Collective Hamiltonian in the Generator Coordinate Method with Local Gaussian Approximation

Tsutomu UNE, Akitsu IKEDA* and Naoki ONISHI**

Department of Physics, Tokyo University of Education
Bunkyo, Tokyo 112
*Department of Physics, Tokyo Institute of Technology
Meguro, Tokyo 152
**Institute of Physics, University of Tokyo, Meguro, Tokyo 158

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A collective Hamiltonian is derived by using the local Gaussian overlap approximation in the generator coordinate method. Three Euler angles $\theta, \phi, \psi$, the gauge angle $\chi$, the gap energy $\Delta$ and the deformation parameters $\beta$ and $\gamma$ are employed as generator coordinates. The metric tensor $g_{\mu\nu}(\xi)$ and the second-moment tensor of energy $a_{\mu\nu}(\xi)$ are calculated by introducing infinitesimal operators for the BCS-type trial wave function. The moments of inertia are obtained for an axially asymmetric rotor.

§ 1. Introduction

The generator coordinate method is based on the variational principle. The trial wave function is characterized by a set of parameters called generator coordinates and the basic wave functions with different values of the parameter are not orthogonal. This non-orthogonality sometimes causes serious difficulties in solving the Hill-Wheeler integral equation. The narrowing kernel, a transformation kernel from the non-orthogonal space to an orthogonal space, can remove the difficulties.

An idea has been proposed to use an approximate narrowing kernel which is obtained by assuming that the overlap kernel behaves like a Gaussian function locally. In order to extend the assumption to a many-dimensional parameter space, the metric tensor is introduced, by which a curvilinear space is defined. The assumption of the local Gaussian overlap leads to an approximate narrowing kernel through which a collective Hamiltonian is derived from the Hill-Wheeler integral equation.

The main purpose of the present paper is to develop the local Gaussian approximation to treat nuclear collective motions realistically. In order to describe the rotational motion Euler angles $\theta, \phi, \psi$ are employed. The deformation parameters $\beta$ and $\gamma$ are used for the shape vibrational motion. By treating these five generator coordinates simultaneously, we can fully take into account the rotation-vibration coupling which is strong in soft nuclei. The BCS-type wave function is used to construct the trial wave function and the pairing vibration is taken
into account by adopting the gap energy $J$ as another generator coordinate. Since the single-particle energy density in the vicinity of Fermi surface changes with deformation, the pairing correlation depends on the deformation. This implies a possible strong coupling between the shape vibration and the pairing vibration. Since the BCS-type wave function is not an eigenfunction for the number operator, the gauge angle $\gamma$ is also used for the number projection.

We provide the infinitesimal one-body operator corresponding to each generator coordinate, which makes the calculation of the metric tensor $g_{\mu\nu}(\xi)$ and the second-moment tensor of energy $a_{\mu\nu}(\xi)$ easy. A compact form of the rotational Hamiltonian is obtained.

§ 2. Metric tensor and second-moment tensor of energy

We expand the logarithmic overlap kernel in powers of the difference of a generator coordinate $\delta^a = \xi^a - \xi'^a$ and define the phase gradient $l_a(\xi)$ and the metric tensor $g_{\mu\nu}(\xi)$ as the coefficient of the linear and bilinear terms, where $\xi^a = (\xi^a + \xi'^a)/2$. The higher-order terms are expected to be small when a generator coordinate represents a collective motion which involves many configurations. The ratio of the fourth-order term to the second, for example, is roughly proportional to the inverse of the number of configurations involved. On the basis of this consideration the higher-order terms than the second are neglected in the local Gaussian approximation. Then the overlap kernel is described by

$$ I(\xi, \xi') = \langle \phi(\xi) | \phi(\xi') \rangle \approx \exp \{ il_a(\xi) \delta^a - g_{\mu\nu}(\xi) \delta^\mu \delta^\nu \}, \quad (2.1) $$

where $|\phi(\xi)\rangle$ is a basic wave function specified by a parameter set $\{\xi^a\}$; $\xi^a$ are restricted to be real. We omit the symbol of summation over Greek letters following Einstein’s contraction rule. The phase gradient $l_a(\xi)$ is real due to the normalization condition of $|\phi(\xi)\rangle$. Similar expansion in terms of the difference of generator coordinate $\delta^a$ is applied to the reduced energy kernel:

$$ h(\xi, \xi') = \frac{\langle \phi(\xi) | H | \phi(\xi') \rangle}{\langle \phi(\xi) | \phi(\xi') \rangle} = V(\xi) + ih_a(\xi) \delta^a - \frac{1}{2} a_{\mu\nu}(\xi) \delta^\mu \delta^\nu + \cdots. \quad (2.2) $$

The tensor $a_{\mu\nu}(\xi)$ is referred to as the second-moment tensor of energy. In the present form of the trial wave function, only one of $l_a(\xi)$ related to the number projection does not vanish and it is independent of $\xi$, as will be shown later. In this case we can approximately construct the narrowing kernel following the procedure of Ref. 5). This approximate narrowing kernel transforms the Hill-Wheeler integral equation into a Schrödinger-type differential equation,

$$ \left[ \frac{1}{2\sqrt{g(\eta)}} \left( \frac{1}{i} \frac{\partial}{\partial \eta^\mu} - l_\mu \right) \sqrt{g(\eta)} B^{\mu\nu}(\eta) \left( \frac{1}{i} \frac{\partial}{\partial \eta^\nu} - l_\nu \right) + \frac{1}{2} \frac{1}{\sqrt{g(\eta)}} \left\{ \left( \frac{1}{i} \frac{\partial}{\partial \eta^\mu} - l_\mu \right), \sqrt{g(\eta)} J^\mu(\eta) \right\}_+ + U(\eta) \right] \phi(\eta) = E\phi(\eta). \quad (2.3) $$
The new collective coordinate \( \eta \) is defined by many-body wave functions \( \varphi(\eta) \) which are related to the original basic functions through the narrowing kernel by

\[
|\varphi(\eta)\rangle = \int d\xi N(\xi, \eta) |\phi(\xi)\rangle
\]

and which constitute an orthogonal set. The reciprocal mass tensor \( B^{\mu\nu}(\eta) \) is defined as

\[
B^{\mu\nu}(\eta) = \sum_{a,b} f_a^\mu(\eta) f_b^\nu(\eta) e^{-\frac{1}{2}g^{\mu\nu}(\eta)} A_{ab}(\eta),
\]

where \( f_a^\mu(\eta) \) stands for the square root matrix of the inverse of the metric tensor, \( g^{\mu\nu}(\eta) \), and

\[
A_{ab}(\eta) = f_a^\mu(\eta) f_b^\nu(\eta) a_{\mu\nu}(\eta),
\]

\( \mathcal{J}(\eta) \) is the Laplacian in the curvilinear space. The second term on the left-hand side of Eq. (2·3) comes from the linear term in Eq. (2·2) and \( J^\mu(\eta) \) is given as

\[
J^\mu(\eta) = \sum_a f_a^\mu(\eta) e^{-\frac{1}{2}g^{\mu\nu}(\eta)} h_\nu(\eta).
\]

The potential energy in Eq. (2·3) is given by

\[
U(\eta) = e^{-\frac{1}{2}g^{\mu\nu}(\eta)} (V(\eta) - \frac{1}{2}g^{\mu\nu}(\eta) a_{\mu\nu}(\eta)) - \frac{1}{8} W(\eta)
\]

with

\[
W(\eta) = \mathcal{F}_\mu \mathcal{F}^\mu B^{\nu\nu}.
\]

Here, \( \mathcal{F}_\mu \) is the covariant differential operator with Christoffel's affine connexion.

In order to determine the collective Hamiltonian of the form of (2·3) we proceed to obtain formal expressions for the coefficients in the expansions (2·1) and (2·2). Since \( |\phi(\xi)\rangle \) is normalized the following relations hold:

\[
\langle \phi | \phi \rangle + \langle \phi | \phi \rangle = 0
\]

and

\[
\langle \phi | \phi \rangle + \langle \phi | \phi \rangle + \langle \phi | \phi \rangle + \langle \phi | \phi \rangle = 0,
\]

where \( \phi \) and \( \phi \) are the derivatives of the wave function

\[
|\phi\rangle = \frac{\partial}{\partial \xi} |\phi(\xi)\rangle
\]

and

\[
|\phi\rangle = \frac{\partial^2}{\partial \xi \partial \xi} |\phi(\xi)\rangle.
\]

Using the expansion (2·1) and the relations (2·10), we obtain forms for the phase gradient and the metric tensor

\[
I^\mu(\xi) = \text{Im} \langle \phi^\mu | \phi \rangle
\]
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and

\[ g_{\mu \nu}(\xi) = g_{\nu \mu}(\xi) = \text{Re}\langle \phi_\mu | \phi_\nu \rangle. \]  

(2.13)

The notation \( \langle | \rangle_c \) stands for the connected part defined as

\[ \langle \phi_\mu | \phi_\nu \rangle_c = \langle \phi_\mu | \phi_\nu \rangle - \langle \phi_\mu | \phi \rangle \langle \phi | \phi_\nu \rangle. \]  

(2.14)

Similarly explicit expressions are obtained for \( h_\mu(\xi) \) and the second-moment tensor of energy as follows:

\[ h_\mu(\xi) = \text{Im}\langle \phi_\mu | H | \phi_\nu \rangle_L \]  

and

\[ a_{\mu \nu}(\xi) = a_{\nu \mu}(\xi) = \frac{1}{2} \text{Re}\{\langle \phi_\mu | H | \phi_\nu \rangle_L - \langle \phi_\nu | H | \phi_\mu \rangle_L\}. \]  

(2.15)

(2.16)

In these expressions, \( \langle | H | \rangle_L \) indicates the linked part of the matrix elements of Hamiltonian \( H_L \) which are defined by

\[ \langle \phi_\mu | H | \phi_\nu \rangle_L = \langle \phi_\mu | H | \phi_\nu \rangle - \langle \phi_\mu | \phi \rangle \langle \phi | \phi_\nu \rangle, \]  

(2.17)

\[ \langle \phi_\mu | H | \phi_\nu \rangle_L = \langle \phi_\mu | H | \phi_\nu \rangle - \langle \phi_\mu | \phi \rangle \langle \phi | \phi_\nu \rangle_L - \langle \phi_\mu | H | \phi_\nu \rangle \langle \phi | \phi_\nu \rangle - \langle \phi | H | \phi_\nu \rangle \langle \phi_\mu | \phi_\nu \rangle \]  

(2.18)

and

\[ \langle \phi | H | \phi_\nu \rangle_L = \langle \phi | H | \phi_\nu \rangle - \langle \phi | H | \phi_\nu \rangle \langle \phi_\mu | \phi_\nu \rangle - \langle \phi | \phi_\mu \rangle \langle \phi | \phi_\nu \rangle H | \phi_\nu \rangle_L. \]  

(2.19)

To calculate matrix elements in equations from (2·12) to (2·19), it is convenient to introduce infinitesimal operators defined as

\[ \frac{\partial}{\partial \xi^\mu} | \phi \rangle = -i \bar{P}_\mu | \phi \rangle. \]  

(2.20)

For the BCS-type wave function, \( \bar{P}_\mu \) are one-body operators and they will be given explicitly in the following sections. The metric tensor \( g_{\mu \nu}(\xi) \) and the second-moment tensor \( a_{\mu \nu}(\xi) \) are then described simply as follows:

\[ g_{\mu \nu}(\xi) = \text{Re}\langle \phi_\mu | \bar{P}_\mu \bar{P}_\nu | \phi \rangle_c \]  

(2.21)

and

\[ a_{\mu \nu}(\xi) = \text{Re}\langle \phi_\mu | \bar{P}_\mu H \bar{P}_\nu | \phi \rangle_L - \frac{1}{4} \langle \phi | [\bar{P}_\mu, \{ H, \bar{P}_\nu \}] | \phi \rangle \]  

\[ - \frac{1}{4} i \langle \phi | [\bar{P}_\nu, H] | \phi \rangle, \]  

(2.22)

where

\[ \bar{P}_{\nu, \mu} = \frac{\partial}{\partial \xi^\mu} \bar{P}_\nu. \]  

(2.23)

The normalization conditions (2·10) guarantee that \( \bar{P}_\mu \) and \( \bar{P}_{\mu, \nu} \) are Hermitian.
§ 3. Rotational Hamiltonian and moment of inertia

We employ Euler angles $\omega(\theta \phi \psi)$ as generator coordinates to describe the nuclear rotation. The basic wave function is generated from a wave function independent of $\omega$ as

$$|\phi(\xi)\rangle = \hat{R}(\omega) |\Phi(\zeta)\rangle,$$

where $\hat{R}(\omega)$ is the rotation operator

$$\hat{R}(\omega) = e^{-iJ_1 e^{-i\theta} J_2 e^{-i\phi} J_3}.$$  

$J_k$ ($k=1, 2$ and $3$) are the total angular momentum operators defined in the space-fixed frame of coordinate. The infinitesimal operators for Euler angles are

$$\hat{P}_\theta = \hat{R}(\omega) (J_1 \sin \psi + J_2 \cos \psi) \hat{R}^{-1}(\omega),$$

$$\hat{P}_\phi = \hat{R}(\omega) \{- (J_1 \cos \psi - J_2 \sin \psi) \sin \theta + J_3 \cos \theta \} \hat{R}^{-1}(\omega)$$

and

$$\hat{P}_\psi = \hat{R}(\omega) J_2 \hat{R}^{-1}(\omega).$$

We assume that the wave function $\Phi(\zeta)$ has $d_2$-symmetry:

$$e^{-iL_4 \Phi(\zeta)} = \Phi(\zeta), \quad k=1, 2 \text{ and } 3.$$  

Owing to this symmetry property, the metric tensor has a reduced form

$$g_{\mu\nu}(\xi) = \begin{pmatrix} g^{(R)}_{ij}(\xi) & 0 \\ 0 & g^{(D)}_{rs}(\zeta) \end{pmatrix}.$$  

The suffixes $i$ and $j$ are for Euler angles, and $r$ and $s$ for the other generator coordinates $\zeta$. Since $\hat{R}(\omega)$ is a unitary operator, $g^{(D)}_{rs}(\zeta)$ does not depend on Euler angles. Now $g^{(R)}_{ij}(\xi)$ is of main interest in this section and can be rewritten to be

$$g^{(R)}_{ij}(\xi) = \sum_{k=1}^{3} L_{ik}^k(\omega) L_{j\ell}^k(\omega) \langle \Phi(\zeta) | J_k \rangle \langle \Phi(\zeta) \rangle_c,$$

where $\{L_{ik}^k(\omega)\}$ is the well-known matrix $^9$

$$L(\omega) = \begin{pmatrix} \sin \psi & \cos \psi & 0 \\ -\sin \theta \cos \psi & \sin \theta \sin \psi & \cos \theta \end{pmatrix}.$$  

The row and column of the matrix $g^{(R)}(\xi)$ are arranged in the order of $(\theta \phi \psi)$. Cross terms such as $\langle \Phi(\zeta) | J_2 J_3 \Phi(\zeta) \rangle_c$ do not appear in (3.6) because of the symmetry property (3.4). The matrix element $\langle \Phi(\zeta) | J_2 \Phi(\zeta) \rangle_c$ does no longer contain Euler angles. The square root matrix of the metric tensor $g^{(R)}(\xi)$ is easily obtained as

$$h_{ik}^{(R)k}(\xi) = L_{ik}^k(\omega) \sqrt{\langle \Phi(\zeta) | J_k \rangle \langle \Phi(\zeta) \rangle_c}.$$  

The inverse matrix of $h^{(R)}(\xi)$ is also given as

$$f_k^{(R)j}(\xi) = \frac{1}{\sqrt{\langle \Phi(\xi) | J^k_i | \Phi(\xi) \rangle_c}} K_k^{ij}(\omega),$$

(3.9)

where $K(\omega)$ is the inverse of $L(\omega)$:

$$K(\omega) = \begin{pmatrix} \sin \phi & -\cos \phi \sin \theta & \cos \phi \cot \theta \\ \cos \phi & \sin \phi \sin \theta & -\sin \phi \cot \theta \\ 0 & 0 & 1 \end{pmatrix}.$$  

(3.10)

The second-moment tensor of energy, $a_{\mu\nu}(\xi)$, is also reduced to two parts as the metric tensor:

$$a_{\mu\nu}(\xi) = \begin{pmatrix} a_{11}^{(R)}(\xi) & 0 \\ 0 & a_{22}^{(R)}(\xi) \end{pmatrix},$$

(3.11)

where

$$a_{11}^{(R)}(\xi) = \sum_{j=1}^{3} L_j^k(\omega) L_j^k(\omega) \langle \Phi(\xi) | H J_j^k | \Phi(\xi) \rangle_L.$$  

(3.12)

Here we have made use of the property that $J_k$ commutes with $H$. Substitution of Eqs. (3.9) and (3.12) into (2.6) leads to the expression

$$A_k^{(R)}(\xi) = \delta_{kj} \langle \Phi(\xi) | H J_j^k | \Phi(\xi) \rangle_L.$$  

(3.13)

It should be noticed that the matrix $A_k^{(R)}(\xi)$ is independent of Euler angles. Then the reciprocal mass tensor (2.5) for Euler angles is found to be

$$B^{(R)}(\xi) = \sum_k f_k^{(R)j}(\xi) f_k^{(R)i}(\xi) e^{-i(\omega)\xi} \langle \Phi(\xi) | H J_j^k | \Phi(\xi) \rangle_L \langle \Phi(\xi) | J_j^k | \Phi(\xi) \rangle_c.$$  

(3.14)

For Euler angles, the coefficients $l_1$ and $h_1(\xi)$ given by Eqs. (2.12) and (2.15) vanish due to the symmetry property (3.4). Then we can show that the rotational part of the collective Hamiltonian given in Eq. (2.3) becomes

$$-\frac{1}{2\sqrt{g(\xi)}} \frac{\partial}{\partial \theta^i} \sqrt{g(\xi)} B^{(R)}(\xi) \frac{\partial}{\partial \theta^i} = \frac{1}{2} \sum_k \bar{Q}_k^2.$$  

(3.15)

Here the operators $\bar{Q}_k$ are defined as

$$\bar{Q}_k = -i K_k^{ij}(\omega) \frac{\partial}{\partial \theta^j},$$

(3.16)

where $(\theta^1, \theta^2, \theta^3) = (\theta, \phi, \psi)$. Three operators $\bar{Q}_k$ satisfy the well-known Lie algebraic relation for the angular momentum

$$[\bar{Q}_k, \bar{Q}_l] = -i \delta_{kl}.$$  

(3.17)

When we derive the right-hand side from the left-hand side in Eq. (3.15), we use the relation
The moment of inertia $J_k(\zeta)$ given here resembles the Peierls-Yoccoz moment of inertia:

$$J_k^{-1}(\zeta) = \frac{1}{\langle \Phi(\zeta) | J_k^2 | \Phi(\zeta) \rangle} e^{-2(\beta)\beta \langle \Phi(\zeta) | H J_k^2 | \Phi(\zeta) \rangle} \langle \Phi(\zeta) | J_k^2 | \Phi(\zeta) \rangle.$$  

(3.19)

It can be proved that the potential $U$ defined by (2.8) is independent of Euler angles.

Thus we have shown that in the local Gaussian approximation the rotational Hamiltonian is derived without making use of any explicit representation of the rotational group.

§ 4. Pairing vibration and number projection

In the low-lying states of nuclei the pairing correlation is very important, so that as an intrinsic wave function we employ the BCS-type wave function

$$\phi(\beta \gamma D) = \exp(-i \hat{\Theta}(\beta \gamma D)) |0\rangle,$$

(4.1)

where $\hat{\Theta}$ is an Hermitian operator,

$$\hat{\Theta}(\beta \gamma D) = \sum_\alpha \theta_\alpha(\beta \gamma D) \hat{S}_\alpha(\beta \gamma).$$

(4.2)

The angles $\theta_\alpha(\beta \gamma D)$ of rotation in the quasi-spin space are determined through the relation

$$\tan \theta_\alpha(\beta \gamma D) = \frac{J}{\varepsilon_\alpha(\beta \gamma) - \lambda(\beta \gamma D)}.$$

(4.3)

Here $\varepsilon_\alpha(\beta \gamma)$ is the energy of single-particle state $|\alpha\rangle$ in a deformed potential specified by $\beta$ and $\gamma$, and $\lambda(\beta \gamma D)$ is the chemical potential. $\hat{S}_\alpha(\beta \gamma)$ is a generator of rotation in the quasi-spin space,

$$\hat{S}_\alpha(\beta \gamma) = \frac{i}{2} (a_\alpha^+ a_\alpha - a_\alpha a_\alpha^+),$$

(4.4)

where $a_\alpha^+(a_\alpha)$ stands for a creation (annihilation) operator for single-particle state $|\alpha\rangle$. The intrinsic wave function (4.1) satisfies the symmetry property (3.4) and is invariant under time reversal operation.

The gap energy $J$ is considered as a generator coordinate and the pairing vibration is described by a superposition of $\phi(\beta \gamma D)$ with various values of $J$. A spurious mode originated from number fluctuation of the BCS-type wave function may creep into the resulting wave function. In order to exclude the spurious mode the gauge angle $\chi$ is used as another generator coordinate for the number projection and then the wave function $\Phi(\zeta)$ of the preceding section takes the form

$$\Phi(\zeta) = e^{-i \chi \hat{S}_0(\beta \gamma D)}.$$

(4.5)
We do not perform the number projection exactly, but apply the local Gaussian approximation for \( \chi \) just as for the other generator coordinates.

The infinitesimal operator for the gauge angle is simply given as the number operator \( \hat{N} \).

\[ \hat{P}_x = \hat{N}, \quad (4.6) \]

and that for the energy gap is\(^{4)}\)

\[ \hat{P}_d (\beta \gamma \mathcal{J}) = \sum_a \frac{\partial \theta_a}{\partial \mathcal{J}} \mathcal{S}_a (\beta \gamma), \quad (4.7) \]

where

\[ \frac{\partial \theta_a}{\partial \mathcal{J}} = \sin \theta_a \cdot \frac{1}{\mathcal{J}} \left( \cos \theta_a + \sin \theta_a \frac{\partial \hat{\lambda}}{\partial \mathcal{J}} \right), \quad (4.8) \]

with

\[ \frac{\partial \hat{\lambda}}{\partial \mathcal{J}} = -\sum_a \sin^2 \theta_a \cos \theta_a \sum_a \sin \theta_a \cdot (4.9) \]

It should be noticed that \( \hat{P}_x \) is odd under time reversal operation.

Inserting the operators \( \hat{P}_x \) and \( \hat{P}_d \) given above into Eq. (2.21), we obtain the metric tensor

\[ g_{xx} = \sum_a \sin^2 \theta_a, \quad (4.10) \]

\[ g_{dd} = \frac{1}{4} \sum_a \left( \frac{\partial \theta_a}{\partial \mathcal{J}} \right)^2 \quad (4.11) \]

and

\[ g_{xd} = \text{Re} \left[ \frac{i}{2} \sum_a \frac{\partial \theta_a}{\partial \mathcal{J}} \sin \theta_a \right] = 0. \quad (4.12) \]

It is more generally proved using the time reversal property of \( \hat{P}_d \) that \( \langle \phi_x | \phi_x \rangle_C \), which is just the quantity in the bracket of (4.12), is pure imaginary; moreover, \( \langle \phi_x | \phi_x \rangle_C \) vanishes when the expectation value of the number operator is fixed to the nucleon number. For energy gap \( L_d \) and \( h_d \) vanish due to this time reversal property, while for gauge angle \( l_x \) is the nucleon number of the system under consideration and \( h_x \) is shown to be \( \langle \phi | H \hat{N} | \phi \rangle_C \). We can also obtain the second-moment tensor of energy by substituting Eqs. (4.6) and (4.7) into Eq. (2.22). In particular the non-diagonal element \( a_{xd} \) vanishes since it contains \( \hat{P}_d \) once.

\section{5. Infinitesimal operators for \( \beta \)- and \( \gamma \)-vibrational modes}

As the intrinsic wave function is generated by such a unitary transformation

\(^{4)}\) The infinitesimal operators for \( \mathcal{J} \), \( \beta \) and \( \gamma \) are defined with respect to the intrinsic wave function. For example, \( \hat{P}_d \) is defined as \( \frac{\partial}{\partial \mathcal{J}} \phi (\beta \gamma \mathcal{J}) = -i \hat{P}_d \phi (\beta \gamma \mathcal{J}) \).
as the one in Eq. (4.1), the infinitesimal operator defined by Eq. (2.20) can be generally represented as

$$\tilde{P}_\mu = \int_0^\beta \exp(-i\epsilon \tilde{\Theta}) \frac{\partial \tilde{\Theta}}{\partial \beta^\mu} \exp(i\epsilon \tilde{\Theta}) d\epsilon$$  \hspace{1cm} (5.1)

for $\beta^\gamma = \beta$ and $\beta^\gamma = \gamma$. In this section the suffix $\mu(\nu)$ is used to indicate $\beta$- or $\gamma$-vibration. In order to calculate the derivatives of $\tilde{\Theta}(\beta \gamma J)$ given in (4.2), we need to differentiate the angle $\theta_a(\beta \gamma J)$ and the operator $\tilde{S}_a(\beta \gamma)$. The derivatives of the angle are

$$\frac{\partial \theta_a}{\partial \beta^\mu} = -\sin^2 \theta_a \cdot \frac{1}{4} \left( \frac{\partial \epsilon_a}{\partial \beta^\mu} - \frac{\partial \epsilon_a}{\partial \beta^\mu} \right)$$  \hspace{1cm} (5.2)

with

$$\frac{\partial \epsilon_a}{\partial \beta^\mu} = \sum_a \frac{\partial \epsilon_a}{\partial \beta^\mu} \frac{\partial \epsilon_a}{\partial \beta^\mu} / \sum_a \frac{\partial \epsilon_a}{\partial \beta^\mu}.$$  \hspace{1cm} (5.3)

In general the derivatives of creation operator are written as follows:

$$\frac{\partial a_a^+}{\partial \beta^\mu} = \sum_{a'} F_{a a'}^{\mu} a_{a'}.$$  \hspace{1cm} (5.4)

Usually we determine the single-particle energy and wave function through a generating Hamiltonian $H_0$,

$$H_0 |\alpha; \beta \gamma\rangle = \epsilon_a(\beta \gamma) |\alpha; \beta \gamma\rangle.$$  \hspace{1cm} (5.5)

Differentiating both sides of this equation with respect to $\beta^\mu$, we obtain

$$\frac{\partial \epsilon_a}{\partial \beta^\mu} = \langle \alpha | \frac{\partial H_0}{\partial \beta^\mu} | \alpha \rangle$$  \hspace{1cm} (5.6)

and

$$F_{a a'}^{\mu}(\beta \gamma) = \langle \alpha' | \frac{\partial H_0}{\partial \beta^\mu} | \alpha \rangle / (\epsilon_a - \epsilon_{a'}).$$  \hspace{1cm} (5.7)

The matrix $F_{a a'}$ is anti-Hermitian,

$$F_{a a'}^{\mu} = -F_{a a'}^{\mu*},$$  \hspace{1cm} (5.8)

and if we assume $H_0$ is time reversal invariant, we have

$$F_{a a'}^{\mu} = F_{a a'}^{\mu*}.$$  \hspace{1cm} (5.9)

The derivatives of $\tilde{S}_a(\beta \gamma)$ are obtained using Eqs. (5.4) and (5.7). Then we can get the explicit form of infinitesimal operators $\tilde{P}_\mu$;

$$\tilde{P}_\mu = \sum_a \frac{\partial \theta_a}{\partial \beta^\mu} \tilde{S}_a(\beta \gamma) + 2 \sum_{a, a'} F_{a a'}^{\mu} \left\{ -\sin \left( \frac{\theta_a - \theta_{a'}}{2} \right) \tilde{S}_{a a'} + \left( 1 - \cos \left( \frac{\theta_a - \theta_{a'}}{2} \right) \right) \tilde{N}_{a a'} \right\},$$  \hspace{1cm} (5.10)
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where

\[ \hat{S}_{n\alpha r} = -i \left( a_{n\alpha} a_{n\alpha}^+ - a_{n\alpha}^+ a_{n\alpha} \right) \]

(5.11)

and

\[ \hat{N}_{n\alpha r} = -i \left( a_{n\alpha} a_{n\alpha}^+ - a_{n\alpha}^+ a_{n\alpha} \right). \]

(5.12)

These infinitesimal operators \( \hat{P}_r \) are odd under time reversal operation just as \( \hat{P}_r \).

The metric tensor \( g_{\mu\nu} \) is given by

\[ g_{\mu\nu} = \frac{1}{4} \sum_{a} \frac{\partial \theta_{a}}{\partial \beta^{\mu}} \frac{\partial \theta_{a}}{\partial \beta^{\nu}}, \]

(5.13)

which makes pairing vibrational mode non-orthogonal to \( \beta \)- and \( \gamma \)-vibrational modes. For \( \beta \)- and \( \gamma \)-vibrational modes \( l_{\mu} \) and \( h_{\mu} \) vanish due to the time reversal property of \( \hat{P}_r \). The metric tensor for these modes is obtained in the form

\[ g_{\mu\nu} = \frac{1}{4} \sum_{a} \frac{\partial \theta_{a}}{\partial \beta^{\mu}} \frac{\partial \theta_{a}}{\partial \beta^{\nu}} + \sum_{a \alpha} F_{a\alpha} F_{a^\alpha*} \sin^2 \left( \frac{\theta_{a} - \theta_{a'}}{2} \right). \]

(5.14)

The second-moment tensor of energy, \( a_{\mu\mu} \) and \( a_{\mu
u} \), can be also obtained substituting Eqs. (4.7) and (5.10) into (2.22).

It is difficult to describe analytically the square root matrix of the metric tensor for three vibrational modes, but it is possible to get this matrix by diagonalizing numerically the metric tensor at each point. Then we could obtain the explicit form of the vibrational Hamiltonian.

Both \( g_{\mu\nu} \) and \( a_{\mu\nu} \) vanish because of the time reversal property of \( \hat{P}_r \). Since \( B_{\mu\nu} \) and \( J_{\mu} \) are independent of \( \chi \), the collective Hamiltonian for \( \chi \) is

\[ H_{\chi} = \frac{1}{2} B_{\chi\chi} \left( \frac{1}{i} \frac{\partial}{\partial \chi} - N \right)^2 + J_{\chi} \left( \frac{1}{i} \frac{\partial}{\partial \chi} - N \right). \]

(5.15)

The general solutions of \( H_{\chi} \), which express the pairing rotation, are easily found to be \( e^{inx} \). Although \( n \) can assume any value formally we should take the solution with \( n = N \), which gives vanishing eigenvalue for \( H_{\chi} \). This corresponds to the procedure of number projection. It is then concluded that the spurious mode originated from number fluctuation of the BCS-type wave function is excluded to the order of local Gaussian approximation.

\section*{§ 6. Concluding remarks}

Since the nuclear collective Hamiltonian was introduced by A. Bohr, many authors have investigated the nuclear collective motion using the time-dependent self-consistent method. In the present paper we have shown a different method to derive the nuclear collective Hamiltonian. The present method is a purely quantum-mechanical method and based on the variational principle.
In the generator coordinate method with the local Gaussian approximation we assumed BCS-type as an intrinsic wave function and showed that the collective Hamiltonian consists of the rotational part and the vibrational part. In particular the collective angular momenta and the Lie algebraic relation for them were naturally derived and the moment of inertia for each axis was shown to have the form similar to the one obtained by Peierls and Yoccoz.

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