting possibility that the Goldstone bosons also become bag states, but this problem is left for future investigations.

§ 3. Gauge invariant bag models

In this section, we shall consider the bag models in which gauge invariantly interacting fields are confined inside the bag. Here, for concreteness, we shall investigate the $SU(2)$ gauge invariant model of a doublet of fermion fields, but we can easily generalize our method to other cases. In order to confine the fermion fields inside the bag, we give large masses to them outside the bag as in Ref. 1). To confine the gauge fields, we introduce a doublet of complex scalar fields $\Phi$ outside the bag, and make the gauge fields massive through Higgs mechanism. We consider the following action.

$$W = \int d^4 x \left[ - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{i}{2} \bar{\psi} \gamma_\nu \partial^\nu \psi + g \bar{\psi} \gamma^\nu A_\nu \cdot \frac{\tau}{2} \psi - \theta (V) \{ B + m \bar{\psi} \psi \} ight.$$

$$+ (1 - \theta (V)) \left\{ - m' \bar{\psi} \psi + \Phi^\dagger (\partial^a + ig A^a \cdot \frac{\tau}{2}) (\partial_a - ig A_a \cdot \frac{\tau}{2}) \Phi ight.$$  

$$\left. - \mu' \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 - \frac{\mu^4}{4 \lambda} \right\},$$  \hspace{1cm} (20)

where $\mu^2 < 0$, and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu$. From the variational principle, we obtain \[\nabla_\nu \] on the bag surface.
\[ n_\sigma \left( \partial^a - ig A^a \cdot \frac{\tau}{2} \right) \Phi = 0 \] (21)

and

\[-B - m \bar{\psi} \psi = -m' \bar{\psi} \psi + \Phi \left( \partial^a + ig A^a \cdot \frac{\tau}{2} \right) \left( \partial - ig A^a \cdot \frac{\tau}{2} \right) \Phi - \mu' \bar{\Phi} \Phi - \lambda \left( \Phi \Phi \right)^2 - \frac{\mu^2}{4\lambda}. \] (22)

Let us put

\[ \Phi = \exp \left( i \frac{\tau \cdot \theta}{2} \right) \left[ \frac{\partial}{\sqrt{2}(\nu + \sigma)} \right], \quad \left( \nu = \sqrt{-\frac{\mu^2}{\lambda}} \right) \]

\[ B = \exp \left( -i \frac{\tau \cdot \theta}{2} \right) A \cdot \frac{\tau}{2} \exp \left( i \frac{\tau \cdot \theta}{2} \right) - \frac{i}{g} \left( \partial_x \exp \left( -i \frac{\tau \cdot \theta}{2} \right) \right) \exp \left( i \frac{\tau \cdot \theta}{2} \right), \]

\[ \chi = \exp \left( -i \frac{\tau \cdot \theta}{2} \right) \psi, \quad G_{\alpha \nu} = \partial_{\alpha} B_{\nu} - \partial_{\nu} B_{\alpha} + g B_{\alpha} \times B_{\nu}. \] (23)

The Lagrangian outside the bag can be written in terms of \( \sigma, B, \) and \( \chi \) as

\[ L = -\frac{1}{4} G_{\alpha \nu} G^{\alpha \nu} + \frac{i}{2} \chi \gamma_{5} \partial \chi + g \chi \gamma_{5} B \cdot \frac{\tau}{2} \chi - m' \chi \chi' \]

\[ + \frac{g^2}{4} \left( \nu + \sigma \right)^2 B_{\alpha} B^{\alpha} + \frac{1}{2} \left( \partial_{\sigma} \sigma \right)^2 + \mu' \sigma' - \lambda \nu \sigma' - \frac{\lambda}{4} \sigma^2. \] (24)

Equation (21) can be written as

\[ n_\sigma \left( \partial^a - ig B^a \cdot \frac{\tau}{2} \right) \left[ \frac{\partial}{(\nu + \sigma)\sqrt{2}} \right] = 0. \] (25)

From Eq. (25), we obtain

\( (\nu + \sigma)n_\sigma B^a = 0 \) (26)

and

\( n_\sigma \partial^a \sigma = 0. \) (27)

Now we consider the limit \( m' \to \infty \) and \( -\mu' \to \infty \). Let us first take \( m' \) to infinity. Proceeding just as in the previous section, we obtain

\[ i n_\sigma \tau^a \psi = 0 \] (28)

and

\[ -B = \frac{1}{2} \left( n \cdot \partial \right) \chi + \frac{g^2}{4} \left( \nu + \sigma \right)^2 B_{\alpha} B^{\alpha} + \frac{1}{2} \left( \partial_{\sigma} \sigma \right)^2 + \mu' \sigma' - \lambda \nu \sigma' - \frac{\lambda}{4} \sigma^2 \] (29)

from Eq. (22).
The next step is to take \(-\mu^2\) to infinity. In this limit, the masses of \(\sigma\) and \(B_s\) become infinite. Since \(\sigma\) does not exist inside the bag, \(\sigma\) entirely disappears from the theory in this limit. From the Lagrangian (24), we obtain the following equations of motion:

\[
(\partial_{\mu} + gB_{\mu} \times \mathcal{G}^{\alpha \beta} + \frac{g^2}{4} (v + \sigma)^2 B^\alpha B^\beta) = 0 \tag{30}
\]

and

\[
(\Box + 2\mu^4)\sigma = 3\lambda v \sigma^3 + \lambda \sigma^2 - \frac{g^2}{4} (v + \sigma) B_{\mu} B^\mu \tag{31}
\]

outside the bag. Again we restrict ourselves to solutions which satisfy the conditions (i) and (ii) of the previous section. We put \(-2\mu^2 = M_s^2\), \(gv/2 = M\), \(M_s r = R_s\) and \(M r = R\), where \(r\) is the coordinate in the direction of outward normal to the bag hypersurface. The equations of motion become in the leading order

\[
\frac{\partial^2}{\partial R^2} \left( B^\alpha + n^\alpha(n^\beta B^\beta) \right) = 0 \tag{32}
\]

and

\[
\frac{\partial^2}{\partial R^2} \sigma = \sigma - \frac{g^2}{4} B_{\mu} B^\mu \frac{1}{\sqrt{2\lambda M_s}} \tag{33}
\]

From Eq. (32), it follows that \(n^\alpha B^\alpha = 0\), which is consistent with Eq. (26). The solution of Eq. (32) can be written as

\[
B^\alpha = e^{-\lambda^2} \beta^\alpha \tag{34}
\]

Let us consider the tensor \(G^{\alpha \beta}\) at an arbitrary point on the surface. In the rest frame of this point, choosing the third coordinate to be in the normal direction, we obtain

\[
G^{\alpha \beta} = \delta_\alpha B_\beta - \delta_\beta B_\alpha + g B_\alpha \times B_\beta \quad (i = 0, 1, 2) \tag{35}
\]

Since \(\delta_\alpha B_\beta = -M \beta_\beta\) (\(i = 0, 1, 2\)) on the surface, and since we have assumed that \(B\) and \(\partial_i B\) are finite in the limit \(M \to \infty\), the physical quantities \(G^{\alpha \beta}\) become infinite on the surface, unless \(\beta_i\) vanishes. Consequently, \(\beta_i\) must vanish as \(O(1/M)\). From this and Eq. (26), we obtain

\[
B_s = 0 \tag{36}
\]

on the surface. Here \(n^\alpha B^\alpha = 0\) strictly, while other components are of order \(1/M\). From Eq. (36), we can write \(\partial_i B^\alpha\), by introducing \(\alpha^\alpha\), as follows:

\[
\partial_\alpha B_\alpha = n^\alpha \alpha^\alpha \tag{37}
\]

Since \(\alpha^\alpha = -(n \cdot \partial) B^\alpha\) and \(n^\alpha B^\alpha\) strictly vanishes on the surface, we obtain the following:
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\[ n_s \mathbf{A}^s = - (n \cdot \partial) n_s \mathbf{B}^s = 0. \]  

(38)

It immediately follows from the condition (36) that

\[ n_s \mathbf{G}^{*sw} = 0, \]  

(39)

where \( G_{sw}^* = \frac{1}{2} \epsilon_{\mu \nu \rho \lambda} \mathbf{G}^{\rho \lambda} \). Since \( F_{sw} \) is linearly related to \( G_{sw} \), the condition (39) is equivalent to

\[ n_s F^{*sw} = 0, \quad F_{sw}^* = \frac{1}{2} \epsilon_{\mu \nu \rho \lambda} F^{\rho \lambda}. \]  

(40)

The behaviour of \( \sigma \) still remains to be investigated. Since \( B_s \leq O(1/M) \), Eq. (33) becomes

\[ \frac{\partial}{\partial x_s} \sigma = \sigma + O \left( \frac{1}{M^2} \right). \]  

(41)

If \( \sigma \geq 1/M_s^2 \), we can neglect the term \( O(1/M_s^2) \) and obtain \( \sigma \sim e^{-M_s \sigma} \cdot \sigma_0 \), but from the condition (27), \( \sigma_0 \) must vanish in this case. This shows that

\[ \sigma \sim O \left( \frac{1}{M^2} \right). \]  

(42)

The above condition guarantees that \( \sigma \) does not contribute to the energy-momentum in the limit. The boundary condition (29) can be written as

\[ -B = \frac{1}{2} (n \cdot \partial) \bar{\psi} \psi - \frac{1}{2} F_{sw} F^{sw}, \]  

(43)

and it completes the derivation of boundary conditions.

We conclude this section by proving the conservation of energy-momentum, and the colour charge. Let us first consider the energy momentum tensor outside the bag \( T^{\nu w}_{\tau} \) before the limit \( m' \rightarrow \infty \) is taken:

\[ T^{\nu w}_{\tau} = \frac{i}{2} \bar{\psi} T^\nu \partial^\tau \phi - F^{\nu \lambda \tau} A^\lambda \left\{ \Phi^\tau \left( \bar{\psi}^\nu + ig A^\nu \cdot \frac{T}{2} \right) \Phi + \text{h.c.} \right\} \]

\[ -g^{\nu \tau} \left[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \Phi^\nu \left( \bar{\psi}^\mu + ig A^\mu \cdot \frac{T}{2} \right) \left( \partial^\mu - ig A^\mu \cdot \frac{T}{2} \right) \Phi \right. \]

\[ -\mu^2 \Phi^\nu \Phi - \lambda (\Phi^\nu \Phi) \frac{\mu^2}{4 \lambda}. \]  

(44)

In the above, we can easily see that the third term diverges on the surface in the limit \( -\mu^2 \rightarrow \infty \). To clarify what happens in the neighborhood of the bag surface, let us note that the divergent term is not gauge invariant (which involves \( \theta \) when written in terms of \( B, \sigma \) and \( \theta \)), and add the following term to make the energy-momentum tensor gauge invariant.

\[ \bar{T}^{\nu w}_{\tau} = T^{\nu w}_{\tau} + \partial^\lambda \left( F^{\nu \lambda} A^\lambda \right) = \left\{ \frac{i}{2} \bar{\psi} T^\nu \left( \partial^\tau - ig A^\tau \cdot \frac{T}{2} \right) \phi + \text{h.c.} \right\} - F^{\nu \lambda} F^{\lambda}_{\tau} \]

\[ + \left\{ \Phi^\nu \left( \bar{\psi}^\mu + ig A^\mu \cdot \frac{T}{2} \right) \left( \partial^\mu - ig A^\mu \cdot \frac{T}{2} \right) \Phi + \text{h.c.} \right\}. \]
We also add the same term to the energy-momentum tensor inside the bag (note that the addition of this term does not influence the energy-momentum),

\[ T_{\mu}^\nu = T_{\mu}^\nu + \partial_\lambda (F_{\mu\nu}^a A^a) \]

\[ = \left\{ \frac{i}{2} \bar{\psi} \gamma^\mu \left( \gamma^\nu - ig A^\nu \cdot \tau \right) \psi + \text{h.c.} \right\} - F_{\mu\lambda}^a F_{\nu}^a + \frac{1}{4} g^{\mu\nu} F_{\lambda}^a F_{\lambda}^a. \]  

(46)

It can be easily shown that the above tensor \( T_{\mu}^\nu \) does not diverge on the bag surface. Let \( V \) be the region of the bag, and let \( \bar{V} \) be a thin skin wrapping the bag surface. Then the energy-momentum \( P^\nu \) can be written as follows:

\[ P^\nu = \int_\mathcal{V} T_{\nu}^\tau d^3x + \int_{\partial V} T_{\nu}^{\tau a} d^2x = \int_\mathcal{V} T_{\nu}^\tau d^3x + \int_{\partial V} \partial_{\nu} (F_{\mu}^a A^a) d^2x. \]  

(47)

Since \( T_{\nu}^\tau, T_{\nu}^{\tau a} \) and \( F_{\mu}^a A^a \) all vanish exponentially outside the bag, the above \( P^\nu \) can be written, in the limit \( -\mu^2, m' \to \infty \) as

\[ P^\nu = \int_\mathcal{V} T_{\nu}^\tau d^3x + \int_{\partial V} T_{\nu}^{\tau a} d^2x \]  

(48)

From the above and the discussion on Eq. (44), we conclude that the surface energy (the second term of the right-hand side of Eq. (48)) is only an effect of unphysical field \( \theta \), and when we consider the gauge invariant energy-momentum tensor, it completely disappears. To show the conservation of the energy-momentum \( P^\nu \), it is sufficient to prove that \( n_t \bar{T}_{t}^\nu = 0 \), but this can be easily done using the previously obtained boundary conditions.

Finally, let us consider the colour currents

\[ J_{t}^\nu = -i g A_{\mu} \times F_{\mu}^a - g \bar{\psi} \gamma^\mu \frac{\tau}{2} \psi \]  

(49)

and

\[ J_{\nu}^\tau = -i g A_{\mu} \times F_{\mu}^a - g \bar{\psi} \gamma^\mu \frac{\tau}{2} \psi + \left\{ ig \Phi^\mu \frac{\tau}{2} \left( \gamma^\nu - ig A^\nu \cdot \tau \right) \right\} \phi + \text{h.c.} \]  

(50)

In Eq. (50), the third term diverges and it is of order \( M \) on the surface. From the equation of motion \( \partial_\alpha F^\alpha = J^\nu \), the charges can be written as

\[ C = \int_{\partial V} J_{t}^\nu d^2x + \int_{\partial V} J_{\nu}^\tau d^2x = \int_{\partial V} \partial_{\nu} F_{\mu}^a A^a d^2x \to 0. \]  

(51)

The above equation shows that the charges inside the bag, \( C_t = \int_{\partial V} J_{t}^\nu d^2x \), are completely shielded by the surface charges \( C_{\nu} = \int_{\partial V} J_{\nu}^\tau d^2x \). Generally \( C_t \) and \( C_{\nu} \) are not
conserved independently, but there exists a particular gauge in which $A_\tau$ coincide with $B_\tau$ on the surface. Then, in this gauge, it can be easily shown that $n_{\mu}J^\tau=0$ on the surface, and hence $C_\tau$ is conserved and $C_\tau=-C_\tau$. Since we have already shown that the surface charges do not carry energy-momentum, we conclude that the surface charges do not correspond to independent freedoms of the bag. From the above discussion, $C_\tau$ in the above gauge ($A_\tau=0$ on the surface) can be interpreted as quantum numbers corresponding to the internal freedoms of the bag. In the next section, we shall discuss our results obtained in this section.

§ 4. Discussion

Let us first generalize the results of the previous section to the case of the coloured quark-gluon model. We denote the quark field by $\psi_{ia}$, where the first index refers to the colour $SU(3)$, and the second to the hadronic symmetry. For the colour gauge fields, we employ similar notations as in the previous section. The boundary conditions are as follows:

\begin{align*}
  i n_{\mu} \tau^a \psi_{ta} &= \psi_{ta}, \quad (52) \\
  B_\tau^a &= 0, \quad (n \cdot \partial) (n_{\mu} B_\tau^a) = 0, \quad (53) \\
  n_{\mu} F_\tau^{\mu \nu} &= 0, \quad F_\tau^{* \mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} F_\tau^{\alpha \beta}, \quad (53') \\
  -B &= \frac{1}{2} (n \cdot \partial) \bar{\psi} \psi - \frac{1}{4} F_\tau^{\mu \nu} F_\tau^{\mu \nu}. \quad (54)
\end{align*}

The boundary condition (53') follows from the condition (53). To compare our model with that of Ref. 1), we present their boundary conditions below.

\begin{align*}
  i n_{\mu} \tau^a \psi_{ta} &= \psi_{ta}, \quad (55) \\
  n_{\mu} F_\tau^{* \mu} &= 0, \quad (56) \\
  -B &= \frac{1}{2} (n \cdot \partial) \bar{\psi} \psi + \frac{1}{4} F_\tau^{\mu \nu} F_\tau^{\mu \nu}. \quad (57)
\end{align*}

The boundary conditions for quarks $\psi$ are of the Dirichlet type in both models, but the essential difference lies in the boundary conditions for the colour gauge fields. When we work with the boundary conditions (55)∼(57), the fields acquire zero modes which make the mass spectrum of the bag states continuous (see the Appendix), while, in our model, the boundary condition (53) fixes the values of $A_\tau^a$ to zero on the bag surface apart from the arbitrariness of the gauge chosen, and we expect that this makes the mass spectrum discrete.

In our model, there can be non-vanishing colour charges inside the bag, but they are completely shielded by the charges on the surface which are vestiges of the Higgs scalars in the original action. As we have shown in the previous section, these surface charges do not correspond to independent freedoms of the bag. In general, the region inside the bag and the bag surface exchange their charges, but if we choose the gauge $A_\tau^a=0$ on the surface, the colour charges inside the bag and on the surface are conserved independently. Thus we interpret
the colour charges inside the bag in the above gauge as quantum numbers which specify the internal freedom of the bag. The problem of quark confinement is to freeze the colour freedom of hadrons. In our model, colour-singlet one-quark states can exist, in which the charges of a quark is completely shielded by the Yang-Mills fields and the Higgs scalars. It is clear from the above that the shielding mechanism does not necessarily lead to quark confinement. One might consider that the quark confinement could be realized in a theory which allows only colour-singlet states, when the Higgs scalars, for example, belong to the regular representations of the gauge group, since we cannot construct a colour singlet state from one fundamental representation and regular representations. However, when the symmetry is spontaneously broken, it is clear from the following example that the above discussion is not correct. Let us consider a single-charged fermion field $\psi$, and a double-charged scalar field $\Phi$. If the vacuum expectation value of $\Phi$ is nonvanishing, we can define a state with one fermion-number,

$$(\Phi^*)^{1/2}\psi = (C + \Phi^*)^{1/2}\psi,$$

$$\langle \phi \rangle = C, \quad \phi = C + \Phi', \quad C \neq 0,$$

which has zero charge. This shows that one unit of charge can be completely shielded by double-charged bosons.

In our model, a bag can be visualized as an extended object which is surrounded by a superconducting medium. The boundary condition (53') shows that (the non-Abelian analog of) the magnetic flux is confined inside the bag. Thus there is a strong connection between our model and that of Nielsen and Olesen.4 The problem of quark confinement is left for future investigations.

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Appendix

The purpose of this Appendix is to prove the following theorem. It follows from the theorem that a bag model with the Neumann boundary conditions has a continuous mass spectrum, if it involves a massless scalar field or Yang-Mills fields.

**Theorem:** If there is a one-parameter non-compact group of transformations, $T_1 = \{t; -\infty < t < \infty\}$, which satisfies the following two conditions, then the Neumann type bag theory (in $n$ space-time dimensions) $W = \int dt d^n x \theta(V) \{\mathcal{L} + B\}$ has a continuous mass spectrum.

i) The Lagrangian $\mathcal{L}$ is invariant under a time independent transformation.
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\( \phi_A \rightarrow \phi_{A'} = t_A \phi_A t_A^{-1} \) (\( \lambda = 0 \)). (We denote the set of all fields in the theory by \( \phi_A \).)

ii) The Lagrangian \( \mathcal{L} \) is not invariant under a time dependent transformation \( (\lambda \neq 0) \).

Proof) Let \( \mathcal{C}V \) be the following set:

\[
\mathcal{C}V = \{ V \subset R^{n-1}; \text{ } V \text{ is simply connected and } \int_V x^i d^{n-1}x = 0 \text{ for } i = 1, 2, \ldots, n-1 \}.
\]

We introduce a coordinate \( \xi = (\xi_1, \xi_2, \ldots) \) in \( \mathcal{C}V \). For each \( V(\xi) \in \mathcal{C}V \), we choose one differentiable bijection \( f_\xi \) from \( V(\xi) \) to the unit ball, \( B^{n-1} \), in \( (n-1) \)-dimensional space.

\[
X = f_\xi(x), \quad x \in V \text{ and } X \in B^{n-1}.
\]

Let \( W \) be the space of differentiable functions on \( B^{n-1} \). We introduce an equivalence relation in \( W \) as follows:

\[
\phi_1 \sim \phi_2 \phi_1, \phi_2 \in W \Leftrightarrow \exists t_\lambda \text{ such that } t_\lambda \phi_1 t_\lambda^{-1} = \phi_2.
\]

In the quotient space \( W/\sim \), we introduce a coordinate, \( \alpha = (\alpha_1, \alpha_2, \ldots) \). Then, an element of \( W \), \( \phi_1 \), can be written as \( \phi = \phi_\alpha \).

Let the region of the bag \( V \) be simply connected with its center at \( \bar{x} \).

\( \int_{V} x^i d^{n-1}x / \int_{V} d^{n-1}x = \bar{x} \).

This region can be obtained from some \( V(\xi) \in \mathcal{C}V \) by translation. We denote \( V = V(\xi, \bar{x}) \). An arbitrary differentiable function on \( V(\xi, \bar{x}) \) can be written as

\[
\phi_\alpha(x) = \phi_\alpha(f_\xi(x - \bar{x})). \tag{A.1}
\]

The derivatives of \( \phi_\alpha \) can be written as follows.

\[
\partial_\lambda \phi_\alpha = \frac{\partial \phi_\alpha}{\partial \lambda} + \sum \alpha \frac{\partial \phi_\alpha}{\partial \xi} \left\{ \sum \frac{\partial f_\xi^i}{\partial \xi^i} - \sum \frac{\partial f_\xi^i}{\partial x} \right\}, \tag{A.2}
\]

\[
\partial_\lambda \phi_\alpha = \frac{\partial \phi_\alpha}{\partial f_\xi^i} \frac{\partial f_\xi^i}{\partial x}, \quad i = 1, 2, \ldots, n-1 \tag{A.3}
\]

Substituting (A.2) and (A.3) in the bag Lagrangian, and carrying out the \( x \)-integration, we obtain the Lagrangian \( L(\hat{\lambda}, \lambda, \hat{\alpha}, \alpha, \bar{x}, \hat{\xi}, \xi) \).

\[
\int dt dx^{n-1} \theta(V) \{ L(\partial \phi, \phi) + B \} = \int dt L(\hat{\lambda}, \lambda, \hat{\alpha}, \alpha, \bar{x}, \hat{\xi}, \xi). \tag{A.4}
\]

From the translation invariance of the theory, \( L \) does not depend on \( \bar{x} \). From the \( T_1 \) invariance (i), we have

\[
L(\lambda(t), \hat{\lambda}(t), \ldots) = L(\lambda(t) + \lambda_0, \hat{\lambda}(t), \ldots). \tag{A.6}
\]

where \( \lambda_0 \) is independent of \( t \). Hence we obtain that

\[
L(\lambda(t), \hat{\lambda}(t), \ldots) = L(\lambda(t) + \lambda_0, \hat{\lambda}(t), \ldots). \tag{A.6}
\]

\[
\mathcal{L}(\partial \phi_{\lambda_0}, \phi_{\lambda_0}) = \mathcal{L}(\partial \phi_{\lambda_0 + \lambda_0}, \phi_{\lambda_0 + \lambda_0}), \tag{A.5}
\]

where \( \lambda_0 \) is independent of \( t \). Hence we obtain that

\[
L(\lambda(t), \hat{\lambda}(t), \ldots) = L(\lambda(t) + \lambda_0, \hat{\lambda}(t), \ldots). \tag{A.6}
\]
The above equation shows that $\lambda$ is also a cyclic coordinate. Consequently, the Lagrangian can be written as $L(\dot{\lambda}, \dot{\alpha}, \alpha, \dot{x}, \dot{\xi}, \xi)$. Since $x$ and $\lambda$ are cyclic coordinates, their canonical conjugates $\pi(x)$ and $\pi(\lambda)$ are conserved.

$$
\pi(x) = \frac{\delta L}{\delta \dot{x}}, \quad \pi(\lambda) = \frac{\delta L}{\delta \dot{\lambda}}, \quad \frac{d}{dt} \pi(x) = \frac{d}{dt} \pi(\lambda) = 0.
$$

(A.7)

Consequently, $\pi(\lambda)$ and $\pi(x)$ and the Hamiltonian $H$ commute with one another and can be diagonalized simultaneously. Since $\lambda$ varies from $-\infty$ to $+\infty$, $\pi(\lambda)$ has continuous eigenvalues. In the subspace $\pi(x) = P$ and $\pi(\lambda) = C$, we can write the Hamiltonian as $H(P, C)$. From the condition (ii), the Hamiltonian contains $\pi(\lambda)$, and hence $H(P, C)$ depends on $C$. This shows that the Hamiltonian has continuous eigenvalues in the subspace with definite momentum $P$, and hence the mass spectrum of bag states is continuous.

From the above theorem, we obtain the following corollaries.

**Corollary 1.** The Neumann type bag theory of a system which involves a massless scalar field with derivative couplings only has a continuous mass spectrum of bag states.

**Proof** Let $\phi$ be the massless scalar field. We define $t\phi(x) = \phi(x) + \lambda$. Then we can apply the theorem.

**Corollary 2.** The Neumann type bag theory of Yang-Mills fields has a continuous mass spectrum.

**Proof** It is easy to find a transformation group $T_1$ for Yang-Mills fields which satisfies the condition (i) of the theorem, but the problem is to verify that this group satisfies the condition (ii), and that $\pi(\lambda)$ corresponds to a physical freedom. For this purpose, we choose the axial gauge ($A_3 = 0$) in which ghosts do not appear. In the following, we shall work in three-space and one-time dimension.

The Euler equation can be written as

$$
\partial_\nu \mathcal{D}^\nu - ig [\mathcal{A}_\nu, \mathcal{D}^\nu] = 0. \quad (\nu = 0, 1, 2)
$$

(A.8)

(Here we use the matrix notation $\mathcal{A}_a = A_a^\nu \lambda^a$, and $\lambda^a$ are the representation matrices of the gauge group.) For $\nu = 0$, Eq. (A.8) becomes

$$
\partial_0 \mathcal{A}_0 + D_i (\mathcal{A}_1, \mathcal{A}_2) D_i (\mathcal{A}_3, \mathcal{A}_4) \mathcal{A}_0 = D_i \partial_0 \mathcal{A}_i, \quad (i = 1, 2)
$$

(A.9)

$$
D_0 \mathcal{A}_0 = \partial_\nu \mathcal{A}_0 - ig [\mathcal{A}_0, \mathcal{A}_i].
$$

(A.10)

We introduce the following Green functions $G^a(\mathcal{A}_1, \mathcal{A}_4)$:

$$
[\partial_0 + D_i D_i] G^a(\mathcal{A}_1, \mathcal{A}_4) = \delta^a(x) \lambda^a.
$$

(A.11)

Using the above Green functions, $\mathcal{A}_0$ can be written as

$$
\mathcal{A}_0(x) = \int dy G^a(\mathcal{A}_1, \mathcal{A}_2)(x - y) (D_0 \partial_\nu \mathcal{A}_i)^a(y),
$$

(A.12)

where $\lambda^a (D_0 \partial_\nu \mathcal{A}_i)^a = D_i \partial_\nu \mathcal{A}_i$. The gauge condition $\mathcal{A}_4 = 0$ does not fix the gauge.
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completely, but we impose the boundary condition on $G^a(x)$ that they tend to zero when $|x| \to \infty$. This fixes the gauge completely. Substituting (A·12) into the original Lagrangian, we obtain the ghost free Lagrangian $\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0, (\mathcal{A}_1, \mathcal{A}_2))$. It can be easily shown that this Lagrangian gives the correct Euler equations for $\mathcal{A}_1$ and $\mathcal{A}_2$.

We define the group $T_1$ as follows:

$$t_1 \mathcal{A}_1(x) t_1^{-1} = e^{i x \mathcal{F}_1} \mathcal{A}_1(x) e^{-i x \mathcal{F}_1} - \frac{1}{g} \lambda C,$$

(A·13)

$$t_2 \mathcal{A}_2(x) t_2^{-1} = e^{i x \mathcal{F}_1} \mathcal{A}_2(x) e^{-i x \mathcal{F}_1},$$

(A·14)

where $C$ is a linear combination of $\lambda^a$. We can easily show that the Lagrangian $\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0, (\mathcal{A}_1, \mathcal{A}_2))$ is invariant under the above transformations. To show that $T_1$ satisfies the condition (ii), we consider $\mathcal{L}(\mathcal{A}_i, \lambda(i)\alpha, \partial \mathcal{A}_i, \lambda(i)\alpha)$ where, as in the proof of the theorem,

$$\mathcal{A}_i, \lambda(i)\alpha = e^{i x \mathcal{F}_1} \mathcal{A}_i, \alpha e^{-i x \mathcal{F}_1} - \frac{1}{g} \lambda(t) C,$$

(A·15)

$$\mathcal{A}_i, \lambda(i)\alpha = e^{i x \mathcal{F}_1} \mathcal{A}_i, \alpha e^{-i x \mathcal{F}_1}.$$  

(A·16)

When $\mathcal{A}_i, \alpha = 0$ ($i=1, 2$), we obtain

$$\mathcal{A}_i, \lambda(i)\alpha = \frac{1}{g} \lambda C, \quad \mathcal{A}_0, \lambda(i)\alpha = 0, \quad \mathcal{A}_0, \mathcal{A}_i, \lambda(i)\alpha = 0,$$

(A·17)

$$\mathcal{L}(\mathcal{A}_i, \lambda(i)\alpha, \partial \mathcal{A}_i, \lambda(i)\alpha) = \frac{1}{2} \text{Tr}(\lambda C)^2.$$  

(A·18)

Equation (A·18) shows that the group $T_1$ satisfies the condition (ii) of the theorem.

Comments: From the above proof, it is clear that the mass spectrum is also continuous in a gauge invariant bag model of Yang-Mills fields and matter fields if the boundary conditions for matter fields are also of the Neumann type. For a gauge invariant model in which the boundary conditions for some of the matter fields are of the Dirichlet type, it can be shown that the mass spectrum is continuous if the bag surface is at rest. From this, we expect that the Neumann boundary conditions for Yang-Mills fields make the mass spectrum of bag states continuous independently of the boundary conditions for matter fields, but we do not have a rigorous proof of this up to now.

References