Two-dimensional induction in a thin sheet of variable integrated conductivity at the surface of a uniformly conducting earth

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Summary. The forward solution of the general two-dimensional problem of induction in a model earth comprising a uniformly conducting half-space covered by a thin sheet of variable integrated conductivity is obtained. Unlike some previous treatments of similar problems, the method presented here does not require the field to be separated into its normal and anomalous parts. Both the $E$- and $B$-polarization modes of induction are considered and in each case the solution is expressed in terms of the horizontal component of the electric field satisfying, on the surface of the conductor, a singular integral equation whose kernel is a well-known analytic function. A recently published solution of the coast effect is included as a special case. The numerical procedure for solving the integral equations is described and some illustrative calculations are presented.

1 Introduction

The difficulties associated with the forward solution of geomagnetic induction problems in which the Earth is allowed to have a laterally varying conductivity are well known. Except in the case of a few simple and highly idealized models, solution by analytical methods is not feasible. Various numerical schemes have been developed for solving two- and three-dimensional models but, especially in the three-dimensional case, they can require large amounts of computer time and storage.

In some models of interest, such as those depicting shallow lying ore bodies, sedimentary layers or an ocean of variable depth near a coastline, the conductivity anomalies are confined to a thin layer near the surface of the Earth. In such cases it is often possible to consider a simplified mathematical model in which the anomalous conducting layer is replaced by an infinitely thin sheet of variable integrated conductivity, underlain by a uniform or horizontally stratified conducting half-space. In this way the lateral variation in conductivity is restricted to the surface plane with the result that a three-dimensional model reduces to one in which the numerical integration need only be taken over the two horizontal

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dimensions, and likewise a two-dimensional model becomes effectively one-dimensional as far as the numerical solution is concerned.

Boundary conditions applicable on the surface of a thin sheet of variable integrated conductivity are based on the fact that the tangential electric field is continuous across the sheet and that the tangential magnetic field is therefore discontinuous by an amount proportional to the sheet current density. They were first formulated for the general problem of electromagnetic induction in thin sheets by Price (1949), in terms of the magnetic scalar potential (the medium on both sides of the sheet was assumed to be non-conducting) and the horizontal gradient of the integrated conductivity. The boundary conditions can also be expressed in terms of the electric field and have been used in this or an equivalent form by Ashour (1971), Weidelt (1971), Schmucker (1971a), and others in recent analytical and numerical work.

When the external inducing field is uniform and horizontal, a vertical magnetic field observed at the surface of the Earth is necessarily associated with an internal anomaly. The corresponding anomalous horizontal field is related to this vertical field by the well-known integral formula for a two-dimensional field devised by Siebert & Kertz (1957). Weaver (1964) showed that this integral was in fact a convolution of the vertical field with a kernel obtained by solving in Fourier transform space the differential equation for the field above the Earth. Schmucker (1971b) was the first to realize that a similar convolution integral also existed on any horizontal plane inside the Earth beneath the conducting anomaly, and that these integrals could be used as boundary conditions in a numerical model calculation. Thus he was able to limit the domain of the numerical solution to the horizontal slab between the surface of the Earth and the level of the deepest part of the anomaly. In the limit, as this slab shrinks to an infinitely thin surface sheet, the boundary conditions reduce to an integral equation to be solved only in the sheet itself. This is the basis of Schmucker's (1971a) method for calculating numerically the field induced in a two-dimensional model containing surface anomalies.

Recently Vasseur & Weidelt (1977) have formulated the three-dimensional problem of electromagnetic induction in thin sheets in terms of a surface integral equation for the electric field and have solved it for a model representing the northern Pyrenean induction anomaly. Their procedure was essentially the same as Weidelt's (1975) surface integral method for a horizontal slab of variable conductivity, reduced to the limiting case of a slab of vanishing thickness. It entailed calculating a tensorial Green's function for the 'normal' structure of the Earth (i.e. the layered structure without the anomalous part of the sheet), and so represented a somewhat different approach to the problem from the one taken by Schmucker (1971a).

In this paper we return to Schmucker's two-dimensional theory, but we have incorporated a number of modifications and extensions which we believe are not only worthwhile improvements in themselves, but which will also facilitate the generalization of the theory to three-dimensional models, thereby providing an alternative method of calculation to the one used by Vasseur & Weidelt. The new features of the theory presented here are as follows. (1) We have not found it useful to separate the field into its normal and anomalous parts: the equations are derived directly in terms of the total field. In particular this means that models in which the conductivity distributions are different at large positive and negative horizontal distances can be handled as a matter of course. (2) The theory for the B-polarization mode is developed alongside that for E polarization. (Schmucker did not publish a method of solution for B-polarization problems.) The B-polarization equations are derived here not, as might be expected, in terms of the magnetic field but, with no extra difficulty, in terms of the horizontal electric field. This is helpful when considering the
generalization to three dimensions, because a three-dimensional model may approach either an $E$- or a $B$-polarization configuration at large horizontal distances. It is clear that both types of behaviour may be accommodated in a three-dimensional theory formulated in terms of the two horizontal components of the electric field. (3) By restricting the model to one in which the thin sheet is underlain by a uniformly conducting half-space, we have been able to express the horizontal magnetic field on the underside of the sheet in terms of a convolution integral whose kernel is given as a well-known analytic function rather than as a Fourier inversion integral which must be evaluated numerically. This permits the derivation to be taken quite far analytically before numerical procedures are introduced. Moreover, the advantages gained by this procedure are not limited to models with a uniform conductor beneath the sheet, for the final integral expressions will also form an essential part of the solution for a model with a layered substructure. However, we do not consider layered substructures in this paper. (4) We have introduced greater flexibility into the numerical calculations by allowing for variable node spacings in the numerical grid, and by including an asymptotic estimation of the contributions to the convolution integrals arising from that part of the surface which lies between the edges of the grid and infinity.

In a very recent paper Fischer, Schnegg \\& Usadel (1978) have solved an idealized model of the coast effect by considering $E$-polarization induction in a perfectly conducting half-plane lying on, and in electrical contact with, a uniform half-space of finite conductivity. Their solution was expressed in the form of an integral equation for the electric field on the surface of the conductor, which they solved numerically. We shall show that in this special case (i.e. when the thin sheet is composed of two half-planes of zero and infinite integrated conductivity respectively), the $E$-polarization integral equation derived in this paper reduces to their solution. Thus in this sense our equations can be regarded as generalizations of the result obtained by Fischer et al. to cover both $E$- and $B$-polarization fields, and also to include sheets with an arbitrarily varying integrated conductivity.

2 Basic equations

Let the electric and magnetic vectors, $E \exp(i\omega t)$ and $B \exp(i\omega t)$ respectively, define a two-dimensional electromagnetic field which is harmonic in time $t$ with angular frequency $\omega$, and which is independent of the coordinate $x$ in a rectangular Cartesian system spanned by the unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$. In a medium of conductivity $\sigma$ and free-space permeability $\mu_0$, the components of the field, denoted in the following form

$$
E = U(y, z)\hat{x} + V(y, z)\hat{y} + W(y, z)\hat{z}
$$

$$
B = X(y, z)\hat{x} + Y(y, z)\hat{y} + Z(y, z)\hat{z}
$$

satisfy the Maxwell equations (in SI units)

$$
\frac{\partial U}{\partial y} = i\omega Z, \quad \frac{\partial U}{\partial z} = -i\omega X
$$

$$
\frac{\partial V}{\partial y} = \frac{\partial W}{\partial z} = i\omega X, \quad \frac{\partial X}{\partial z} = \mu_0\sigma V
$$

$$
\frac{\partial Z}{\partial y} = -\frac{\partial Y}{\partial z} = \mu_0\sigma U, \quad \frac{\partial X}{\partial y} = -\mu_0\sigma W
$$

in which displacement currents have been neglected for the quasi-static approximation. The equations in the left-hand column define the $E$-polarization field; those in the right-hand column define the $B$-polarization field. It follows that in a region of constant conductivity the components all satisfy the same differential equation

$$
\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = i\alpha^2 F
$$

where $\alpha^2 = \mu_0\sigma\omega$ and $F$ denotes any component of the electromagnetic field.
We shall represent the Earth by a conducting half-space \( z > 0 \), and assume that the external magnetic inducing field is uniform and horizontal in the non-conducting region \( z < 0 \). If the lateral variations in the Earth's conductivity are confined to a thin surface layer of depth \( \delta \) we may take the conductivity to be defined by the functions \( \kappa(y, z) \) for \( 0 < z < \delta \), and \( \sigma(z) \) for \( z > \delta \). For simplicity we assume in this paper that \( \sigma(z) \) has the constant value of \( \sigma \) as shown in Fig. 1. It is not difficult to extend the theory to allow for a layered structure beneath the surface layer.

From equation (2.2) it is readily deduced that

\[
[X\hat{x} + Y\hat{y}]_z \bigg|_{z = -\delta}^{z = +\delta} = \mu_0 \int_{-\delta}^{+\delta} \kappa(V\hat{x} - U\hat{y})dz - \frac{i\omega}{\sigma} \frac{\partial^2}{\partial y^2} \int_{-\delta}^{+\delta} Udz.
\]

Since the horizontal electric field components are continuous across the surface \( z = 0 \) we obtain in the limit as \( \delta \to +0 \)

\[
Y(y, +0) - Y(y, -0) = -\mu_0 \tau(y) U(y, 0),
\]

\[
X(y, +0) - X(y, -0) = \mu_0 \tau(y) V(y, 0)
\]

where the integrated conductivity \( \tau(y) \) is defined by

\[
\tau(y) = \lim_{\delta \to +0} \int_{-\delta}^{+\delta} \kappa(y, z)dz.
\]

These are mathematical results which are applicable in practice to a surface layer of finite thickness whenever the horizontal electric field suffers little attenuation through the layer. In many such applications the surface layer (e.g. an ocean) usually has constant conductivity \( k \) but variable depth \( d(y) \): thus in these cases the integrated conductivity, defined mathematically by equation (2.5), is actually calculated from the formula

\[
\tau(y) = kd(y).
\]

For convenience in the algebra to follow we normalize the integrated conductivity to the conductivity of the half-space by defining the quantity

\[
\lambda(y) = \tau(y)/\sigma
\]

which has the dimensions of length.
The conditions under which the mathematical thin-sheet theory may be applied have been examined by Schmucker (1970). He concluded that the thickness of the surface layer should be several times smaller than the skin depth of the surface material and also that the depth of penetration of the field into the underlying conductor should be large compared with the layer thickness. This latter condition can be violated if a good conductor, such as the upper mantle, is located less than a skin depth or two beneath the surface layer. These conditions impose upper bounds on the range of frequencies for which the thin-sheet approximations can be used. Schmucker (1971a) has estimated the limiting frequency values for typical examples in oceanic induction and exploration geophysics.

As \( y \to \pm \infty \) we assume that \( \lambda (y) \to \lambda^* \). Thus the field at infinity can be found by solving the one-dimensional problem of induction by a uniform, horizontal field in a conducting half-space with a surface layer of constant integrated conductivity. It is not difficult to show that

\[
U(\pm \infty, z) \hat{x} + V(\pm \infty, z) \hat{y} = \frac{\omega \sqrt{i} (Y_0 \hat{x} - X_0 \hat{y})}{\alpha (1 + \alpha^2 \sqrt{i})} \begin{cases} 
1 - z \alpha \sqrt{i} (1 + \alpha \lambda^* \sqrt{i}) & (z < 0) \\
\exp (- \alpha z \sqrt{i}) & (z > 0)
\end{cases}
\]  

(2.8)

where \( X_0 \hat{x} + Y_0 \hat{y} \) is the constant magnetic field in the region \( z < 0 \) at infinity.

The \( B \)-polarization equations (2.2) show that the two-dimensional magnetic field is constant when \( \sigma = 0 \). Thus \( X = X_0 \) in the entire region \( z < 0 \). In the \( E \)-polarization case the magnetic field \( Y - Y_0 \) is clearly of internal origin since the constant field \( Y_0 \) contains the \( y \)-component of the inducing field. It is well known (see, e.g. Weaver 1964) that on the surface of the Earth the horizontal field of internal origin is related to the corresponding vertical field by a negative Hilbert transform, i.e.

\[
Y(y, -\infty) - Y_0 = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Z(s, 0)}{y - s} \, ds.
\]

Here the bar on the integral sign denotes a Cauchy principal value. Expressing \( Z \) in terms of \( \partial U/\partial y \) by the Maxwell equation (2.2), and integrating by parts, we obtain

\[
Y(y, 0) - Y_0 = \frac{1}{\pi \omega} \int_{-\infty}^{\infty} \frac{U(s, 0) - U(y, 0)}{(y - s)^2} \, ds
\]

(2.9)

where the constant of integration \(- U(y, 0)\) has been included to ensure the existence of the integrated part at \( s = y \). When these results are combined with equations (2.4), the boundary conditions can be stated, with the aid of Maxwell’s equations (2.2), in terms of the electric-field components as follows

\[
U_2(y, +\infty) = -i \omega Y_0 + i\alpha^2 \lambda(y) U(y, 0) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{U(s, 0) - U(y, 0)}{(y - s)^2} \, ds
\]

(2.10)

\[
V_2(y, +\infty) = \frac{1}{\pi \omega} \int_{-\infty}^{\infty} \frac{U(s, 0) - U(y, 0)}{(y - s)^2} \, ds
\]

(2.11)

Here, and henceforth, we use a numerical subscript on the function symbol to indicate a partial derivative, e.g. \( U_2(y, +\infty) = (\partial U/\partial z)_z = +\infty \). It remains only to solve for the field in the region \( z > 0 \) in order to find expressions for \( U_2(y, +\infty) \), \( V_2(y, +\infty) \) and \( W_1(y, +\infty) \) for substitution in equations (2.10) and (2.11).

3 The field in the conducting half-space

We consider the \( U \)-field first. Suppose for the moment that \( \lambda^* = \lambda^- \) so that \( U(+\infty, z) = U(-\infty, z) = U(\infty, 0) \exp (-\alpha z \sqrt{i}) \), by equation (2.8), and consider the function

\[
u(y, z) = U(y, z) - U(\infty, 0) \exp (-\alpha z \sqrt{i}).
\]

(3.1)
This clearly vanishes as \(|y| \to \infty\) and satisfies the differential equation (2.3). Thus we may take a Fourier transform in \(y\) defined by

\[
\hat{u}(\eta, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(y, z) \exp(\imath \eta y) \, dy
\]

and solve the resulting differential equation to obtain

\[
\hat{u}(\eta, z) = \hat{u}(\eta, 0) \exp(-\gamma z)
\]

as the solution which vanishes as \(z \to \infty\), where

\[
\gamma = (\eta^2 + i\alpha^2)^{1/2},
\]

with \(\Re \gamma > 0\). Since (Erdélyi 1954, section I.4, 26)

\[
\exp(-\gamma z - i\eta y) \, d\eta = \frac{\sqrt{2\pi}}{\Gamma(\gamma)(\gamma^2 + \alpha^2)^{1/2}}
\]

where \(K_1\) is the first-order modified Bessel function of the second kind, the Fourier inversion of (3.3) is given by the convolution theorem as

\[
\hat{u}(\eta, z) = \frac{z \alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} \{U(s, 0) - U(\infty, 0)\} \hat{P}(y - s, z) \, ds
\]

where we have defined

\[
\hat{P}(y, z) = (y^2 + z^2)^{-1/2} K_1(\alpha \sqrt{i}(y^2 + z^2)^{1/2})
\]

A substitution for \(u\) from equation (3.1) yields the final solution

\[
U(y, z) = \frac{z \alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} \{U(s, 0) - U(\infty, 0)\} \hat{P}(y - s, z) \, ds.
\]

The inverse of the transform (3.5) evaluated at \(\eta = 0\) and with the variable \(y\) replaced by \(y - s\) is

\[
\frac{z \alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} P(y - s, z) \, ds = \exp(-\alpha \sqrt{i}).
\]

Thus the solution (3.8) can be written more concisely as

\[
U(y, z) = \frac{z \alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} U(s, 0) \hat{P}(y - s, z) \, ds.
\]

Simple as it may be, this form of the solution is not particularly helpful in evaluating \(U_2(y, +0)\) because the integral is not convergent at \(z = +0\). In fact, it is clear from equation (3.10) that

\[
\lim_{z \to +0} \frac{z \alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} \ldots P(y - s, z) \, ds \equiv \int_{-\infty}^{\infty} \ldots \delta(y - s) \, ds
\]

where \(\delta\) is the Dirac delta function. However, by virtue of the identity (3.9), equation (3.10) can also be rewritten as

\[
U(y, z) = U(y, 0) \exp(-\alpha \sqrt{i}) + \frac{z \alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} \{U(s, 0) - U(y, 0)\} \hat{P}(y - s, z) \, ds.
\]

\[\text{(3.11)}\]
This is the most useful form of the solution. The integral is convergent at \( z = 0 \) when interpreted as a Cauchy principal value, and it can be shown that the integral obtained from it, by differentiating the integrand with respect to \( z \), is uniformly convergent on \( 0 < z < \infty \). Thus \( U_2(y, 0) \) can be obtained from equation (3.11) without difficulty, a straightforward differentiation yielding

\[
U_2(y, 0) = -\alpha \sqrt{i} U(y, 0) + \frac{\alpha \sqrt{i}}{\pi} \lim_{s \to -\infty} \left\{ U(s, 0) - U(y, 0) \right\} P(y - s, 0) ds.
\]  

(3.12)

In addition we note that as \( y \to \pm \infty \) in equation (3.11), \( U(y, z) \) approaches the limiting values \( U(\pm \infty, 0) \exp(-\alpha \sqrt{i}) \) which are the required one-dimensional solutions as given by equation (2.8). This remains true even if \( U(+\infty, 0) \) and \( U(-\infty, 0) \) are unequal, so that it is no longer necessary to assume that \( \lambda^+ = \lambda^- \) as we did in deriving the solution (3.8). This confirms that equation (3.11) is the general solution we have been seeking: it satisfies the correct differential equation (2.3) in \( z > 0 \) since (3.10), from which it is obtained, does; it reduces to \( U(y, 0) \) as \( z \to 0 \), it vanishes as \( z \to \infty \) and it approaches the correct (and possibly different) one-dimensional solutions at \( y \approx \infty \). It follows that the expression for \( U_2(y, 0) \) given by equation (3.12) is also quite generally valid. (In the limit as \( \sigma \), and hence \( \alpha \), becomes vanishingly small in equation (3.12) we retrieve the integral in (2.9) applicable on a surface bordering an insulator.)

The derivation of equation (3.11) can be based on a more formal mathematical treatment either by appealing to generalized function theory, or by considering the function

\[ f(y, z) = U(y, z) - U(y, 0) \exp(-\alpha \sqrt{i}) \]

which tends to zero as \( y \to \pm \infty \) and as \( z \to 0 \), and solving the differential equation it satisfies in \( z > 0 \). However, the heuristic arguments given above serve well enough for our purposes here.

Since \( V \) satisfies the same differential equation and boundary conditions in \( z > 0 \) as \( U \) does, its solution is the same. For \( V \), however, we prefer to use the alternative, and simpler, form of solution given by (3.10) because we shall not be evaluating its normal derivative on \( z = 0 \) directly. Differentiating (3.10) with respect to \( z \) and using the well-known recurrence relation for modified Bessel functions

\[
wK_1(w) = -wK_0(w) - K_1(w)
\]

(3.13)

we find that

\[
V_2(y, z) = \frac{\alpha \sqrt{i}}{\pi} \lim_{s \to -\infty} \frac{(y - s)^2 - z^2}{(y - s)^2 + z^2} P(y - s, z) - z^2 \alpha \sqrt{i} Q(y - s, z)\]

\[
\int_{-\infty}^{\infty} V(s, 0) ds
\]

(3.14)

where

\[
Q(y, z) = K_0[\alpha \sqrt{i} (y^2 + z^2)^{1/2}]
\]

(3.15)

In addition, an expression for the \( y \) derivative of the vertical electric field \( W \) is required for substitution in equation (2.11). Since \( W \to 0 \) as \( y \to \pm \infty \) we may immediately take a Fourier transform of the differential equation (2.3) satisfied by \( W \). The solution of the transformed equation is

\[
W(\eta, z) = -\hat{W}_2(\eta, +0) \exp(-\gamma \eta) / \gamma.
\]

(3.16)

Using the fact that the Fourier inverse of \( \exp(-\gamma \eta) / \gamma \) is \( \sqrt{2/\pi} Q(y, z) \) (Erdélyi 1954, section 1.4, (27)) and applying the convolution theorem we can invert equation (3.16) to obtain

\[
W(y, z) = \frac{1}{\pi} \int_{-\infty}^{\infty} W_2(s, +0) Q(y - s, z) ds.
\]

(3.17)
It follows from the $B$-polarization equations (2.2) that, in a medium of constant conductivity,

$$V_1(y, z) = - W_2(y, z).$$

Substituting this result in equation (3.17), integrating by parts, and noting that

$$Q_1(y, z) = - y \alpha \sqrt{i} P(y, z)$$

we deduce that

$$W(y, z) = - \frac{\alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} (y - s) V(s, 0) P(y - s, z) ds.$$  (3.18)

It is not difficult to show by differentiating this equation with respect to $y$ and by again using the relation (3.13) that

$$W_1(y, z) = V_2(y, z) + \frac{i \alpha^2}{\pi} \int V(s, 0) Q(y - s, z) ds$$  (3.19)

where $V_2(y, z)$ is given by the expression (3.14). Even though we started with the form of solution for $V$ which is not convergent on $z = +0$, it is nevertheless now possible to proceed to the limit $z \to +0$ in equation (3.19) because the integral appearing therein does exist on $z = +0$ and is uniformly convergent. The resulting equation is

$$W_1(y, +0) - V_2(y, +0) = \frac{i \alpha^2}{\pi} \int_{-\infty}^{\infty} V(s, 0) Q(y - s, 0) ds.$$  (3.20)

## 4 Integral equations for the surface field

Integral equations for the electric field in the surface sheet are found by substituting equations (3.12) and (3.20) into equations (2.10) and (2.11) respectively. For $E$ polarization we obtain, after some rearrangement and substitution for $P$ from equation (3.7),

$$\{1 + \alpha \sqrt{i} \lambda(y)\} U(y, 0) = \frac{\omega Y_0 \sqrt{i}}{\alpha} + \frac{1}{\pi} \int_{-\infty}^{\infty} \{U(s, 0) - U(y, 0)\} G(y - s) ds$$  (4.1)

where

$$G(s) = \frac{i + \frac{K_1(|s| \alpha \sqrt{i})}{s^2 \alpha \sqrt{i}}}{|s|}.$$  (4.2)

The horizontal magnetic field on the upper surface of the thin sheet is then given by $Y(y, -0)$ in equation (2.9), while on the underside of the sheet it can be expressed, according to equation (3.12) and the second of Maxwell’s equations (2.2), in the form

$$Y(y, +0) = \frac{\alpha(1 - i)}{\omega \sqrt{2}} \left[ U(y, 0) - \frac{1}{\pi} \int_{-\infty}^{\infty} \{U(s, 0) - U(y, 0)\} \frac{K_1(\alpha \sqrt{i} |y - s|)}{|y - s|} ds \right].$$  (4.3)

Alternatively, given one of the fields, the other may be found from equation (2.4). The vertical magnetic field can be obtained from the computed values of $U$ simply by numerical differentiation with respect to $y$ according to the relevant Maxwell equation (2.2). On the other hand the numerical differentiation can be avoided altogether by noting that $Z$ satisfies...
the differential equation (2.3) and vanishes as \( y \to \pm \infty \) so that its solution in \( z > 0 \) takes the same form as that for \( W \) in equation (3.17). Moreover, since

\[
Y_1(y, z) = -Z_2(y, z)
\]  

(4.4)

the solution can be developed through the same steps that led before to equation (3.18), and hence here to the analogous result (at \( z = 0 \))

\[
Z(y, 0) = -\frac{\alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} \text{sgn} (y - s) Y(s, 0) K_1(\alpha \sqrt{i} |y - s|) \, ds
\]  

(4.5)

where \( \text{sgn} y = y/|y| \). Equations (4.1), (2.9), (4.3) and (4.5) comprise a complete solution of the \( E \)-polarization field on both surfaces of the thin sheet. It is, of course, a simple matter to obtain the field in the half-space \( z > 0 \) also, should this be desired.

After substitution for \( Q \) from equation (3.15) the corresponding integral equation for the surface electric field in the \( B \)-polarization mode reduces to

\[
\lambda(y) V(y, 0) = -\frac{\omega X_0}{\alpha^2} \frac{1}{\pi} \int_{-\infty}^{\infty} V(s, 0) K_0(\alpha \sqrt{i} |y - s|) \, ds.
\]  

(4.6)

Expressions for other field components on the upper and lower surfaces of the thin sheet follow from equations (2.4) and (3.18), namely

\[
X(y, -0) = X_0, \quad X(y, +0) = X_0 + (\alpha^2/\omega) \lambda(y) V(y, 0)
\]  

(4.7)

\[
W(y, +0) = -\frac{\alpha \sqrt{i}}{\pi} \int_{-\infty}^{\infty} \text{sgn} (y - s) V(s, 0) K_1(\alpha \sqrt{i} |y - s|) \, ds.
\]  

(4.8)

The \( W \) field can also be calculated by numerically differentiating the expression for \( X \) in (4.7) according to the final Maxwell equation in (2.2). Equations (4.6) to (4.8) define the complete \( B \)-polarization field except for the vertical electric field on the top surface \( (z = 0) \) which is of little practical interest anyway since it merely indicates the surface density of electric charge in the sheet.

The anomalous field considered by Schmucker (1971a) corresponds to our function \( u(y, z) \) defined by equation (3.1). Following his analysis we would introduce the transfer function,

\[
C(\eta) = -\dot{u}(\eta, 0)/\dot{u}_2(\eta, +0)
\]

which, by equation (3.3), is simply \( 1/\gamma \) for the uniform half-space in our model, and express the anomalous electric field as a convolution of the anomalous sheet current density with a kernel given by the inverse Fourier transform of \( (\gamma + |\eta|^{-1}) \). The integral equation obtained by this procedure is therefore quite different from equation (4.1) and is in a form suitable only for model calculations, whereas equation (4.1) can also be used for the inverse problem of finding the integrated conductivity given the surface profile of the electric field. Equation (4.1) is, in fact, more closely related to the solution obtained by Fischer et al. (1978) for the particular problem of \( E \)-polarization induction in a uniformly conducting earth with a perfectly conducting surface sheet occupying the half-plane \( y > 0 \). In our notation this is equivalent to putting \( \lambda(y) = 0 \) for \( y < 0 \) and \( U(y, 0) = 0 \) for \( y > 0 \). Now by a simple transformation of the variables, equation (4.1) can be written in the form

\[
\{ 1 + \alpha \sqrt{i} \lambda(y) \} U(y, 0) = \frac{\omega Y_0 \sqrt{i}}{\alpha} + \frac{1}{\pi} \int_{-\infty}^{\infty} \{ U(y + s, 0) + U(y - s, 0) - 2U(y, 0) \} G(s) \, ds.
\]
When the conditions stated above for the perfectly conducting sheet in \( y > 0 \) are inserted in this equation we find that the electric field on the half-plane \( y < 0 \) satisfies

\[
U(y, 0) = \frac{\omega Y_0 \sqrt{\mu}}{\alpha \sigma} + \frac{1}{\pi \mu_0 \sigma} \int_{y}^{\infty} \{U(y + s, 0) + U(y - s, 0) - 2U(y, 0)\} G(s) \, ds
\]

\[
+ \frac{1}{\pi \mu_0 \sigma} \int_{-\infty}^{y} \{U(y - s, 0) - 2U(y, 0)\} G(s) \, ds
\]

which is the solution obtained by Fischer et al.

The same model can also be solved in the B-polarization mode, equation (4.6) reducing in this case to \( V(y, 0) = 0 \) on \( y > 0 \) and

\[
\int_{0}^{\infty} V(-s, 0) K_0(\alpha \sqrt{i} |s - |y||) \, ds = -\frac{\pi \chi_0}{\mu_0 \sigma}
\]

on \( y < 0 \). This integral equation can be solved analytically by the Wiener–Hopf technique and leads to the same solution as that obtained by Bailey (1977) and Nicoll & Weaver (1977).

5 The numerical procedure

In order to solve equation (4.1) it is first necessary to express it in discrete form. For this purpose a one-dimensional grid with variable node spacings is defined on the surface \( z = 0 \) by the points \( y_m (m = 1, \ldots, M) \) with \( y_{m+1} - y_m = \kappa_m \) as shown in Fig. 1. The end points of the grid are chosen to be sufficiently remote from the region of variable integrated conductivity that \( y_1 \) and \( y_M \) lie well within the regions of constant conductivity \( \tau^+ \) and \( \tau^- \) respectively. The variation of the integrated conductivity \( \tau(y) \) is represented by the \( M - 1 \) discrete values \( \tau_m (m = 1, \ldots, M - 1) \) assigned to the corresponding \( M - 1 \) regions \( y_m < y < y_{m+1} \) between the grid points. Following Brewitt-Taylor & Weaver (1976) we then define the integrated conductivity at the grid points themselves by the weighted average

\[
\tau(y_m) = \frac{(k_m \tau_m + k_{m-1} \tau_{m-1})/(k_m + k_{m-1})}{(k_m + k_{m-1})}
\]

for \( m = 2, \ldots, M - 1 \), with \( \tau_1 = \tau^- \) and \( \tau_M = \tau^+ \). For simplicity we also write \( F_m \equiv F(y_m, 0) \) for the value of the field component \( F \) at the grid point \( y_m \).

At the node \( y = y_{\mu} (2 < \mu < M - 2) \) the integral in equation (4.1) can be written as the sum

\[
\int_{y_{\mu-1}}^{y_{\mu+1}} \{U(s, 0) - U_{\mu}\} G(y - s) \, ds + \left[ \int_{-\infty}^{y_1} + \int_{y_M}^{\infty} + \sum_{m=1}^{M-1} \int_{y_m}^{y_{m+1}} \right] U(s, 0) G(y - s) \, ds
\]

\[
- U_{\mu} \left[ \int_{-\infty}^{y_{\mu-1}} + \int_{y_{\mu+1}}^{\infty} \right] G(y - s) \, ds.
\]

The main contribution to the sum comes from the first integral embracing the kernel singularity. Our method of dealing with the Cauchy principal value there is based on the original work of Hartmann (1963). Thus, over the range of integration \( y_{\mu-1} < s < y_{\mu+1} \), we approximate the field by the quadratic function represented by the parabola passing through the three points defined by the values \( U_{\mu-1}, U_{\mu} \) and \( U_{\mu+1} \) respectively of the field at \( y_{\mu-1}, y_{\mu} \) and \( y_{\mu+1} \), whereas in every other interval \( y_m < s < y_{m+1} \) \((m \neq \mu - 1, \mu)\)
the field is assumed to have a linear variation from its value \( U_m \) at \( y_m \) to \( U_{m+1} \) at \( y_{m+1} \). Finally, in the regions out to infinity covered by the second and third integrals in (5.2) the asymptotic representation of the electric field derived by Weaver & Brewitt-Taylor (1978) is used. This representation, evaluated at \( y_1 \) and \( y_M \), also yields equations relating \( U_1 \) and \( U_2 \) to \( U(-\infty,0) \), and \( U_M \) and \( U_{M-1} \) to \( U(+\infty,0) \), both of which are known boundary values given by (2.8). The unknowns \( U_1 \) and \( U_M \) can therefore be expressed by these two equations in terms of \( U_2 \) and \( U_{M-1} \) respectively.

When the various approximate expressions for the field \( U(s,0) \) are substituted in the component integrals (5.2) of the integral equation (4.1) we obtain a linear algebraic equation in the \( M-2 \) field values \( U_m \) \((m = 2, \ldots, M-1)\). The coefficients of \( U_m \) are expressed in terms of integrals which can be either integrated directly into logarithmic and Bessel functions, or transformed into a definite integral of the Bessel function \( K_0 \) which has well-known power series expansions suitable for numerical computation (Abramowitz & Stegun 1964, p. 480). The calculation of the coefficients is straightforward but lengthy. We omit the details here.

A linear equation is obtained in this way at each node \( y_m \) \((m = 2, \ldots, M-1)\). The system of \( M-2 \) such equations in the \( M-2 \) unknowns \( U_m \) \((m = 2, \ldots, M-1)\) can then be solved by standard methods such as Gaussian elimination.

The corresponding integral equation (4.6), giving the \( B \)-polarization solution, is much simpler because the integrand does not involve a difference of field values and because a Cauchy principal value of the integral is not required. In fact, for numerical solution it would be sufficient to regard \( V \) as constant in each interval between the grid points. However, it is no more difficult to follow exactly the same procedure as before in order to take advantage of the more accurate quadratic representation of the field around the kernel singularity, where the major contribution to the integral arises. Thus we replace \( V \) by a quadratic function in \( y_m \) \(< s < y_m \) and by a linear function in every other interval. The anomalous field dies off very rapidly in the \( B \)-polarization mode and \( V \) is therefore regarded as constant beyond the edges of the grid. Substituting the appropriate expressions into the various parts of the integral in equation (4.6), broken up as indicated in (5.2), we again obtain a system of \( M-2 \) linear algebraic equations in the \( M-2 \) unknowns \( V_m \) \((m = 2, \ldots, M-1)\). (The values \( V_1 \) and \( V_M \) respectively are given by \( V(-\infty,0) \) and \( V(+\infty,0) \) in this case.)

The magnetic-field components (2.9), (4.3) and (4.5) in \( E \) polarization and the vertical electric field (4.8) in \( B \) polarization are calculated in similar fashion using the numerical solutions for \( U \) and \( V \).

6 Model calculations

In the numerical work it is convenient to use the skin-depth \( \sqrt{2/\alpha} \) of the half-space \( z > 0 \) as the scale of measurement for the length parameters \( y, z, s \) and \( \lambda \) appearing in the integral equations, and to calculate the dimensionless fields \((2\mu_0\sigma/\omega)^{1/2}U/Y_0\) and \((2\mu_0\sigma/\omega)^{1/2}V/X_0\). The computations are then independent of the frequency \( \omega \) and the conductivity \( \sigma \).

As a simple illustration of the theory we have considered two thin sheets of integrated conductivities \( 5 \times 10^3 \) S and \( 10^4 \) S respectively meeting at the origin and underlain by a half-space of conductivity \( 10^{-2} \) S/m. The inducing field is assumed to have a period of \( 10^3 \) s. In dimensionless units this corresponds to choosing the values of \( \lambda^+ \) and \( \lambda^- \) to be \( \pi \) and \( 2\pi \) skin depths respectively. Two forms of the model are considered. In the first (model 1) the transition from \( \lambda^- \) to \( \lambda^+ \) is assumed to be an abrupt change at \( y = 0 \) while in the other (model 2) a linear variation from \( y = -\lambda^- \) skin-depth to \( y = \lambda^+ \) skin-depth is assumed. The
Figure 2. Real and imaginary parts of the (dimensionless) surface fields for E-polarization induction in two thin sheets: (a) the horizontal electric field \((2\mu_0\sigma/\omega)U(y, 0)/\gamma_o\), (b) the horizontal magnetic field \((Y(y, 0) - \gamma_i)/\gamma_o\), (c) the vertical magnetic field \(Z(y, 0)/\gamma_o\). In model 1 the integrated conductivity changes abruptly at \(y = 0\); in model 2 the transition is linear over the interval between \(y = \frac{1}{4}\) skin depths. For an inducing field of period \(2\pi/\omega = 10^3\) s the values \(\lambda^- = 2\pi\) skin depths and \(\lambda^+ = \pi\) skin depths correspond to integrated conductivities of \(10^4\) and \(5 \times 10^3\) S respectively above a half-space of conductivity \(\sigma = 10^{-2}\) S/m.

The curves in Fig. 2 show the horizontal electric, the horizontal magnetic and the vertical magnetic components of the E-polarization field at the surface \(z = -0\). At the sharp boundary in model 1 the real part of the magnetic field behaves in a manner which suggests...
a discontinuity in its horizontal component and a singularity in its vertical component, both features being well-known properties of analytic solutions for models of this type (Weidelt 1971). (Actually, the numerical solution gives a value for the real part of the horizontal magnetic field at the origin, which lies somewhere between the markedly different values on either side of the origin, indicating a steep and sudden change in the field rather than an actual mathematical discontinuity. This is, of course, a necessary consequence of the fact that in a finite numerical grid there is always a small interval between points of different integrated conductivity, over which the field must change continuously.) As is evident from the corresponding curves for model 2, this unrealistic behaviour of the magnetic field near the origin completely disappears when the integrated conductivity varies in a more natural manner from one sheet to the other, a problem which cannot be solved analytically but which is easily handled by the numerical method.

In Fig. 3 we have plotted the $B$-polarization field at the surface $z = 0$, comprising the horizontal electric field together with the horizontal magnetic and the vertical electric fields on the underside ($z = +0$) of the thin sheet. Again the apparent discontinuity in the electric field at the sharp boundary in model 1 is replaced by a smooth variation when the more realistic model 2 is used.

As a more practical example we have considered $E$-polarization induction for the model shown in Fig. 4. It represents an idealized cross-section of Vancouver Island with the sloping continental shelf out to the Pacific Ocean on one side, and the shallow strait separating the island from the mainland on the other. The frequency of the inducing field was taken to be 1 cycle/hr. The problem was solved in two ways. Firstly a full two-dimensional model of the conductivity distribution incorporating the atmosphere, the oceans of variable depth and uniform conductivity $4 \, \text{S/m}$, and the underlying medium of conductivity $10^{-3} \, \text{S/m}$, was solved by the finite-difference method of Brewitt-Taylor & Weaver (1976). Secondly, the method described in this paper was used in which the ocean was replaced by a thin sheet of variable integrated conductivity calculated according to equation (2.6), and underlain by a uniform half-space of conductivity $10^{-3} \, \text{S/m}$. The same horizontal grid points were used in both methods of calculation.

The amplitudes of the horizontal and vertical components of the surface magnetic field computed by the thin-sheet method on a $59 \times 1$ grid are plotted in Fig. 4. Curves obtained
Figure 3. Real and imaginary parts of the dimensionless surface fields for B-polarization induction in models 1 and 2 described in Fig. 2: (a) the horizontal electric field $(2\mu_0\sigma/\omega)^{1/2} V(y, 0)/X_0$, (b) the horizontal magnetic field $X(y, +0)/X_0$ (note that the negative imaginary part is plotted), (c) the vertical electric field $(2\mu_0\sigma/\omega)^{1/2} W(y, +0)/X_0$ (note that the negative real part is plotted).
Induction in a thin sheet

\[ \text{by the finite-difference calculation on a } 59 \times 40 \text{ grid are shown in the same diagram, and they clearly confirm the accuracy of the results obtained by the method developed in this paper. Thus the use of the thin-sheet approximation in oceanic induction problems of this type is well justified by the considerable saving in computing time (a reduction from 15 to 1 min for this particular calculation on an IBM 370/148 computer) which is gained without an appreciable loss of accuracy.}

The amplitude of the horizontal magnetic field on the underside of the thin sheet is also plotted in Fig. 4. This represents the field that would be recorded by a sea-floor magnetometer. The characteristic attenuation of the horizontal magnetic field by the seawater, even in shallow parts of the ocean, which is in sharp contrast to the negligible attenuation of the electric and vertical magnetic fields is clearly visible in the diagram. Enhancements of the vertical magnetic field, familiar from coast effect problems, can also be seen in Fig. 4. The major coast effect is some distance offshore over the continental shelf, but there are secondary effects on the left-hand side of the island bordering the deep ocean, and also at the mainland coast.

It is hoped that this paper is a forerunner of a complete three-dimensional treatment of the thin-sheet problem based on the methods developed here. Work on this extension is in progress.

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References


