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Analogous to the famous Euler angle parametrization in three-dimensional Euclidean space, a reflection-free Lorentz transformation in $(2 + 1)$ -dimensional Minkowski space can be decomposed into three simple parts. Applying this decomposition to the Wigner rotation problem, we are able to show that the related mathematics becomes much simpler and the physical meanings more comprehensible and enlightening. © 2023 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).

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I. INTRODUCTION

First discovered by Silberstein, then rediscovered by Thomas,^{1,2} the phenomenon that two successive non-parallel boosts (i.e., Lorentz transformations that contain neither rotation nor reflection) lead to a boost and a rotation is generally called the Wigner rotation.³ It has been studied by many authors for almost a century,⁴⁻⁹ yet the conclusion is still “paradoxical” for most people. “The spatial rotation resulting from the successive application of two non-parallel Lorentz transformations has been declared every bit as paradoxical as the more frequently discussed apparent violations of common sense, such as the so-called ‘twin paradox’...” said Goldstein in his classic work *Classical Mechanics*.¹⁰

To explain how the puzzlement arises, we start with the classical counterpart of a boost transformation, the so-called Galilean transformation between two inertial frames with relative velocity $\vec{V} = [V_x, V_y, V_z]$,

$$\begin{aligned}x' &= x - V_x t, \\y' &= y - V_y t, \\z' &= z - V_z t, \\t' &= t.\end{aligned}\tag{1}$$

From a geometrical point of view, the first three lines of Eq. (1) constitute a time-dependent passive translation. It implies that a Galilean transformation follows the rules of a three-dimensional translation, e.g., the composition of two translations leads to another one, and this net translation does not depend on the order in which the original two are undertaken. Being the relativistic version of Galilean transformation, a boost is often believed to possess this same property.

As a matter of fact, relativistic boosts do not commute unless their directions are parallel, i.e., their composition is order-dependent in general. Moreover, as mentioned earlier, the composition of two non-parallel boosts is not a single boost, but a boost along with a rotation. This is quite counter-intuitive since the rotation seems to emerge out of nowhere from the three-dimensional point of view. Getting to the bottom of the matter, it is inappropriate to analogize boost to translation since the former is essentially a sort of rotation (or more precisely a pseudo-rotation) in four-dimensional spacetime. Therefore, the Wigner rotation may be regarded as a geometric problem that involves both rotation and pseudo-rotation, and the mathematical complexity is enough to cloud those subtle physical meanings.

If the mathematics could be substantially simplified, we believe the physical meanings of the Wigner rotation would become clear, and people would find this phenomenon is not so counter-intuitive as usually thought. To achieve this goal, we develop a formulation analogous to the Euler angle parametrization of $SO(3)$, i.e., decomposing a reflection-free Lorentz transformation in $(2 + 1)$ -dimensional Minkowski space into a product of two rotations and one pseudo-rotation (Sec. II). As a demonstration of its effectiveness, we show how simple it is to derive important rules about the Wigner rotation problem (Sec. III) and how little mathematical knowledge is needed to calculate the most general Wigner angle (Sec. IV). Physical insights into Wigner rotation via this decomposition are discussed in Sec. III, and a rigorous proof of this decomposition is provided in the Appendix.

II. PRELIMINARIES

A. $(2 + 1)$ -dimensional Minkowski space

Although physical spacetime is the $(3 + 1)$ -dimensional Minkowski space $\mathbb{R}^{3,1}$, physicists work on its subspace in many cases without losing generality. For example, when discussing a pure (i.e., rotation-free and reflection-free) Lorentz transformation between two inertial frames with the Minkowski coordinates (x, y, z, ct) and (x', y', z', ct') , where c is the light speed, we may assume the relative velocity \vec{V} is along the x -direction and consider just the transformation between (x, ct) and (x', ct') , which is represented by the formula

$$\begin{aligned}\begin{pmatrix} x' \\ ct' \end{pmatrix} &= \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \\ &= \begin{pmatrix} \cosh \eta & -\sinh \eta \\ -\sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix},\end{aligned}\tag{2}$$

where $\beta = \tanh \eta = V/c$ and $\gamma = \cosh \eta = 1/\sqrt{1 - \beta^2}$. This transformation is usually called a boost along the x -direction or an x -direction boost. From a geometric point of view, it is a pseudo-rotation around $(0, 0)$ in the $(1 + 1)$ -dimensional Minkowski space $\mathbb{R}^{1,1}$.

Similarly, since the Wigner rotation problem involves only two relative velocities, it is legitimate to put them in the xy -plane so that none of the z -components show up in the calculations. Therefore, our discussion will be restricted to the $(2 + 1)$ -dimensional Minkowski space $\mathbb{R}^{2,1}$, which is sufficient for us to derive all of the related results.



It is apparent $x^2 - c^2t^2$ is an invariant under the transformation represented in Eq. (2). When we use this invariant as the criterion for the (1 + 1)-dimensional Lorentz transformation, the reflections such as $x \rightarrow -x$ or $ct \rightarrow -ct$ will be included as well. It is straightforward to generalize this criterion to (2 + 1)-dimensional Minkowski space, i.e., we may define the Lorentz transformation in this space as the one that preserves $x^2 + y^2 - c^2t^2$. Clearly, both of the x -direction and y -direction boosts as well as the xy -plane rotation are special cases of this (2 + 1)-dimensional Lorentz transformation.

There are many similarities between $\mathbb{R}^{2,1}$ and the three-dimensional Euclidean space \mathbb{R}^3 . For example, the three-dimensional rotation by the angle θ about the axis $[-\sin \phi, \cos \phi, 0]$ takes the form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \times \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (3)$$

while the transformation

$$\begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cosh \eta & 0 & -\sinh \eta \\ 0 & 1 & 0 \\ -\sinh \eta & 0 & \cosh \eta \end{pmatrix} \times \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ ct \end{pmatrix} \quad (4)$$

represents a pseudo-rotation around the corresponding axis $[-\sin \phi, \cos \phi, 0]$ in $\mathbb{R}^{2,1}$, and it meets the criterion of being a (2 + 1)-dimensional Lorentz transformation. However, note that the product of the three transformation matrices in Eq. (3) is an orthogonal matrix, i.e., its transpose is equal to its inverse, while that of Eq. (4) is a symmetric one, i.e., its transpose is itself.

Finally, if we express $x^2 + y^2 - c^2t^2$ as a matrix product

$$\begin{pmatrix} x & y & ct \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ ct \end{pmatrix} =: (x \ y \ ct)g \begin{pmatrix} x \\ y \\ ct \end{pmatrix}, \quad (5)$$

it is easy to see that the necessary and sufficient condition for a 3×3 matrix L to represent a (2 + 1)-dimensional Lorentz transformation is $L^\top g L = g$, where the superscript \top represents taking transpose of the matrix. Accordingly, if we have two (2 + 1)-dimensional Lorentz transformation matrices, say L_1 and L_2 , then both $L_1 L_2$ and $L_2 L_1$ are also Lorentz

transformations of this kind. (For the sake of brevity, we will henceforward not distinguish between the transformation and its matrix representation.)

B. Euler angles and their Minkowski counterparts

The theory of Euler angles guarantees that any proper (i.e., reflection-free) rotation in \mathbb{R}^3 can be decomposed into three simple proper rotations with each of them keeping one coordinate axis fixed. Among those commonly used Euler angle parametrizations, the one suits us most is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \times \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (6)$$

where $0 \leq \phi, \psi < 2\pi$ and $0 \leq \theta \leq \pi$.

Based on the analogy between Eqs. (3) and (4), it is reasonable to assume that the (2 + 1)-dimensional counterpart of Eq. (6) takes the form

$$\begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cosh \eta & 0 & -\sinh \eta \\ 0 & 1 & 0 \\ -\sinh \eta & 0 & \cosh \eta \end{pmatrix} \times \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ ct \end{pmatrix}, \quad (7)$$

where $\eta \geq 0$ and the ranges of ϕ and ψ are the same as those of Eq. (6). For an obvious reason, Eq. (7) will also be called an Euler decomposition in this paper.

In the Appendix, we will prove that the (2 + 1)-dimensional reflection-free Lorentz transformation

$$\begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ ct \end{pmatrix} \quad (8)$$

can always be expressed as Eq. (7). If we further demand $L_{33} \neq 1$, then the decomposition is unique and the parameters are determined by the following formulas:

$$\begin{aligned} \cos \phi &= -L_{31} / \sqrt{L_{33}^2 - 1}, \\ \sin \phi &= -L_{32} / \sqrt{L_{33}^2 - 1}, \\ \cosh \eta &= L_{33}; \\ \cos \psi &= -L_{13} / \sqrt{L_{33}^2 - 1}, \\ \sin \psi &= -L_{23} / \sqrt{L_{33}^2 - 1}. \end{aligned} \quad (9)$$

C. (2 + 1)-dimensional velocity

A boost transformation takes place between two inertial frames; hence, each boost is defined by a constant velocity that is the relative velocity between the frames. When a (2 + 1)-dimensional velocity undergoes a boost $B(\vec{V})$ with \vec{V} being the relative velocity, the formula $W' = B(\vec{V})W$ is analogous to the boost transformation of spacetime coordinates, where the column matrices W and W' represent the (2 + 1)-dimensional velocities in the old and the new frames, respectively. Conversely, the inverse boost transformation $W = B(\vec{V})^{-1}W'$ allows us to calculate the (2 + 1)-dimensional velocity in the old frame from that in the new one.

Note that for an object with the spacetime coordinates (x, y, ct) , its (2 + 1)-dimensional velocity W is defined as

$$W^\top = \begin{pmatrix} dx & dy & dt \\ d\tau & d\tau & c d\tau \end{pmatrix} = \frac{dt}{d\tau} \begin{pmatrix} dx & dy \\ dt & c \end{pmatrix} =: \gamma(\vec{v})(v_x \ v_y \ c), \quad (10)$$

where τ is the proper time of this object, $\vec{v} = [v_x, v_y]$ is the two-dimensional velocity with respect to the coordinate time, and

$$\gamma(\vec{v}) = \frac{1}{\sqrt{1 - (v_x^2 + v_y^2)/c^2}}. \quad (11)$$

Assume the object at rest in the new frame. Since its two-dimensional velocity relative to the old frame equals the relative velocity \vec{V} between those two frames, transforming its (2 + 1)-dimensional velocity in the new frame back to that in the old reveals the information of the boost velocity,

$$W = B(\vec{V})^{-1} \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \gamma(\vec{V}) \begin{pmatrix} V_x \\ V_y \\ c \end{pmatrix}. \quad (12)$$

Taking Eq. (4) as an example, interpreting this pseudo-rotation as a boost gives us the corresponding (2 + 1)-dimensional velocity

$$\begin{aligned} & \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \eta & 0 & \sinh \eta \\ 0 & 1 & 0 \\ \sinh \eta & 0 & \cosh \eta \end{pmatrix} \\ & \times \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \\ & = c \begin{pmatrix} \sinh \eta \cos \phi \\ \sinh \eta \sin \phi \\ \cosh \eta \end{pmatrix} = \cosh \eta \begin{pmatrix} c \tanh \eta \cos \phi \\ c \tanh \eta \sin \phi \\ c \end{pmatrix}. \end{aligned} \quad (13)$$

This reveals that the boost velocity $\vec{V} = [V_x, V_y] = c \tanh \eta [\cos \phi, \sin \phi]$. Accordingly, Eq. (4) enumerates all possible boosts in $\mathbb{R}^{2,1}$.

When a problem involves three inertial frames and two successive boosts, say first $B(\vec{V}_1)$ then $B(\vec{V}_2)$, the (2 + 1)-dimensional velocity of a rest object in the third frame can be transformed to that in the first by

$$W = [B(\vec{V}_2)B(\vec{V}_1)]^{-1} \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}. \quad (14)$$

The two-dimensional velocity contained in this (2 + 1)-dimensional velocity is the composition of the two boost velocities in that order and can be denoted by $\vec{V}_{1\oplus 2}$. Therefore, we rewrite Eq. (14) as

$$W_{1\oplus 2} = [B(\vec{V}_2)B(\vec{V}_1)]^{-1} \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \gamma(\vec{V}_{1\oplus 2}) \begin{pmatrix} (V_{1\oplus 2})_x \\ (V_{1\oplus 2})_y \\ c \end{pmatrix}. \quad (15)$$

III. WIGNER ROTATION

A. Three rules

By employing the Euler decomposition, we can derive three important rules about the Wigner rotation problem systematically and effortlessly. To be concise, the following shorthand notations will be used:

$$R(\phi) = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (16)$$

$$B_x(-\eta) = \begin{pmatrix} \cosh \eta & 0 & -\sinh \eta \\ 0 & 1 & 0 \\ -\sinh \eta & 0 & \cosh \eta \end{pmatrix}, \quad (17)$$

and

$$B_\phi(-\eta) = R(-\phi)B_x(-\eta)R(\phi). \quad (18)$$

Thereupon, Eqs. (4) and (7) can be, respectively, expressed as

$$\begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} = B_\phi(-\eta) \begin{pmatrix} x \\ y \\ ct \end{pmatrix}$$

and

$$\begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} = R(-\psi)B_x(-\eta)R(\phi) \begin{pmatrix} x \\ y \\ ct \end{pmatrix}. \quad (19)$$

Rule 1. Two successive boosts are equivalent to a boost followed or preceded by a rotation, which may be taken as the definition of Wigner rotation.

Proof: As discussed in Sec. II A, for any two boosts B_1 and B_2 in $\mathbb{R}^{2,1}$, their product B_2B_1 must be a Lorentz transformation in this space. Since these boosts contain no reflection according to Eq. (4), neither does their product. Hence, we may apply the Euler decomposition to this product and demand that

$$B_2B_1 = R(-\psi)B_x(-\eta)R(\phi). \quad (20)$$

Note that the corresponding coordinate transformation is

$$\begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} = B_2 B_1 \begin{pmatrix} x \\ y \\ ct \end{pmatrix}, \quad (21)$$

i.e., the boost B_1 takes place before B_2 .

Referring to Eq. (18), Eq. (20) can be rearranged in two different ways,

$$B_2 B_1 = R(\phi - \psi) B_\phi(-\eta) = B_\psi(-\eta) R(\phi - \psi), \quad (22)$$

wherein the product $R(\phi - \psi) B_\phi(-\eta)$ corresponds to a boost followed by a Wigner rotation, while $B_\psi(-\eta) R(\phi - \psi)$ to a boost preceded by the same rotation.

If the boosts B_1 and B_2 are parallel, i.e., their directions are the same or differ by 180° , then it is easy to prove $\psi = \phi$ and $R(\phi - \psi) = I$. On the other hand, $\psi \neq \phi$ implies B_1 and B_2 are not parallel. In other words, the non-parallelism of boosts B_1 and B_2 is a necessary condition for the existence of a non-trivial Wigner rotation. It is also a sufficient condition as will be proved in Sec. IV B.

Rule 2. If the order of the boosts in Rule 1 is exchanged, then the sense of Wigner rotation is reversed.

Proof: Since the boost matrices are all symmetric and rotation matrices all orthogonal, this rule can be proved by taking the transpose of Eq. (22),

$$B_1 B_2 = B_\phi(-\eta) R(\psi - \phi) = R(\psi - \phi) B_\psi(-\eta). \quad (23)$$

For convenience sake, $\psi - \phi$ will be called Wigner angle from now on.

Rule 3. The two-dimensional velocities corresponding to $B_2 B_1$ and $B_1 B_2$ in the previous rules differ only by a Wigner angle.^{8,9}

Proof: According to Eq. (15), the two-dimensional velocities $\vec{V}_{1\oplus 2}$ and $\vec{V}_{2\oplus 1}$ are generated by $B_2 B_1$ and $B_1 B_2$, respectively, via the following formulas:

$$W_{1\oplus 2} = (B_2 B_1)^{-1} \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}; \quad (24)$$

$$W_{2\oplus 1} = (B_1 B_2)^{-1} \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}. \quad (25)$$

Substituting Eq. (20) into Eq. (24) gives us

$$W_{1\oplus 2} = R(-\phi) B_x(\eta) R(\psi) \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} \sinh \eta \cos \phi \\ \sinh \eta \sin \phi \\ \cosh \eta \end{pmatrix}. \quad (26)$$

Similarly, substituting the transpose of Eq. (20) into Eq. (25) leads to

$$W_{2\oplus 1} = R(-\psi) B_x(\eta) R(\phi) \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} \sinh \eta \cos \psi \\ \sinh \eta \sin \psi \\ \cosh \eta \end{pmatrix}. \quad (27)$$

Thus, we find $\vec{V}_{1\oplus 2} = c \tanh \eta [\cos \phi, \sin \phi]$ and $\vec{V}_{2\oplus 1} = c \tanh \eta [\cos \psi, \sin \psi]$, i.e., their magnitudes are the same while their directions differ by a Wigner angle $\psi - \phi$.

The physical meaning of Rule 3 is as follows: Consider three inertial frames K_A , K_B , and K_C . If K_B results from boosting K_A by B_1 and K_C from boosting K_B by B_2 , then a rest observer in K_A finds the two-dimensional velocity of K_C is $c \tanh \eta [\cos \phi, \sin \phi]$. On the other hand, if K_B is from boosting K_A by B_2 and K_C from boosting K_B by B_1 , the same observer will find K_C moving with the same speed but toward the direction $[\cos \psi, \sin \psi]$.

B. Two kinds of Wigner rotations

Assume the spacetime coordinates of an inertial frame K are (x, y, ct) . Applying Eq. (23) to this frame yields two equivalent results. Both of them correspond to the same transformed frame K' with the coordinates (x', y', ct') ,

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} &= B_\phi(-\eta) R(\psi - \phi) \begin{pmatrix} x \\ y \\ ct \end{pmatrix} \\ &= R(\psi - \phi) B_\psi(-\eta) \begin{pmatrix} x \\ y \\ ct \end{pmatrix}. \end{aligned} \quad (28)$$

Although being mathematically equivalent, they convey significantly different physical meanings. According to a rest observer in the original frame K , the first $R(\psi - \phi)$ operates on a rest frame, i.e., K itself, while the second is responsible for rotating a moving frame, which is K boosted by $B_\psi(-\eta)$.

In comparing them, the first rotation is more intriguing. This is because

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} &= B_\phi(-\eta) R(\psi - \phi) \begin{pmatrix} x \\ y \\ ct \end{pmatrix} \\ &\iff B_\phi(\eta) \begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} = R(\psi - \phi) \begin{pmatrix} x \\ y \\ ct \end{pmatrix}, \end{aligned} \quad (29)$$

where $B_\phi(\eta) = R(-\phi) B_x(\eta) R(\phi) = R(-\phi - 180^\circ) B_x(-\eta) \times R(\phi + 180^\circ)$.

The physical meaning of the second line of Eq. (29) is the following. When we apply $B_\phi(\eta)$ to K' to obtain a rest frame, the new frame will differ from the original rest frame by a Wigner angle $\psi - \phi$. In other words, it is possible to engineer a Wigner rotation in two-dimensional Euclidean space since the rotation of a rest frame does not involve the temporal dimension.

IV. WIGNER ANGLE

In principle, for any two $(2+1)$ -dimensional boosts B_1 and B_2 , we can always use Eq. (9) to calculate the corresponding (ϕ, η, ψ) and obtain Wigner angle $\psi - \phi$ for the product $B_2 B_1$. In practice, there is an easier way as shown in the following two examples.

A. Perpendicular case

If the directions of B_1 and B_2 are perpendicular to each other, it is legitimate to specify

$$B_1 = B_x(-\eta_1)$$

and

$$B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \eta_2 & -\sinh \eta_2 \\ 0 & -\sinh \eta_2 & \cosh \eta_2 \end{pmatrix} =: B_y(-\eta_2), \quad (30)$$

where η_1 and η_2 are positive. As discussed in Sec. III A, we may write

$$B_2 B_1 = R(-\psi_p) B_x(-\eta_p) R(\phi_p), \quad (31)$$

where the subscript p denotes ‘‘perpendicular.’’

Now instead of using Eq. (9) to express (ϕ_p, η_p, ψ_p) in terms of those elements in $B_2 B_1$, we calculate the following $(2+1)$ -dimensional velocities:

$$W_{1\oplus 2} = (B_2 B_1)^{-1} \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = B_x(\eta_1) B_y(\eta_2) \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} \sinh \eta_1 \cosh \eta_2 \\ \sinh \eta_2 \\ \cosh \eta_1 \cosh \eta_2 \end{pmatrix}; \quad (32)$$

$$W_{2\oplus 1} = (B_1 B_2)^{-1} \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = B_y(\eta_2) B_x(\eta_1) \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} \sinh \eta_1 \\ \cosh \eta_1 \sinh \eta_2 \\ \cosh \eta_1 \cosh \eta_2 \end{pmatrix}. \quad (33)$$

Comparing these results with Eqs. (26) and (27), we obtain what we are looking for, namely,

$$\begin{aligned} \cos \phi_p &= \sinh \eta_1 \cosh \eta_2 / \sinh \eta_p, \\ \sin \phi_p &= \sinh \eta_2 / \sinh \eta_p; \\ \cosh \eta_p &= \cosh \eta_1 \cosh \eta_2; \\ \cos \psi_p &= \sinh \eta_1 / \sinh \eta_p, \\ \sin \psi_p &= \cosh \eta_1 \sinh \eta_2 / \sinh \eta_p. \end{aligned} \quad (34)$$

The benefit of this method is there is no need to perform any 3×3 matrix multiplication.

Once obtaining Eq. (34), with the help of the identity

$$\sinh^2 \eta_p = (\cosh \eta_1 \cosh \eta_2 + 1)(\cosh \eta_1 \cosh \eta_2 - 1), \quad (35)$$

we are able to calculate the sine and cosine functions of $\psi_p - \phi_p$,

$$\sin(\psi_p - \phi_p) = \frac{\sinh \eta_1 \sinh \eta_2}{\cosh \eta_1 \cosh \eta_2 + 1} > 0; \quad (36)$$

$$\cos(\psi_p - \phi_p) = \frac{\cosh \eta_1 + \cosh \eta_2}{\cosh \eta_1 \cosh \eta_2 + 1} > 0, \quad (37)$$

and find the range of $\psi_p - \phi_p$ is $(0, \pi/2)$.

If we adopt the $\beta - \gamma$ notation from Sec. II A,

$$\beta_i = \tanh \eta_i; \quad \gamma_i = 1/\sqrt{1 - \beta_i^2} = \cosh \eta_i, \quad (38)$$

the results in Eqs. (36) and (37) become

$$\sin(\psi_p - \phi_p) = \frac{\beta_1 \beta_2 \gamma_1 \gamma_2}{\gamma_1 \gamma_2 + 1}$$

and

$$\cos(\psi_p - \phi_p) = \frac{\gamma_1 + \gamma_2}{\gamma_1 \gamma_2 + 1}, \quad (39)$$

and the results in Eq. (34) can be expressed elegantly via the Euler decomposition,

$$B_2 B_1 = \begin{pmatrix} \frac{\beta_1}{\beta_p \gamma_2} & \frac{-\beta_2}{\beta_p} & 0 \\ \frac{\beta_2}{\beta_p} & \frac{\beta_1}{\beta_p \gamma_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_p & 0 & -\beta_p \gamma_p \\ 0 & 1 & 0 \\ -\beta_p \gamma_p & 0 & \gamma_p \end{pmatrix} \times \begin{pmatrix} \frac{\beta_1}{\beta_p} & \frac{\beta_2}{\beta_p \gamma_1} & 0 \\ \frac{-\beta_2}{\beta_p \gamma_1} & \frac{\beta_1}{\beta_p} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (40)$$

where $\gamma_p = \gamma_1 \gamma_2$, and

$$\beta_p = \sqrt{1 - \gamma_p^{-2}} = \sqrt{\beta_1^2 + \beta_2^2 - \beta_1^2 \beta_2^2}. \quad (41)$$

B. General case

If the directions of the two boosts differ by an arbitrary angle $\Theta = 2\theta \in [0, \pi]$, the calculation of Wigner angle becomes complicated and is usually performed using advanced mathematical tools.⁶⁻⁸ With the aid of Euler decomposition, however, it is not necessary to introduce any new tool and the derivation is just a little longer than that of the perpendicular case.

In order to make the most of the symmetry, we consider the following two boosts without losing generality:

$$B_1 = B_{-\theta}(-\eta_1) = R(\theta) B_x(-\eta_1) R(-\theta); \quad (42)$$

$$B_2 = B_{\theta}(-\eta_2) = R(-\theta) B_x(-\eta_2) R(\theta), \quad (43)$$

where η_1 and η_2 are positive. It is obvious that their products are

$$B_2 B_1 = R(-\theta) B_x(-\eta_2) R(2\theta) B_x(-\eta_1) R(-\theta) \quad (44)$$

and

$$B_1 B_2 = R(\theta) B_x(-\eta_1) R(-2\theta) B_x(-\eta_2) R(\theta). \quad (45)$$

We will denote the Euler decompositions of these two products by $R(-\psi) B_x(-\eta) R(\phi)$ and $R(-\phi) B_x(-\eta) R(\psi)$,

respectively. There is no need to provide a subscript for the parameters because the current case is no less general than the one discussed in Sec. III A.

The perpendicular example suggests it might be wise to use the $(2 + 1)$ -dimensional velocities corresponding to B_2B_1 and B_1B_2 as shortcuts, so our calculation begins by writing down the inverses of these two products,

$$\begin{aligned} (B_2B_1)^{-1} &= R(\theta)B_x(\eta_1)R(-2\theta)B_x(\eta_2)R(\theta) \\ &= R(-\phi)B_x(\eta)R(\psi); \end{aligned} \quad (46)$$

$$\begin{aligned} (B_1B_2)^{-1} &= R(-\theta)B_x(\eta_2)R(2\theta)B_x(\eta_1)R(-\theta) \\ &= R(-\psi)B_x(\eta)R(\phi). \end{aligned} \quad (47)$$

Combining Eqs. (46) and (26) leads to

$$\begin{aligned} W_{1\oplus 2} &= R(\theta)B_x(\eta_1)R(-2\theta)B_x(\eta_2)R(\theta) \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \\ &= c \begin{pmatrix} \sinh \eta \cos \phi \\ \sinh \eta \sin \phi \\ \cosh \eta \end{pmatrix}, \end{aligned} \quad (48)$$

which generates the relations

$$\begin{aligned} \sinh \eta \cos \phi &= \cos \theta (\cos 2\theta \cosh \eta_1 \sinh \eta_2 \\ &\quad + \sinh \eta_1 \cosh \eta_2) + \sin \theta \sin 2\theta \sinh \eta_2, \end{aligned} \quad (49)$$

$$\begin{aligned} \sinh \eta \sin \phi &= -\sin \theta (\cos 2\theta \cosh \eta_1 \sinh \eta_2 \\ &\quad + \sinh \eta_1 \cosh \eta_2) + \cos \theta \sin 2\theta \sinh \eta_2, \end{aligned} \quad (50)$$

$$\cosh \eta = \cosh \eta_1 \cosh \eta_2 + \cos 2\theta \sinh \eta_1 \sinh \eta_2. \quad (51)$$

Similarly, Eqs. (47) and (27) together give us

$$\begin{aligned} W_{2\oplus 1} &= R(-\theta)B_x(\eta_2)R(2\theta)B_x(\eta_1)R(-\theta) \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \\ &= c \begin{pmatrix} \sinh \eta \cos \psi \\ \sinh \eta \sin \psi \\ \cosh \eta \end{pmatrix}. \end{aligned} \quad (52)$$

The similarity between Eqs. (48) and (52) enables us to obtain the following two relations from Eqs. (49) and (50) by change of notation,

$$\begin{aligned} \sinh \eta \cos \psi &= \cos \theta (\cos 2\theta \sinh \eta_1 \cosh \eta_2 \\ &\quad + \cosh \eta_1 \sinh \eta_2) + \sin \theta \sin 2\theta \sinh \eta_1, \end{aligned} \quad (53)$$

$$\begin{aligned} \sinh \eta \sin \psi &= \sin \theta (\cos 2\theta \sinh \eta_1 \cosh \eta_2 \\ &\quad + \cosh \eta_1 \sinh \eta_2) - \cos \theta \sin 2\theta \sinh \eta_1. \end{aligned} \quad (54)$$

Now we are equipped to find out $\sin(\psi - \phi)$ and $\cos(\psi - \phi)$, but it is better to make a detour to calculate $\tan[(\psi - \phi)/2]$ first. This is because the identity

$$\tan\left(\frac{\psi - \phi}{2}\right) = \frac{\sin \psi - \sin \phi}{\cos \psi + \cos \phi} \quad (55)$$

eliminates the common $\sinh \eta$ term when calculating the ratio, and the calculation will be simpler.

Substituting Eqs. (49), (50), (53), and (54) to the right hand side of Eq. (55) and ignoring those $\sinh \eta$'s, we find the numerator is proportional to

$$\begin{aligned} &\sin \theta (\cos 2\theta + 1) (\cosh \eta_1 \sinh \eta_2 + \sinh \eta_1 \cosh \eta_2) \\ &\quad - \cos \theta \sin 2\theta (\sinh \eta_1 + \sinh \eta_2) \\ &= \cos \theta \sin 2\theta (\cosh \eta_1 \sinh \eta_2 + \sinh \eta_1 \cosh \eta_2 \\ &\quad - \sinh \eta_1 - \sinh \eta_2), \end{aligned} \quad (56)$$

and the denominator proportional to

$$\begin{aligned} &\cos \theta (\cos 2\theta + 1) (\cosh \eta_1 \sinh \eta_2 + \sinh \eta_1 \cosh \eta_2) \\ &\quad + \sin \theta \sin 2\theta (\sinh \eta_1 + \sinh \eta_2) \\ &= \cos \theta \cos 2\theta (\cosh \eta_1 \sinh \eta_2 + \sinh \eta_1 \cosh \eta_2 \\ &\quad - \sinh \eta_1 - \sinh \eta_2) + \cos \theta (\cosh \eta_1 \sinh \eta_2 \\ &\quad + \sinh \eta_1 \cosh \eta_2 + \sinh \eta_1 + \sinh \eta_2). \end{aligned} \quad (57)$$

Thus, Eq. (55) can be expressed as

$$\tan\left(\frac{\psi - \phi}{2}\right) = \frac{\sin \Theta}{\cos \Theta + X(\eta_1, \eta_2)}, \quad (58)$$

where $\Theta = 2\theta \in [0, \pi]$, and

$$\begin{aligned} X(\eta_1, \eta_2) &= \frac{(\cosh \eta_1 + 1) \sinh \eta_2 + \sinh \eta_1 (\cosh \eta_2 + 1)}{(\cosh \eta_1 - 1) \sinh \eta_2 + \sinh \eta_1 (\cosh \eta_2 - 1)} \\ &= \frac{\coth \frac{\eta_1}{2} + \coth \frac{\eta_2}{2}}{\tanh \frac{\eta_1}{2} + \tanh \frac{\eta_2}{2}} = \coth \frac{\eta_1}{2} \coth \frac{\eta_2}{2}. \end{aligned} \quad (59)$$

Since both η_1 and η_2 are positive, X is always finite and Eq. (58) leads to the conclusion that $\psi - \phi = 0$ implies $\Theta = 0$ or π . Therefore, the non-parallelism of B_1 and B_2 is a sufficient condition for the existence of a non-zero Wigner angle. Once the range of Θ is restricted to $(0, \pi)$, we can deduce from $\infty > X > 1$ that $0 < \psi - \phi < \Theta$.

Next step is to use Eq. (58) to derive the sine and cosine functions of the half angle,

$$\begin{aligned} \sin\left(\frac{\psi - \phi}{2}\right) &= \frac{\sin \Theta}{\sqrt{1 + 2X \cos \Theta + X^2}} \\ &= \frac{\sinh \eta_1 \sinh \eta_2 \sin \Theta}{\sqrt{2(\cosh \eta + 1)(\cosh \eta_1 + 1)(\cosh \eta_2 + 1)}}; \end{aligned} \quad (60)$$

$$\begin{aligned} \cos\left(\frac{\psi - \phi}{2}\right) &= \frac{\cos \Theta + X}{\sqrt{1 + 2X \cos \Theta + X^2}} \\ &= \frac{\cosh \eta + \cosh \eta_1 + \cosh \eta_2 + 1}{\sqrt{2(\cosh \eta + 1)(\cosh \eta_1 + 1)(\cosh \eta_2 + 1)}}, \end{aligned} \quad (61)$$

where $\cosh \eta$ is given in Eq. (51), and

$$X = \frac{\cosh \eta_1 + 1}{\sinh \eta_1} \frac{\cosh \eta_2 + 1}{\sinh \eta_2} \quad (62)$$

has been used in the derivation.

Switching to the $\beta - \gamma$ notation, we obtain the following results:^{8,11}

$$\begin{aligned} \sin\left(\frac{\psi - \phi}{2}\right) &= \frac{\beta_1 \beta_2 \gamma_1 \gamma_2 \sin \Theta}{\sqrt{2(\gamma + 1)(\gamma_1 + 1)(\gamma_2 + 1)}} \\ &= \sqrt{\frac{(\gamma_1 - 1)(\gamma_2 - 1)}{2(\gamma + 1)}} \sin \Theta, \end{aligned} \quad (63)$$

$$\cos\left(\frac{\psi - \phi}{2}\right) = \frac{\gamma + \gamma_1 + \gamma_2 + 1}{\sqrt{2(\gamma + 1)(\gamma_1 + 1)(\gamma_2 + 1)}}; \quad (64)$$

$$\sin(\psi - \phi) = \frac{\beta_1 \beta_2 \gamma_1 \gamma_2 (\gamma + \gamma_1 + \gamma_2 + 1)}{(\gamma + 1)(\gamma_1 + 1)(\gamma_2 + 1)} \sin \Theta, \quad (65)$$

$$\begin{aligned} \cos(\psi - \phi) &= 1 - \frac{(\gamma_1 - 1)(\gamma_2 - 1)}{\gamma + 1} \sin^2 \Theta \\ &= \frac{(\gamma + \gamma_1 + \gamma_2 + 1)^2}{(\gamma + 1)(\gamma_1 + 1)(\gamma_2 + 1)} - 1, \end{aligned} \quad (66)$$

where $\gamma = \gamma_1 \gamma_2 (1 + \beta_1 \beta_2 \cos \Theta)$.

V. CONCLUSION

The Euler decomposition introduced in this paper is the most natural and powerful tool for studying the Wigner rotation problem. Once the mathematics is substantially simplified, the physical meanings are easier to comprehend even for the beginners.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

APPENDIX: VALIDITY OF EULER DECOMPOSITION

In this appendix, we provide a rigorous proof for the validity of Euler decomposition of the reflection-free Lorentz transformation in the $(2+1)$ -dimensional Minkowski space $\mathbb{R}^{2,1}$. Our proof begins with considering a general Lorentz transformation in this space,

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}. \quad (A1)$$

From the discussion in Sec. II A, we know the matrix L is neither symmetric nor orthogonal in general. Instead, it

satisfies the condition $L^\top g L = g$, which implies $\det(L) = \pm 1$ and L is invertible. This condition and its equivalent $L g L^\top = g$ together generate 12 relations among the elements in L (not totally independent of course), wherein the following four will be used in our proof:

$$L_{13}^2 + L_{23}^2 = L_{33}^2 - 1, \quad (A2)$$

$$L_{31}^2 + L_{32}^2 = L_{33}^2 - 1, \quad (A3)$$

$$L_{11}L_{31} + L_{12}L_{32} = L_{13}L_{33}, \quad (A4)$$

$$L_{21}L_{31} + L_{22}L_{32} = L_{23}L_{33}. \quad (A5)$$

To exclude reflections from the transformation Eq. (A1), we first rule out those with $\det(L) = -1$, e.g.,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cosh \eta & 0 & -\sinh \eta \\ 0 & -1 & 0 \\ -\sinh \eta & 0 & \cosh \eta \end{pmatrix},$$

and

$$\begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (A6)$$

Then among the L 's with positive determinant, we have to rule out those with negative L_{33} , which correspond to the transformations containing both spatial and temporal reflections, e.g.,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -\cosh \eta & 0 & \sinh \eta \\ 0 & 1 & 0 \\ \sinh \eta & 0 & -\cosh \eta \end{pmatrix},$$

and

$$\begin{pmatrix} \cos \theta & \sin \theta & 0 \\ \sin \theta & -\cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (A7)$$

Since $L_{33}^2 \geq 1$ according to Eq. (A2) or Eq. (A3), once the negative part is ruled out, the range of L_{33} is reduced to $[1, \infty)$.

In summary, for Eq. (A1) to contain no reflection, the matrix L has to satisfy: (i) $\det(L) = 1$ and (ii) $L_{33} \geq 1$.¹²

Equations (A2) and (A3) also tell us that $L_{33} = 1$ implies $L_{13} = L_{23} = L_{31} = L_{32} = 0$, and accordingly L degenerates to a rotation which corresponds to a special case of Eq. (7),

$$\begin{pmatrix} x' \\ y' \\ ct' \end{pmatrix} = \begin{pmatrix} \cos(\psi - \phi) & -\sin(\psi - \phi) & 0 \\ \sin(\psi - \phi) & \cos(\psi - \phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ ct \end{pmatrix}. \quad (A8)$$

It is obvious that the Euler decomposition is valid for this degenerate case, but with the result that the parameters ϕ and ψ are not unique.

After we exclude all reflections and the degenerate case, a one-to-one correspondence can be built between Eq. (A1) and the transformation matrix in Eq. (7)

$$\begin{pmatrix} \cosh \eta \cos \psi \cos \phi + \sin \psi \sin \phi & \cosh \eta \cos \psi \sin \phi - \sin \psi \cos \phi & -\sinh \eta \cos \psi \\ \cosh \eta \sin \psi \cos \phi - \cos \psi \sin \phi & \cosh \eta \sin \psi \sin \phi + \cos \psi \cos \phi & -\sinh \eta \sin \psi \\ -\sinh \eta \cos \phi & -\sinh \eta \sin \phi & \cosh \eta \end{pmatrix}. \quad (\text{A9})$$

This correspondence may be achieved through the following three steps.

- (1) Since $L_{33} > 1$, we can always find a unique positive η such that $L_{33} = \cosh \eta$, and accordingly $\sinh \eta = \sqrt{L_{33}^2 - 1} > 0$.
(2) Since Eq. (A2) is the same as $(-L_{13})^2 + (-L_{23})^2 = L_{33}^2 - 1$, a unique $\psi \in [0, 2\pi)$ exists such that

$$-L_{13} = \sqrt{L_{33}^2 - 1} \cos \psi = \sinh \eta \cos \psi; \quad (\text{A10})$$

$$-L_{23} = \sqrt{L_{33}^2 - 1} \sin \psi = \sinh \eta \sin \psi. \quad (\text{A11})$$

For the same reason, Eq. (A3) guarantees there is a unique $\phi \in [0, 2\pi)$ such that

$$-L_{31} = \sqrt{L_{33}^2 - 1} \cos \phi = \sinh \eta \cos \phi; \quad (\text{A12})$$

$$-L_{32} = \sqrt{L_{33}^2 - 1} \sin \phi = \sinh \eta \sin \phi. \quad (\text{A13})$$

- (3) So far we have obtained Eq. (9) which allows us to determine a unique set (ϕ, η, ψ) by the matrix elements $L_{13}, L_{23}, L_{31}, L_{32}$, and L_{33} . What remains is to verify that Eq. (9) is consistent with the other four relations,

$$\begin{aligned} L_{11} &= \cosh \eta \cos \psi \cos \phi + \sin \psi \sin \phi, \\ L_{12} &= \cosh \eta \cos \psi \sin \phi - \sin \psi \cos \phi, \\ L_{21} &= \cosh \eta \sin \psi \cos \phi - \cos \psi \sin \phi, \\ L_{22} &= \cosh \eta \sin \psi \sin \phi + \cos \psi \cos \phi. \end{aligned} \quad (\text{A14})$$

This is equivalent to proving the following four equalities:

$$\begin{aligned} L_{11} &= (L_{33}L_{13}L_{31} + L_{23}L_{32})/(L_{33}^2 - 1), \\ L_{12} &= (L_{33}L_{13}L_{32} - L_{23}L_{31})/(L_{33}^2 - 1), \\ L_{21} &= (L_{33}L_{23}L_{31} - L_{13}L_{32})/(L_{33}^2 - 1), \\ L_{22} &= (L_{33}L_{23}L_{32} + L_{13}L_{31})/(L_{33}^2 - 1). \end{aligned} \quad (\text{A15})$$

Taking Eqs. (A3)–(A5) into account, these four equalities can be reduced to the following two:

$$L_{21}L_{32} - L_{22}L_{31} = -L_{13}, \quad (\text{A16})$$

$$L_{31}L_{12} - L_{32}L_{11} = -L_{23}. \quad (\text{A17})$$

In order to prove Eqs. (A16) and (A17), we express the determinant of L as (adopting the summation convention)

$$\det(L) = \frac{1}{3!} \epsilon_{ijk} \epsilon_{pqr} L_{ip} L_{jq} L_{kr} = 1, \quad (\text{A18})$$

then transpose one of the two Levi-Civita symbols to the other side of the equal sign¹³

$$\epsilon_{pqr} L_{ip} L_{jq} L_{kr} = \epsilon_{ijk}. \quad (\text{A19})$$

Since $L^T g L = g$ is equivalent to $g L g = (L^T)^{-1}$, the above equality can be transformed to

$$\epsilon_{pqr} L_{ip} L_{jq} = \epsilon_{ijk} (g L g)_{kr}. \quad (\text{A20})$$

Both Eqs. (A16) and (A17) are special cases of Eq. (A20), the former corresponds to $(r, i, j) = (3, 2, 3)$ and the latter to $(r, i, j) = (3, 3, 1)$. Q.E.D.

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