Letters to the Editor

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Linear and Nonlinear Critical Slowing Down in the Kinetic Ising Model

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Recently, it was presented that the critical singularity of the linear and nonlinear relaxation time may be different in the kinetic Ising model by using the mean field approximation (MFA). Both critical singularities have been asserted to be identical in ergodic systems so far by an intuitive expectation. In this letter, as a further example we consider the kinetic Ising model on a Bethe lattice. Such a condition makes the high-temperature-expansion method simple, and furthermore the equilibrium properties are well understood, ,\alpha=0, \beta=1/2, \gamma=1, for usual critical indices. It will be desirable for our purpose to use the large coordination number z. In our case we set z=6.

From now on we adopt the same notations as in Ref. 4) except that \tau=1, m=1. Thus we write the master equation as

\[
\frac{d}{dt} P(\sigma_1, \ldots, \sigma_N, t) = - \sum_j W_j(\sigma_j) \times P(\sigma_1, \ldots, \sigma_N, t) + \sum_j W_j(-\sigma_j) \times P(-\sigma_1, \ldots, t). \tag{1}
\]

For our Hamiltonian \( H_0 = -J \sum_{\text{nearest-neighbors}} \sigma_i \sigma_j \), the transition probability \( W_j(\sigma_j) \) is assumed to be

\[
W_j(\sigma_j) = (1/2) \{1 - \sigma_j \tanh(K \sum_{\text{nearest-neighbors}} \sigma_k)\} \tag{2}
\]

with \( K=J/kT \). Here \( \sum_k \) denotes the sum over the nearest-neighbors of the j-th site.

First, consider the linear relaxation time of the magnetization \( \tau_{(l.2)} \) defined by

\[
\tau_{(l.2)} = \int_0^\infty \frac{\langle MM(t) \rangle}{\langle M^2 \rangle} dt
\]

\[
= \int_0^\infty \frac{\langle Me^{-\tau_1 M} \rangle}{\langle M^2 \rangle} dt
\]

\[
= \frac{\langle M e^{-\tau_1 M} \rangle}{\langle M^2 \rangle} \equiv \frac{N}{\langle M^2 \rangle} A, \tag{3}
\]

where \( M=\sum_j \sigma_j, \langle \cdots \rangle \) is the canonical average over the Hamiltonian \( H_0 \), and the time-displacement operator \( L \) is written as \( L=\sum_j W_j(\sigma_j) (1-P_j) \) with \( P_j \) denoting the spin-flip operator: \( P_j \sigma_h = -\sigma_j \delta_{jk} + \sigma_h \times (1-\delta_{jk}) \). Note that \( \langle M^2 \rangle \sim e^{-\tau}, \tau=1 \) for the critical region. Now, as in the case of the square lattice, we can use the high-temperature-expansion method. From the property of the Bethe lattice, it may be easily obtained that \( \langle \sigma \sum_{j=1}^N \sigma_j \rangle = 1 + 6 \sum_{n=1}^\infty 5^{n-1} \kappa^n, \langle \sigma \sum_{j=1}^N \sigma_j \sigma_k \sigma_h \rangle = 3\kappa^2 + 16\kappa^3 + \cdots \), etc., where \( \kappa=\tanh K, j_1, j_2, j_3 \) the nearest-neighbors of the j-th site. Therefore it follows that \( A \) on the right-hand side of Eq. (3) can be expanded as

\[
A = 1 + 12\kappa + 102\kappa^2 + 732\kappa^3 + 4782\kappa^4 + \cdots
\]

\[
+ 29532\kappa^5 + (527666/3)\kappa^6 + \cdots. \tag{4}
\]

By using the ratio method as the successive approximation we obtain

\[
A_{(l.1)} \sim 1.40, 1.40, 1.31, 1.23, 1.18, 1.15, \ldots
\]

where \( \tau_{(l.1)} \sim \varepsilon^{-\mu_{(l.1)}} \). We have used \( \kappa_c=1/5 \).
for the critical temperature. Then we conjecture that $\Delta^{t_{\mathcal{L}}} = 1.0 \sim 1.1$, and so that $\Delta^{t_{\mathcal{L}}} \approx \gamma$.\(^7\)

Next, consider the relaxation time $\tau^{(n_{\mathcal{L}})}$ for the nonlinear response in which the system is in equilibrium described by an initial Hamiltonian $H_i$ and at $t=0$, the system is abruptly changed into a condition described by the Hamiltonian $H_0$.

$$
\tau^{(n_{\mathcal{L}})} = \int_0^{\infty} \frac{\langle M(t) \rangle_i}{\langle M \rangle_i^t} dt
$$

$$
= \langle \int_0^{\infty} \frac{M(t)}{\langle M \rangle_i^t} dt \rangle_i \equiv B, \quad (6)
$$

where $\langle \cdots \rangle_i$ denotes the canonical average over initial states with the Hamiltonian $H_i$.\(^2\) For simplicity, the initial state is assumed to be completely ferromagnetic.\(^5\) In a similar manner, we then obtain

$$
B = 1 + 6\kappa + 36\kappa^2 + (518/3)\kappa^3 + 776\kappa^4
$$

$$
+ (159734/45)\kappa^5 + (2235764/135)\kappa^6 + \cdots \quad (7)
$$

and

$$
\Delta^{(n_{\mathcal{L}})} \approx 1.20, 1.40, 0.88, 0.60, 0.57, 0.60, \cdots \quad (8)
$$

with $\tau^{(n_{\mathcal{L}})} \sim e^{-\Delta^{(n_{\mathcal{L}})}}$, i.e., we conjecture that $\Delta^{(n_{\mathcal{L}})} \approx 0.5 \sim 0.6$. From (5) and (8), we get $\Delta^{t_{\mathcal{L}}} > \Delta^{(n_{\mathcal{L}})}$ beyond the range of error. The exponents coincide nearly with the exponents of the MFA,\(^1\) as is expected.

Furthermore, it was also shown quite recently that the difference $\Delta^{t_{\mathcal{L}}} - \Delta^{(n_{\mathcal{L}})}$ is equal to $\beta$ from the dynamic scaling hypothesis.\(^8\) In the MFA, the difference $\Delta^{t_{\mathcal{L}}} - \Delta^{(n_{\mathcal{L}})}$ exactly coincides with the dynamic scaling prediction. The present result is nearly consistent with the dynamic scaling prediction, i.e., $\Delta^{t_{\mathcal{L}}} - \Delta^{(n_{\mathcal{L}})} \approx 0.5 = \beta$. In the square lattice with small $\beta (=1/8)$, on the other hand, it is difficult to ascertain the dynamic scaling prediction. In fact, the results of the high-temperature-expansion method\(^2\) and the Monte Carlo method\(^9\) show that $\Delta^{t_{\mathcal{L}}} \approx \Delta^{(n_{\mathcal{L}})}$. The cubic-lattice case may be, however, hopeful, since $\beta \approx 1/3$.

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2) M. Suzuki, Inter. J. Magnetism 1 (1971), 123.\(^2\)
3) R. J. Glauber, J. Math. Phys. 4 (1963), 294.\(^3\)