Canonical Quantization of Non-Abelian Gauge Fields

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A canonical quantization is applied in an unambiguous way to a non-abelian gauge field under a condition of the Coulomb gauge. The ambiguity with respect to the order of quantized field variables is avoided by performing a transition from the classical to the quantum theory before removing the constraint imposed on the state vectors. The constraint which causes a big trouble with the violation of the unitarity of the S-matrix is transformed into a simple one which shows that the physical state vectors should be functionals of the transversal field variables. The Hamiltonian derived after this transformation becomes a complicated functional of the transversal fields.

The S-matrix is easily derived from this Hamiltonian by following the conventional procedure. The well-known trouble due to the time derivatives of field variables in the decomposition of T-products in the S-matrix is overcome by introducing a strange singular term to the Hamiltonian which is identical with the DeWitt term in the Coulomb gauge. A brief discussion is made on the relationship between our result and those already published in other papers.

§ 1. Introduction

Since Yang and Mills wrote an article on the classical non-abelian gauge fields, a large number of papers have been published on the quantization of the fields of this kind. Among these papers, DeWitt's article is worth notice in the sense that he exploited the third method of quantization which looks quite suitable for the system with the invariance under transformations of a non-abelian gauge group even though there remain several unclarified points in the details of his theory. Among many merits of his approach, it is very interesting that his theory showed the necessity of adding to the S-matrix of the gauge fields an infinite set of strange terms which corresponds to a set of closed loops in a graphical representation.

Popov and Faddeev succeeded to derive the DeWitt term by making use of the 2nd method of quantization, namely, Feynman's method of functional integration. They showed that the DeWitt term is closely related with the functional integration with respect to the redundant components of field variables which do not give any essential contribution to the action-integral owing to the gauge-invariance. Their results suggest that it seems not easy to derive the DeWitt term in the framework of the first method of quantization, namely, the conventional quantum field theory.

Even nowadays the canonical quantization is still regarded as a most well-established formalism that gives an unambiguous prescription for the transition from
the classical to the quantum physics at least when we deal with classical linear fields having a non-singular Hamiltonian. If the field equations are, however, non-linear, namely, the coefficients of time-derivatives of field variables in the field equations depend upon the field variables, an ambiguity always occurs with respect to the order of field operators when we try to quantize this system. This ambiguity is one of the obstacles which make it difficult to apply the conventional method of quantization to the non-abelian gauge fields. It is seen, however, that this ambiguity prohibits also applications of both Feynman's and DeWitt's methods of quantization in an unambiguous way.

Besides the ambiguity mentioned above, the gauge invariance of the action integral gives rise to the well-known mathematical difficulty. Namely, it is impossible to define a canonical set of independent field variables owing to some functional relationship between canonically conjugate momenta. In order to apply the method of canonical quantization to the gauge fields, this relationship is treated as the constraints imposed on the state vectors. These constraints, however, cause another trouble connected with the violation of unitarity of the $S$-matrix. Because of these difficulties, almost all the articles so far published which were founded on the canonical quantization are more or less unsatisfactory and cannot be compared with the results given by DeWitt or with those of Popov and Faddeev in order to examine the relationship between DeWitt's approach and the conventional one or to confirm the equivalence of Feynman's formalism with the ordinary quantum theory in case of the non-abelian gauge fields.

The aim of the present paper is to present a reliable quantum theory of a non-abelian gauge field based on the canonical quantization and to investigate the equivalence of the traditional method of quantization with other two formalisms.

The following is the outline of our approach. We begin with the familiar classical Lagrangian (2.1). In course of transition from the classical to the quantum theory we do not have any ambiguity at all with respect to the order of operators. The difficulty of the vanishing canonical conjugate momenta is avoided by introducing to the Lagrangian a new additional term with a Lagrangian multiplier $\phi_n$. In order to eliminate this additional field $\phi_n$, some constraints are imposed on state vectors. By means of a non-singular operator, the above-mentioned constraints are transformed into new ones which show that the physical states depend only upon the transversal components of field variables. Corresponding to the transformation stated above, the total Hamiltonian changes its form into a complicated one which is expressed as a sum of spatial multiple-integrals of non-local and non-linear functions of the transversal components.

The unitary $S$-matrix can be easily derived in a form of a series of the $T$-products of the interaction Hamiltonian. The non-linearity of the Hamiltonian gives rise to some singular extra terms when the $T$-products are decomposed by Wick's theorem. These extra terms, however, can be absorbed by a change of the coefficients of some particular terms of the Hamiltonian together with an addition of
a new singular term to the Hamiltonian. This new term corresponds to a set of closed loops and is nothing but the DeWitt term.

The result obtained in the present article seems to suggest that there may be some inequivalence between the method of quantization of the canonical form and that of Feynman and, probably, that of DeWitt, when these methods are applied to non-abelian gauge fields.

The Lorentz-invariance of the whole theory and the discussion on the renormalization will be investigated in other places in the near future.

§ 2. Classical Lagrangian and quantization

As a representative of the non-abelian gauge fields, let us consider the field $A_{\mu a}(x), (\mu = 0, 1, 2, 3; a = 1, 2, 3)^\dagger$ associated with the group $SU(2)$. The Lagrangian of this field is given by

$$ L = -\frac{1}{4} F_{\mu \nu a} F^{\mu \nu a} - \phi_a \cdot \partial_\mu A^{\mu a}, \quad (2.1) $$

where the last term has been added in order to avoid the difficulty of vanishing canonical-conjugate momenta. The scalar field $\phi_a$ plays a role of the Lagrangian multiplier. The field equations derived from (2.1) are

$$ (F_\nu)_{ab} F^\nu_b = \delta^a_\nu \phi_a \quad (2.2) $$

and

$$ \partial_\mu A^{\mu a}_a = 0 \quad (2.3) $$

(cf., (b) and (c) in Appendix I).

The action integral

$$ I = \int L d^4 x $$

is invariant under an infinitesimal gauge transformation:

$$ A_{\mu a}(x) \rightarrow A'_{\mu a}(x) = A_{\mu a}(x) + (F_\mu)_{ab} \lambda_b, \quad \phi_a \rightarrow \phi'_a(x) = \phi_a + g(\phi)_{ab} \lambda_b, $$

provided that the infinitesimal arbitrary function $\lambda_a(x)$ satisfies

$$ (F_\nu)^a_{ab} \partial_\nu \lambda_b = 0. \quad (2.4) $$

The gauge invariance of $I$ leads to the relation

$$ (F_\nu)_{ab} \partial_\nu \phi_b = 0, \quad (2.4) $$

which can also be derived directly from (2.2) and the identity

$$ (F_\nu F^\nu)_{ab} F^{\mu \nu a} = 0. \quad (2.5) $$

\dagger The explanations or the definitions of the notation are given in Appendix I. About the indices used above, cf., (a) in Appendix I.
(2.4) is rewritten as
\[ \square \phi_a + g(\hat{A}_\alpha \partial^\xi \phi)_a = 0 \] 
Therefore, if we put
\[ \phi_a = 0, \quad \partial \phi_a / \partial x^a = 0 \] 
at some instant, say \( t=0 \), then (2.4)' leads to
\[ \phi_a(x, t) = 0. \quad (t > 0) \]
This result changes (2.2) into the conventional form
\[ (\mathcal{F}_a \mathcal{F}^a)_a = 0. \]

Among the canonically conjugate variables \( H_a^a, A_{\alpha a}, \) and \( p_a, \phi_a \), there are relations:
\begin{align*}
H_a^k &= \delta L / \partial (\partial A_{k a} / \partial x^a) = F_{\theta r a}, \quad (k=1, 2, 3) \\
H_a^\theta &= \delta L / \partial (\partial A_{\theta a} / \partial x^a) = \phi_a, \\
p_a &= \delta L / \partial (\partial \phi_a / \partial x^a) = 0.
\end{align*}

The classical Hamiltonian is given by the well-known prescription:
\[ H = \int \left[ \frac{1}{2} H_a^k H_a^k + \frac{1}{4} F_{\theta r a} F_{\theta r a} - \Lambda_{\alpha a} \{ \partial_i H_a^k + g(\hat{A}_\alpha)_a \} H_b^k \right] \\
+ H_a^\theta \partial x A_{\theta a} + p_a \xi_a \right] d^3 x, \quad (2.7)\]
where \( \xi_a(x) \) is an undetermined function of \( x \) which stands for the time derivative of \( \phi_a \) and may depend upon other canonical variables. But if we put \( \xi_a = 0 \), the field \( \phi_a \) completely disappears from the objects of our consideration. The remaining terms of (2.7) have no ambiguities with respect to the order of field variables even if the \( H \)'s and the \( A \)'s are regarded as a canonical set of quantum operators which are assumed to satisfy the following commutation relations:
\[ [H_a^k(x, t), A_{\alpha b}(x', t)] = -i \hbar \delta_{\alpha}^c \partial_{\beta} \delta(x - x'), \]
\[ [\text{all other commutators}]. = 0. \]
From now on, all the quantities in the present section are assumed to be Heisenberg operators governed by the commutation relations written above.

The Heisenberg equations derived from (2.7) are transformed into the following form
\[ (\mathcal{F}_a)_b F_{b}^a = \partial^\xi H_a^\theta, \quad (2.8) \]
\[ \partial_a A_{\alpha a} = 0. \quad (2.9) \]
Since the identity \((2\cdot5)\) holds also in the quantum theory, the operator equation \((2\cdot8)\) gives the equation
\[
(\Gamma^a)_{ab} \partial_b \Pi^0_a = \Box \Pi^0_a + g (\hat{A}^a)_{ab} \partial_b \Pi^0_a = 0.
\]
\((2\cdot8)\) and \((2\cdot10)\) show that \(\Pi^0_a\) in the quantum theory just corresponds to the \(\phi_a\) in the classical theory. Therefore it is necessary to impose the following constraints on state vectors at \(t=0\):
\[
\Pi^0_a \cdot e^0 = 0, \quad (2\cdot11)
\]
\[
\partial \Pi^0_a \cdot e^0 = 0. \quad (2\cdot12)
\]
These constraints\(^*\) are the quantum counterparts of the classical constraints \((2\cdot6)\).
It is inevitable to impose these constraints in order to guarantee the positive definiteness of the Hamiltonian which is indispensable for the existence of the vacuum state. \((2\cdot12)\) can be rewritten as
\[
(\Gamma_a)_{ab} \Pi^0_b \cdot e^0 = 0
\]
owing to the field equation. It is easily seen that Eq. \((2\cdot10)\) maintains the validity of \((2\cdot11)\) and \((2\cdot12)'\) at any time \(t>0\). The compatibility of these constraints can be confirmed by a direct computation of the commutation relations of the left-hand side of these constraints or by taking into account the group property of the gauge transformations and the fact that
\[
J = \int (\Pi^0_a \cdot (\Gamma_a)_{ab} \phi^0_b - \lambda_a \cdot (\Gamma_a)_{ab} \Pi^0_b) \, dx
\]
is the generator of the infinitesimal gauge transformation.

§ 3. Constraints and gauge condition

In the present section and § 4 we shall employ the Schrödinger representation for the sake of convenience. Therefore the constraints \((2\cdot11)\) and \((2\cdot12)'\) are replaced by
\[
\Pi^0_a (x) \varphi^0 (t) = 0, \quad (3\cdot1)
\]
\[
(\Gamma_a)_{ab} \Pi^0_b (x) \varphi^0 (t) = 0. \quad (3\cdot2)
\]
It may be needless to notice the compatibility of these constraints with the Schrödinger equation:
\[
\ii \hbar \partial \varphi^0 (t) / \partial t = \overline{H} \cdot \varphi^0 (t),
\]
where the total Hamiltonian \(\overline{H}\) is

\(^*\) There is no contradiction between these constraints and the commutation relations. Cf., Appendix III.
Let it be assumed that the operator $F_{k\ell,a}$ is given a diagonal representation. Then, (3·1) can be written as
\[ -i\hbar \delta T(t)/\delta A_{b,a}(x) = 0 , \tag{3·1}' \]
which shows that the physical state $\Psi$ does not depend on $A_{b,a}$. The constraint (3·2) has, however, a too complicated appearance to learn a simple meaning from this constraint.

In order to change (3·2) into a simple expression, let us transform $\Psi(t)$ into $\Psi'(t) = T^{-1}\Psi(t)$ (3·4) by means of a non-singular operator $T$. Before giving the definition of $T$, however, some preliminary arrangements should be made. Let $\Pi^k_a$ and $A_{k,a}$ be decomposed into a sum of longitudinal and transversal parts:
\[ \Pi^k_a = \pi^k_a + \partial_k V_a, \quad A_{k,a} = a_{k,a} + \partial_k \tilde{A}_a , \]
where $\pi^k_a$ and $a_{k,a}$ are transversal vector components. The commutation relations between various quantities are given by
\[ [\pi^k_a(x), a_{\ell,b}(x')] = -i\hbar \delta_{ab} G_{k\ell}(x-x') , \]
\[ [V_a(x), A_b(x')] = -i\hbar \delta_{ab} \mathcal{J}^{-1}(x-x') , \tag{3·5} \]
\[ \text{[all other combinations]} = 0 . \]
The definitions of $\mathcal{J}^{-1}$ and $G_{k\ell}$ are given in (d) in Appendix I.

Let us give a diagonal representation to the operator $A_a(x)$ and define the operator $T$ as follows:
\[ \mathcal{G}_a(x) \equiv \left\{ i\hbar \int d^2x' (a, x) 1 + g\tilde{A} \partial \mathcal{J}^{-1} b, x' \right\} \frac{\partial}{\partial A_b(x')} + g(\tilde{A}_a(x))_{ab} \pi^k_a(x) \right\} T = 0 , \tag{3·6} \]
and
\[ \frac{\partial T}{\partial A_{b,a}(x)} = 0 , \tag{3·7} \]
where the following notation has been employed:
\[ (a, x) 1 + g\tilde{A} \partial \mathcal{J}^{-1} b, x' \]
\[ \equiv \delta_{ab} \delta^4(x-x') g(\tilde{A}_a(x))_{ab} \frac{\partial}{\partial x^4} \mathcal{J}^{-1}(x-x') . \]

By making use of (3·6) and (3·7), the constraints (3·1) and (3·2) are transformed into the following ones.
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\[ \partial \Psi'(t) / \partial A_{\varphi}(x) = 0, \quad (3.8) \]
\[ \partial \Psi'(t) / \partial A_{\varphi}(x) = 0. \quad (3.9) \]

(3.8) and (3.9) show that the physical states are represented by functionals of the transversal vector \( a_{k,a}(x) \) and do not depend upon the scalar or longitudinal components. In other words, the transformation of \( \Psi \) by the operator \( T \) defined by (3.6) and (3.7) means an introduction of the Coulomb gauge condition.

The new state vector \( \Psi'(t) \) satisfies the equation
\[ i\hbar \partial \Psi'(t) / \partial t = (T^{-1}\overline{H}T) \cdot \Psi'(t). \]

In order to derive a concrete expression of the new Hamiltonian \( \overline{H}' = T^{-1}\overline{H}T \), we shall prove the following two facts: (i) Equation (3.6) satisfies the integrability condition, and (ii) \( \overline{H}' \) does not depend on \( \Lambda_{\varphi,a} \) and \( A_{\varphi} \).

Proof of (i): The operator \( \mathcal{C}_a(x) \) defined by (3.6) satisfies the following commutation relation:
\[ [\mathcal{C}_a(x), \mathcal{C}_b(y)] = i\hbar \epsilon_{abc} \delta^4(x - y) \cdot \mathcal{C}_c(y) - h^2(a, x|1 + gA\partial J^{-1}|a'x') \]
\[ \times (b, y|1 + gA\partial J^{-1}|b', y'). \]

Let this relation operate on the solution \( T \) of (3.6). Then, there remains a relation
\[ \frac{\delta^2T}{\partial A_{a'}(x') \partial A_{\varphi}(y')} - \frac{\delta^2T}{\partial A_{\varphi}(y') \partial A_{a'}(x')} = 0, \]
which shows that the integrability condition is surely satisfied.

Proof of (ii): Since \( T \) is independent of \( \Lambda_{\varphi,a} \) [cf., (3.7)], the following relation holds:
\[ (\partial \overline{H}' / \partial A_{\varphi,a}) \Psi' = T^{-1}(\partial \overline{H} / \partial A_{\varphi,a}) T \Psi' \]
\[ = i/\hbar \cdot T^{-1}[\mathcal{F}_a'(x), \overline{H}]. \]

The last expression vanishes due to (3.2).

In order to show that \( \overline{H}' \) does not depend on \( A_{\varphi} \), let us rewrite Eq. (3.6) in a form
\[ \partial T / \partial A_{\varphi}(x) = i/\hbar \cdot K_a(x) \cdot T, \quad (3.6)' \]
where
\[ K_a(x)^d = g \int d^4y (a, x|1 + gA\partial J^{-1}|b, y) (\hat{A}_k(y)_{b,\epsilon e^e}(y) \]
(cf., (e) in Appendix I).

The definition of \( K_a \) gives a rewriting of \( (\mathcal{F}_a''')_a \):
\[ (\mathcal{F}_a''')_a = (a|1 + gA\partial J^{-1}|b) \cdot (J V_b + K_a). \]

This expression and (3.6)' lead to the following relations:
\[
\frac{\partial \tilde{H}'}{\partial A_a(x)} \cdot \Phi' = i/\hbar \cdot T^{-1} \left[ \tilde{H}, \int V_a + K_a(x) \right] \cdot T \Phi'
\]
\[
= i/\hbar \cdot T^{-1} \left\{ \int d^3 y \left[ \tilde{H}, (a, x) \cdot (1 + g \partial J^a)^{-1} \cdot b, y \right] \cdot (F_s \Pi^b(y))_a \cdot \Phi' + \int d^3 y (a, x) \cdot (1 + g \partial J^a)^{-1} \cdot b, y \cdot \left[ \tilde{H}, (F_s \Pi^b(y))_a \right] \cdot \Phi' \right\}.
\]

The first term of the right-hand side of the above relation vanishes due to (3.2), while the second term also vanishes for the same reason because the commutator is decomposed into a sum of (3.1) and (3.2).

The above result gives us a convenient recipe for obtaining \( \tilde{H}' \). In the computation of \( T^{-1} \tilde{H} T \), \( V_a \) and \( H_a^0 \) should be moved rightwards so as to make it possible to employ the constraints (3.8) and (3.9). After such movements and after the constraints have been used, \( A_a \) and \( A_{0,a} \) still left in the Hamiltonian should be put equal to zero. The result thus obtained is the Hamiltonian we desired.

\section*{§ 4. Physical Hamiltonian}

The prescription stated in the end of the previous section enables us to obtain a concrete expression of \( \tilde{H}' \) even though we do not have a complete knowledge of the solution of (3.6).

Let us put

\[
T = \exp \cdot (\Phi).
\]

then \( \tilde{H}' \) becomes

\[
\tilde{H}' = \left\{ \tilde{H} - \left[ \Phi, \tilde{H} \right] \cdot \frac{1}{2!} \left[ \Phi, \left[ \Phi, \tilde{H} \right] \right] \cdot \ldots \right\}_{\Phi = 0, \Phi = 0} \cdot \Phi'. \tag{4.1}
\]

In addition, let \( \Phi \) be expanded into a functional power series with respect to \( A_a \):

\[
\Phi = \Phi_1 + \Phi_2 + \cdots,
\]

where \( \Phi_n \) is a polynomial functional of \( A_a \) of the \( n \)-th order. The fact that \( \tilde{H} \) has a quadratic term

\[
-\frac{1}{2} \int V_a dV_a d^3 x
\]

when the \( H_a^s \)'s in \( \tilde{H} \) are decomposed into longitudinal and transversal parts, gives a result that the only non-vanishing terms of the right-hand side of (4.1) are the following ones after the constraints have been employed and \( A_{0,a} \) and \( A_a \) have been put equal to zero:

\[
\tilde{H}' = \left\{ \tilde{H} - \left[ \Phi_2, \frac{1}{2} \int V_a dV_a d^3 x \right] \right\}
\]
\[
+ \frac{1}{2!} \left[ \Phi_1, \left[ \Phi_1, \frac{1}{2} \int V_a dV_a d^3 x \right] \right] \right\}_{\Phi = 0, \Phi = 0} \cdot \Phi'. \tag{4.2}
\]
Let us also expand $K_a(x)$ into a power series with respect to $A_a$. Then Eq. (3·6)' is rewritten in a form of a power series:

$$\frac{\partial T}{\partial A_a}T^{-1} = \partial \mathcal{G}_a/\partial A_a + \partial \mathcal{G}_a/\partial A_a + \frac{1}{2!} [\mathcal{G}_a, \partial \mathcal{G}_a/\partial A_a]_+ + \cdots$$

$$= i/\hbar \cdot \{K_{a(0)} + K_{a(1)} + \cdots \}, \quad (4·3)$$

where $K_{a(n)}$ is a polynomial functional of $A$ of the $n$-th order. $(4·3)$ gives

$$\partial \mathcal{G}_a/\partial A_a(x) = i/\hbar \cdot K_{a(0)}(x), \quad (4·4)$$

$$\partial \mathcal{G}_a/\partial A_a(x) + i/\hbar \cdot [\mathcal{G}_a, \partial \mathcal{G}_a/\partial A_a(x)]_+ = i/\hbar \cdot K_{a(1)}(x), \quad (4·5)$$

The commutation relation (3·5) and Eqs. (4·4) and (4·5) change the right-hand side of (4·2) into the following form:

$$\overline{H}' = \frac{1}{2} \int \pi_a^k \cdot \pi_a^k d^3x + \frac{1}{4} \int \mathcal{F}_{k,l,a} \mathcal{F}_{k,l,a} d^3x$$

$$+ i \frac{\hbar}{2} \int J^{-1}(x-x') \cdot \partial K_{a(1)}(x)/\partial A_a(x') d^3xd^3x'$$

$$- \frac{1}{2} \int K_{a(0)}(x) J^{-1}(x-x') K_{a(0)}(x') d^3xd^3x', \quad (4·6)$$

where $\mathcal{F}_{k,l,a}$ stands for

$$\mathcal{F}_{k,l,a} = \delta_{s,k} a_{l,a} - \partial_s a_{k,a} + g_{\epsilon ab} a_{k,a} \cdot a_{\epsilon, c}.$$

The Hamiltonian derived above is, however, not hermitian and cannot be accepted as a physical Hamiltonian. This unwanted result is due to the fact that $K_a(x)$ is not hermitian and $T$ is a non-unitary operator.

In order to recover the hermitian property of the Hamiltonian let us transform $\mathcal{F}'(t)$ again into $\mathcal{F}''(t)$ by a non-singular operator:

$$\mathcal{F}'(t) \rightarrow \mathcal{F}''(t) = e^{-M} \mathcal{F}'(t), \quad (4·7)$$

where $M$ is assumed to be an hermitian operator and to depend only on the transversal vector $a_{k,a}$.

Corresponding to (4·7), the Hamiltonian is changed into

$$\overline{H}'' = e^{-M} \overline{H}' e^M = \overline{H}' - [M, \overline{H}']_+ + \frac{1}{2!} [M, [M, \overline{H}']]_+ - \cdots. \quad (4·8)$$

Now $\overline{H}'$ is written in the forms

$$\overline{H}' = \text{Re} \overline{H}' + i \text{Im} \overline{H}', \quad (4·9)$$

$$\text{Re} \overline{H}' = \frac{1}{2} \int \pi_a^k (k, a | 1 + g^i W | l, b) \pi_b^l d^3x d^3y.$$
The definitions of $W$ and $Q$ in the above expressions which are some hermitian functionals of the operator $a_{k,a}$ are given in Appendix I (f). The important point of the expression (4.9) is that $\text{Re} \overline{H}'$ is quadratic with respect to $\pi_a^k$ and $\text{Im} \overline{H}'$ is a linear functional of $\pi_a^k$. This fact and the fact that $M$ is a functional of $a_{k,a}$ alone leave a few non-vanishing terms in the right-hand side of (4.8):

$$
\overline{H}'' = \text{Re} \overline{H}' - i[M, \text{Im} \overline{H}'] - \frac{1}{2}[M, \text{[M, Re} \overline{H}'] - ] - i \text{Im} \overline{H}' - [M, \text{Re} \overline{H}'] - .
$$

If $M$ is chosen as to satisfy

$$
i \text{Im} \overline{H}' = [M, \text{Re} \overline{H}'] - ,
$$

the physical hermitian Hamiltonian becomes

$$
\overline{H}'' = \text{Re} \overline{H}' - i[M, \text{Im} \overline{H}'] - .
$$

Let us put

$$
[M, \pi_a^k(x)] - = i\hbar M_a^k(x),
$$

then (4.10) is written in terms of $\pi_a^k$ as follows:

$$
\frac{i}{4} \hbar g^z \int \{Q_a^k, \pi_a^k\} d^3x = \frac{i}{2} \int \{M_b^l(y) (l, b, y|1 + g^s W|k, a, x), \pi_a^k(x)\} d^3x d^3y.
$$

By making use of the relation

$$
\pi_a^k(x) = \int G_{kl}(x - x') II_a^l(x') d^3x',
$$

and taking into account the fact that the three components of $II_a^l$ are independent of each other, the above equation can be written in such a simplified form as

$$
\frac{g^z}{2} G_{kl}Q_a^l = G_{kl}(l, a|1 + g^s W|b, b) M_b^b.
$$

The solution of this equation is

$$
M_a^k(x) = g^z \frac{2}{2} \int G_{kl}(x - x') (l, a, x'|1 + g^s W G)^{-1}|h, b, y) Q_b^b(y) d^3x d^3y,
$$

$$
(4.13)^*\)

$^*G$ in $(1 + g^s W G)^{-1}$ is the abbreviated notation for $G_{kl}(x - x')$. 

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or is written in an abbreviated form as follows:

\[ M_{\alpha}^k = \frac{g^2}{2} \{ G (1 + g^2 W G)^{-1} Q \}_{\alpha}^k. \]  

(4.13)'

To obtain the solution (4.13), some care should be taken in the singular feature of the right-hand side of (4.12) which is closely connected with the following relations:

\[ G_{kl} G_{il} = G_{kl}, \quad G_{kl} M_{l}^i = M_{l}^k. \]

Substituting the solution (4.13) for the second term of (4.8)', we have the following final expression:

\[ \bar{H}'' = \frac{1}{2} \int \pi_{\alpha}^k (k, a) [1 + g^2 W l, b) \pi_{\beta}^l d^3 x d^3 y \]

\[ + \frac{1}{4} \int \mathcal{F}_{kl, a} \mathcal{F}_{kl, a} d^3 x + \frac{i}{4} g^2 \int [Q_{\alpha}^k, \pi_{\alpha}^k] d^3 x \]

\[ + \frac{g^4}{8} \hbar^2 \int Q_{\alpha}^k (k, a) G (1 + g^2 W G)^{-1} l, b) Q_{\alpha}^k d^3 x d^3 y. \]  

(4.14)

The fact that the resultant of \( T \) and \( \exp \cdot M \) transforms \( \bar{H}'' \) into an hermitian Hamiltonian \( \bar{H}'' \) does not necessarily indicate the unitarity of this operator, but in the present case this resultant is shown to be really a unitary operator in the physical Hilbert space. (Cf., Appendix II.)

§ 5. S-matrix

The S-matrix of the present system is easily derived by following the well-known procedure. For this purpose let us introduce the interaction representation. The Tomonaga equation of the state vector \( \chi (t) \) in the interaction representation is

\[ i \hbar \partial \chi (t) / \partial t = \bar{H}_{\text{int}} (t) \cdot \chi (t), \]

where

\[ \bar{H}_{\text{int}} = \frac{1}{2} g^2 \int \pi_{\alpha}^k (k, a) W l, b) \pi_{\beta}^l d^3 x d^3 y \]

\[ + \frac{1}{4} \int (\mathcal{F}_{kl, a} \mathcal{F}_{kl, a} - f_{kl, a} f_{kl, a}) d^3 x \]

\[ + \frac{i}{4} g^2 \hbar \int [Q_{\alpha}^k, \pi_{\alpha}^k] d^3 x \]

\[ + \frac{g^4}{4} \hbar^2 \int Q_{\alpha}^k G_{kl} (l, a) (1 + g^2 W G)^{-1} h, b) Q_{\alpha}^k d^3 x d^3 y, \]

\[ f_{kl, a} = \partial_k a_{l, a} - \partial_l a_{k, a}. \]  

(5.1)
$a_{k,a}$ and $\pi_a^k$ appearing in (5.1) satisfy the free-field equations
\[ \frac{\partial a_{k,a}}{\partial x^0} = \pi_a^k, \quad \Box a_{k,a} = 0. \]

The commutation relation and the propagator are given by
\[ [a_{k,a}(x), a_{l,b}(x')] = -i\hbar \delta_{ab} \int G_{kl}(x-y) D(y-x', t-t') d^3y \]
and
\[ \langle 0 | T(a_{k,a}(x), a_{n,b}(x')) | 0 \rangle = i\hbar \delta_{ab} \int G_{kl}(x-y) D^r(y-x', t-t') d^3y \] respectively. Here $D$ and $D^r$ are defined by
\[ \Box D = 0, \quad (\partial D/\partial t),_{t=t} = \delta^t(x), \]
\[ \Box D^r = \delta^r(x). \]

For the definition of the propagator it is necessary to confirm the existence of the vacuum state. But in the present case we have no trouble with this problem because all the redundant field variables have been eliminated and the positive definiteness of the free Hamiltonian $H_0$ is self-evident.

The $S$-matrix is written in terms of the ordinary $T$-product of $H_{\text{int}}$ in the following form:
\[ S = \sum_{n=0}^{\infty} (-i/\hbar)^n/n! \int T \{ H_{\text{int}}(t_1), \cdots H_{\text{int}}(t_n) \} dt_1 \cdots dt_n. \] (5.3)

The unitarity of the $S$-matrix given above is self-evident because of the Tomonaga equation and the fact that no constraints are imposed on $\chi(t)$ which is a functional of $a_{k,a}(x)$ alone.

The fact that $H_{\text{int}}$ is a quadratic functional of $\pi_a^k$ gives rise to the well-known trouble in computing the $T$-products. This trouble comes from the singular term on the right-hand side of the relation below:
\[ \langle 0 | T \left( \frac{da_{k,a}(x)}{dt}, \frac{da_{l,b}(x')}{dt'} \right) | 0 \rangle = \frac{d^2}{dt dt'} \langle 0 | T(a_{k,a}(x), a_{l,b}(x')) | 0 \rangle - i\hbar \delta_{ab} \delta(t-t') G_{kl}(x-x'). \] (5.4)

Lee and Yang showed that the contributions from this singular term to the $S$-matrix can be eliminated by a suitable modification of the coefficients of the terms having the $\pi_a^k$'s in the interaction Hamiltonian and an addition of a strange singular term to the Hamiltonian.

In our case, the Hamiltonian $H_{\text{int}}$ should be replaced with the following one:
\[ \mathcal{H} = \frac{g^2}{2} \int \pi_a^k(k, a) (1 + g^2 W G)^{-1} W | l, b \rangle \pi_b^l d^3x d^3y + \frac{1}{4} \int (\mathcal{F} \mathcal{F} - ff) d^3x \]
The theorem of Lee and Yang shows that the S-matrix given by (5·3) is equivalent to

\[ S \sum_{n=0}^{\infty} \frac{(-i/\hbar)^n}{n!} \int \cdots \int T^* \{ \mathcal{H}(t_1) \cdots \mathcal{H}(t_n) \} dt_1 \cdots dt_n, \]

where the operator \( T^* \) is identical with the conventional \( T \) with one exception that \( T^* \) is defined to satisfy

\[ T^* \left( \frac{da}{dt}, \frac{da'}{dt'} \right) = \frac{d^2}{dt dt'} T^* (a, a') = \frac{d^2}{dt dt'} T (a, a'). \]

The last singular term on the right-hand side of (5·5) is nothing but the DeWitt term in the Coulomb gauge.

It may be interesting to compare the present result with those already published. Fradkin and Tyutin also derived the physical Hamiltonian. They eliminated the redundant fields from the Hamiltonian under the condition of the Coulomb gauge within a framework of the classical canonical theory. Their expression of the Hamiltonian differs from our expression (4·14), even if the differences due to the order of operators were ignored. This difference is caused by the fact that an important quantum effect was missed in course of the elimination of the redundant components.

Let us suppose that a transformation similar to (4·7) is performed in order to make their Hamiltonian be an hermitian operator. Then their interaction Hamiltonian (3·33) is given two additional terms by this transformation. Their resultant Hamiltonian expressed in terms of our notation is what is given by simply replacing all the \( Q^a_k \)'s in (5·1) by \( gR^a_k \) (cf., (f) in the Appendix). The difference \( B^a_k = Q^a_k - gR^a_k \) (cf., (f) in the Appendix) which represents some quantum effects, has never appeared in any classical theory. It may be impossible to remove the disagreement between (5·1) and the Fradkin-Tyutin Hamiltonian by devising a clever rearrangement of the non-commutable operators in the Hamiltonian.

The disagreement stated above also suggests that our S-matrix (5·3) will probably differ from that which will be given by the method of Popov and Faddeev under the condition of the Coulomb gauge.

In order to observe the relationship between our result and that derived from Feynman's method of quantization, let us obtain the q-number Lagrangian corresponding to the physical Hamiltonian (4·14). For this purpose, it is better to employ the Heisenberg representation in which we have an equation
\[ \frac{\partial a_{k,\alpha}}{\partial x^0} = \frac{i}{\hbar} [H^\omega, a_{k,\alpha}] = \frac{1}{2} \int \{G_{kl}(x - y) (l, a, y | 1 + g^2 W | h, b, y'), \pi^b(y') \} \cdot d^2 y d^2 y' \]

or

\[ \dot{a} = \frac{1}{2} \{G(1 + g^2 W), \pi \} = G(1 + g^2 W) \pi - \frac{1}{2} [G(1 + g^2 W), \pi] \]

in an abbreviated form. This equation can be solved with respect to \( \pi \) by the same method as that employed in the derivation of (4.13). The solution is

\[ \pi = G(1 + g^2 W)^{-1} \{ \dot{a} + \frac{1}{2} [G(1 + g^2 W), \pi] \} \]

Substituting the above result for the \( \pi \)'s in the conventional definition of the Lagrangian, we have

\[ L = \frac{1}{2} \int (\dot{a} + \dot{\pi}) d^3 x - H^\omega \]

\[ = \frac{1}{2} \int \dot{a}_{\alpha,\alpha} (k, a | G(1 + g^2 W) \pi^{-1} | h, b) \dot{a}_{\beta,\beta} d^2 x d^2 y \]

\[ - \frac{1}{4} \int \frac{\partial G}{\partial \lambda} d^3 x - \frac{i}{2} g^2 \int [{Q}, \pi]_\pm d^3 x \]

\[ - \frac{g^2}{8} \int Q^h G_{kl} (l | (1 + g^2 W)^{-1} | h) Q^l d^2 x d^2 y \]

\[ - \frac{g^2}{8} \int [GW, \pi]_\pm (k, a | (1 + g^2 W)^{-1} | l, b) [GW, \pi]_\pm d^2 x d^2 y, \]

where the following abbreviation has been employed:

\[ [GW, \pi]_\pm \equiv \int G_{kl}(x - x') \cdot [(l, a, x' | W | h, b, y'), \pi^b(y')] \cdot d^2 x' d^2 y. \]

The interaction part of the above Lagrangian is written in an abbreviated form as follows:

\[ L_{\text{int}} \equiv L - \frac{1}{2} \int \dot{a} d^3 x + \frac{1}{4} \int \dot{ff} d^3 x. \]

By making use of this result, the modified interaction Hamiltonian \( \tilde{H} \) is rewritten as

\[ \tilde{H} = - L_{\text{int}} - \frac{i\hbar}{2} \partial (0) \text{Tr} \log (1 + g^2 W) \]

\[ - \frac{g^2}{8} \int [GW, \pi]_\pm (1 + g^2 W)^{-1} [GW, \pi]_\pm d^2 x d^2 y. \]

The last term of the above expression is due to the non-commutability of the
operators and has never appeared in other papers.

The result obtained above suggests that the S-matrix which will be given by Feynman's method of functional integration under the condition of the Coulomb gauge may be also different from our expression given by (5.3). As to these comparisons, a detailed explanation will be given in other papers by one of the present authors.

**Appendix I**

---Notations---

(a) The small Latin indices, $a, b, c, \ldots$, taken from the beginning part of the alphabet express the components with respect to the internal freedom (isospin) and are assumed to take values 1, 2, 3.

The Greek letters are used to represent the space-time components of vectors or tensors and take values 0, 1, 2, 3. The space-time metric is given by a tensor $\eta_{\mu\nu} = (-, +, +, +)$.

The spatial components of vectors are denoted by Latin letters, $j, k, l, \ldots$, taken from the middle part of the alphabet.

The summation convention is used throughout this paper for every kind of indices.

(b) $F_{\nu a, \alpha} = \partial_{\mu} A_{\nu, a} - \partial_{\nu} A_{\alpha, a} + g \epsilon_{abc} A_{\alpha, b} A_{\nu, c}$, where $\epsilon_{abc}$ is the structure constant of the group $SU(2)$ and is completely antisymmetric with respect to any transposition of $a, b, c$.

(c) $\langle F_\nu \rangle_{ab} = \partial_{\alpha} \partial_{\sigma} / \partial x^\alpha + g (\tilde{A}_{\nu}(x))_{ab}$,

$\langle \tilde{A}_{\nu} \rangle_{ab} = \epsilon_{abc} A_{\nu, c}(x) = - (\tilde{A}_{\nu})_{ba}$.

(d) $J \cdot J^{-1}(x-x') = \delta^4(x-x')$, $J^{-1}(x-x') = -(4\pi)^{-1/2} |x-x'|$, $G_{kk}(x-x') = \delta_{kk} \delta^4(x-x') - \delta^3 J^{-1}(x-x') / \partial x^k \partial x^k$.

(e) $(a, x | (1 + g \tilde{A} \partial J^{-1})^{-1} | b, y)$

$= \partial_{\alpha} \partial_{\sigma} \delta^4 (x-y) - g (\tilde{A}_{k}(x))_{ab} \partial J^{-1}(x-y) / \partial x^k$

$+ g^2 \int (\tilde{A}_{k}(x))_{ab} \partial J^{-1}(x-z) / \partial x^k \cdot (\tilde{A}_{k}(z))_{cb} \partial J^{-1}(z-y) / \partial z^c d^4 z - \cdots$.

Fradkin and Tyutin employed $\nabla_{ab}$ which is defined by

$(\nabla_k)_{ab} = \partial_{\nu} \partial_{\mu} (x, y, \tilde{A}) / \partial x^\nu = \delta_{ab} \delta^4 (x-y)$.

The relation between $\nabla$ and $(1 + g \tilde{A} \partial J^{-1})^{-1}$ is

$\Delta_x \nabla_{ab}(x, y, \tilde{A}) = (a, x | (1 + g \tilde{A} \partial J^{-1})^{-1} | b, y)$.
If \( \hat{A} \) is replaced by \( \hat{b} \), \( \mathcal{D} \) becomes symmetric
\[
\mathcal{D}_{ab}(x, y, \hat{a}) = \mathcal{D}_{ba}(y, x, \hat{a}),
\]
and satisfies
\[
\Gamma_x \partial_x \mathcal{D} = \partial_x \Gamma_x \mathcal{D} = 1
\]
owing to the transversal property of \( \hat{a} \).

(f) The definition of \( B_a^k(x) \):
\[
\phi \int B_a^k(x) \pi_a^k(x) d^3x = \int \mathcal{D}^{-1}(x-x') \partial K_a \mathcal{D}(x') dxdx' = \int \mathcal{D}^{-1}(x-x') \{ \partial K_a(x) / \partial A_a(x') \} dxdx'.
\]
\[
C_{eb}(x, x') \triangleq \langle a, x | (1 + g\hat{a} \mathcal{D}^{-1})^{-1} \hat{a}^k | b, x' \rangle = \mathcal{D}_e \mathcal{D}_{ac}(x, x', \hat{a}) (\hat{a}^k(x'))_{eb}.
\]
\[
gR_a^k(x) \triangleq \frac{i}{\hbar} \int [C_{eb}(y', y), \pi_b^e(y)] \mathcal{D}^{-1}(y'-x') b_c^e(x', x) d^3x' d^3y' d^3y.
\]
\[
Q_a^k(x) \equiv B_a^k(x) + gR_a^k(x).
\]
(k, a, x | W | l, b, y)
\[
\equiv - \int C_{ca}^e(x', x) \mathcal{D}^{-1}(x'-y') C_{eb}(y', y) d^3x' d^3y' = \int (\hat{a}^k(x))_{ac} \mathcal{D}_e \mathcal{D}_{ac}(x, z, \hat{a}) \mathcal{D}_e \mathcal{D}_{ac}(z, y, \hat{a}) (\hat{a}^i(y))_{eb} d^3z = (l, b, y | W | k, a, x).
\]

(g) Tr log \((1 + g^2 W G)\)
\[
\equiv g^2 \int (k, a, x | W | l, a, x') G_{lk}(x'-x) d^3x d^3x' - \frac{g^4}{2} \int (k, a | W | l, b) G_{lm}(m, b | W | n, a) G_{nn} d^3x d^3y d^3z d^3u + \cdots.
\]

**Appendix II**

---Unitarity of \( T \cdot \exp M ---

The relation between \( \bar{H}^n \) and the original Hamiltonian \( H \) is
\[
\bar{H}^n \mathcal{F}^n(t) = e^{-\mathcal{S}T^{-1}\bar{H} T e^{\mathcal{S}T^{-1}}} \mathcal{F}^n(t).
\]
The hermitian property of \( \bar{H} \) and \( \bar{H}^n \) leads to
\[
(\mathcal{F}_{a''}^{**} \cdot [\bar{H}^n, Z] \mathcal{F}_{b''}) = 0.
\]
for any couple of the physical states $A$ and $B$, where

$$Z = (T \cdot e^\eta)^* \cdot (T \cdot e^\eta).$$

Now let us show that

$$(T''^* (t) \cdot ZZB'' (t)) = (T''^* (t)ZB'' (t))$$

at any time $t$.

Proof: The Schrödinger equation gives the following relation

$$T'' (t) = \exp \{- i (t-t_0) \hat{H}' / h \} \cdot T'' (t_0).$$

Using this relation, we have

$$(T''^* (t) \cdot ZZB'' (t)) = (T''^* (t_0) \cdot ZZB'' (t)).$$

Let $t_0$ tend to $-\infty$, and let us assume the adiabatic switching-off of the coupling constant $g$ in the infinitely remote past. Then the definition of $T$ and $e^\eta$ in § 4 gives

$$\lim_{g \to 0} Z = 1.$$

Therefore, we have

$$(T''^* (t) \cdot ZZB'' (t)) = (T''^* (t) \cdot ZZB'' (t)).$$

The proof given above shows that the operator $T \cdot e^\eta$ can be regarded as a unitary operator in the physical Hilbert space. This result gives

$$(T^* \cdot T) = (T''^* \cdot T''), \quad (T^* \cdot \hat{H} T) = (T''^* \cdot \hat{H} T).$$

These relations justify our interpretation that $T''$ is a physical probability amplitude.

Appendix III

Let us show that there is no contradiction between the constraint (2·11) and the commutation relation. For the sake of simplicity we shall drop all the suffices in what follows.

When $A(x)$ is given a diagonal representation in the Schrödinger picture, the eigenstate of $\Pi (x)$ with the eigenvalue $\lambda (x)$ is given by

$$T_1 = \exp \left\{ i \int d^3 x \lambda (x) A (x) \right\} \cdot \phi,$$

where $\phi$ does not depend on $\lambda (x)$. The inner product of an eigenstate $T_1$ with another $T_x$ takes the following form:

$$(T_x^* \cdot T_1) = \int \prod_x d A (x) \exp \left[ i \int d^3 x \{ \mu (x) - \lambda (x) \} A (x) \right] \cdot (\phi^*_i \cdot \phi_x)$$

$$= \text{const} \times \prod \delta \{ \mu (x) - \lambda (x) \} \cdot (\phi^*_i \cdot \phi_x).$$

(AIII·2)
Let us calculate a matrix element \( \langle \mathcal{F}_i \rangle \cdot [\Pi(x), A(y) \mathcal{F}_\mu] \) in the following two ways:

(i) The direct employment of the commutation relation leads to

\[
\langle \mathcal{F}_i \rangle \cdot [\Pi(x), A(y) \mathcal{F}_\mu] = -i \delta^3(x-y) \prod_z \delta \{ \lambda(z) - \mu(z) \} \cdot (\mathcal{F}_1 \cdot \mathcal{F}_2) \times \text{const.} \quad (\text{AIII} \cdot 3)
\]

(ii) Making use of the fact that \( \mathcal{F}_i \) and \( \mathcal{F}_\mu \) are both the eigenstates of \( \Pi(x) \), we have

\[
\langle \mathcal{F}_i \rangle \cdot [\Pi(x), A(y) \mathcal{F}_\mu] = \{ \lambda(x) - \mu(x) \} \langle \mathcal{F}_i \rangle \cdot A(y) \mathcal{F}_\mu.
\]

As the expression (AIII \cdot 1) gives the relation

\[
A(x) \mathcal{F}_\lambda = -i \frac{\partial}{\partial \lambda(x)} \mathcal{F}_\lambda,
\]

the above result is rewritten as

\[
\langle \mathcal{F}_i \rangle \cdot [\Pi(x), A(y) \mathcal{F}_\mu] = - \{ \lambda(x) - \mu(x) \} \frac{i \theta}{\delta \mu(y)} (\mathcal{F}_i \cdot \mathcal{F}_\mu)
\]

\[
= -i \frac{\partial}{\partial \mu(y)} \{ \lambda(x) - \mu(x) \} \prod_z \delta \{ \lambda(z) - \mu(z) \} \cdot (\mathcal{F}_i \cdot \mathcal{F}_\mu) \times \text{const}
\]

\[
- i \delta^3(x-y) \prod_z \delta \{ \lambda(z) - \mu(z) \} \cdot (\mathcal{F}_1 \cdot \mathcal{F}_2) \times \text{const}.
\]

The first term of the right-hand side identically vanishes. Consequently we have

\[
\langle \mathcal{F}_i \rangle \cdot [\Pi(x), A(y) \mathcal{F}_\mu] = -i \delta^3(x-y) \prod \delta \{ \lambda(z) - \mu(z) \} \cdot (\mathcal{F}_1 \cdot \mathcal{F}_2) \times \text{const.} \quad (\text{AIII} \cdot 4)
\]

The final expression coincides with the right-hand side of (AIII \cdot 3), showing the consistency between the commutation relation and the constraint (2 \cdot 11) which corresponds to the case \( \lambda(x) = \mu(x) = 0 \).

A similar argument can be also valid for the constraint (2 \cdot 12).

References