Collective Instabilities of Self-Gravitating Systems. III
—Disk-Halo System—

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The instability modes of the infinitesimally thin self-gravitating disk, which is composed of rotating and non-rotating parts, are investigated. By solving the dispersion relation numerically under various physical situations, two growing modes are found. They are the modified Jeans mode of rotating component or non-rotating one and the Landau mode by wave-particle (non-rotating component) interactions. Possible relevances to spiral arms are discussed briefly.

§ 1. Introduction

In relation to the density wave theory of spiral arms, much work has been done on the waves and their instabilities of self-gravitating systems. We have also studied this problem and clarified some instability modes and their properties in comparison with electron oscillations in plasma.1,2

In this paper, on the basis of the local analysis we investigate the instability modes of the infinitesimally thin self-gravitating disk, when it consists of two kinds of stellar systems, that is, rotating one (disk component) and non-rotating one (halo component). This composite model seems to represent flat galaxies well, because the galactic halo with large mass has been recently suggested both theoretically3 and observationally.4

In § 3 we solve the dispersion relation numerically for various physical situations and obtain the growing modes and their growth rates. By virtue of the composite nature of this system, the features of the instability mode are different from the one-component system5 in two points: One is that the Jeans instability mode is not purely growing, but is modified to be overstable. The other is that the Landau growing mode, which is proper to the two-component system with relative velocity, appears. The former growth time turns out to be about a rotational period and the latter one is some hundred times longer, in general. The Landau growing mode has been already considered by Marochnik and Suchkov6 and they concluded that the wave can be excited in a sufficiently short period. As is shown later, their growth rate was, however, overestimated about an order of magnitude.

On the basis of these results, the possibility of wave excitation in these model galaxies is discussed in § 4.
§ 2. Basic equations and dispersion relation

The system under consideration is the infinitesimally thin self-gravitating stellar disk consisting of the rotating system and the non-rotating one. Hitherto, we call the former the disk component (suffix d) and the latter the halo component (suffix h), respectively.

Basic equations are the perturbed Vlasov equations for the above two systems coupled with the perturbed Poisson equation. In the cylindrical coordinate, these equations are written as

\[
\frac{\partial f_d}{\partial t} + v_r \frac{\partial f_d}{\partial r} + \left( \frac{v_\theta}{r} + \Omega \right) \frac{\partial f_d}{\partial \theta} + \frac{\partial f_d}{\partial \varphi} \frac{\partial f_{\phi d}}{\partial r} + \frac{\partial \varphi}{\partial \theta} \frac{\partial f_{\phi d}}{\partial \varphi} \frac{1}{r} \frac{\partial \varphi}{\partial \varphi} \frac{\partial f_{\phi d}}{\partial \varphi} = 0,
\]

(1)

\[
\frac{\partial f_h}{\partial t} + v_r \frac{\partial f_h}{\partial r} + \frac{v_\theta}{r} \frac{\partial f_h}{\partial \theta} - \frac{\partial \varphi}{\partial \varphi} \frac{\partial f_{\phi h}}{\partial \varphi} \frac{1}{r} \frac{\partial \varphi}{\partial \varphi} \frac{\partial f_{\phi h}}{\partial \varphi} = 0
\]

(2)

and

\[
\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = 4\pi G (\sigma_d + \sigma_h) \delta (z),
\]

(3)

where

\[
\sigma_d = \int f_d \, d^2v \quad \text{and} \quad \sigma_h = \int f_h \, d^2v,
\]

(4)

and G is the gravitational constant. In the above, \(v_r, v_\theta, f, \sigma \) and \(\varphi\) are radial and azimuthal components of velocity, velocity distribution function, surface density and gravitational potential, respectively. The quantities with suffix 0 represent zero-th order ones and others are perturbed ones. The angular velocity \(\Omega (r)\) and the epicyclic frequency of the disk component \(\kappa (r) = 2\Omega [1 + (r/2\Omega) (d\Omega/dr)]^{1/2}\) should be deduced from the resultant gravitational field of the disk and the halo components in an equilibrium state.

In the present local analysis the zero-th order distribution functions, \(f_{0d}\) and \(f_{0h}\) are assumed to be the Schwarzschild and the Maxwell distributions, respectively, i.e.,

\[
f_{0d} = \frac{\sigma_d}{2\pi c_d^2} \left( \frac{2\Omega}{\kappa} \right) \exp \left[ -\frac{v_r^2 + (2\Omega/\kappa)^2 v_\theta^2}{2c_d^2} \right]
\]

(5)

and

\[
f_{0h} = \frac{\sigma_h}{2\pi c_h^2} \exp \left[ -\frac{v_r^2 + v_\theta^2}{2c_h^2} \right],
\]

(6)

where \(c_d\) and \(c_h\) are velocity dispersions. Therefore, it should be understood that
the disk and the halo components in the equilibrium state, which are described by Eqs. (5) and (6), respectively, are coupled merely through \( \Omega (r) \) or \( \kappa (r) \).

In the usual WKB approximation, perturbed quantities, \( A(r, \theta, t) \), are assumed to be written as

\[
A(r, t) = \tilde{A}(r, t) e^{i(kr + \omega t)},
\]

where \( \tilde{A}(r, t) \) is slowly varying function of \( r \) and \( t \), and \( m \) is an integer, usually taken to be 2.

The dispersion relation is derived in the first WKB approximation as

\[
|K| \beta = 1 - I_n(x) e^{-x} - 2 \sum_{n=1}^{\infty} \frac{v^2}{p^2 - n^2} I_n(x) e^{-x} - \frac{1}{2} \frac{D}{S^2} Z' \left( \frac{\nu + R}{\sqrt{2} KS} \right),
\]

where

\[
x = K^2 = \left( \frac{k c_d}{\kappa} \right)^2, \quad D = \frac{\sigma_b}{c_d}, \quad S = \frac{c_b}{c_d}, \quad R = m \frac{\Omega}{\kappa}, \quad \beta = \frac{\kappa c_d}{2\pi G\sigma_d}, \quad \nu = \frac{\omega - m \Omega}{\kappa}, \quad (9)
\]

\( I_n(x) \) is the modified Bessel function of the \( n \)-th order, and \( Z(\zeta) \) and \( Z'(\zeta) \) are the plasma dispersion function and its first derivative, i.e.,

\[
Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - \zeta} dx \quad \text{and} \quad Z'(\zeta) = \frac{dZ}{d\zeta}, \quad (10)
\]

respectively.

This dispersion relation is the same as Marochnik and Suchkov obtained. Assuming \( D \ll 1 \) and \( \nu \ll \nu_r \), they obtained the approximate solution for the kinetic branch (Landau mode) as

\[
\omega_r = m \Omega \pm \sqrt{\kappa^2 - 2\pi G\sigma_d |k| F_v(x)} \quad (11)
\]

and

\[
\omega_i = \frac{2\pi^3 G^2 \sigma_d \sigma_h F_v(x)}{\omega_r - m \Omega} f_h \left( \frac{\omega_r}{k} \right), \quad (12)
\]

where

\[
F_v(x) = \frac{(1 - \nu^2)}{x} \left[ 1 - I_n(x) e^{-x} - 2 \sum_{n=1}^{\infty} \frac{v^2}{p^2 - n^2} I_n(x) e^{-x} \right]. \quad (13)
\]

Here, suffixes \( r \) and \( i \) denote real and imaginary part, respectively. Moreover, they approximated \( F_v(x) \) by \( (2/x) I_1(x) e^{-x} \). However, the imaginary part \( \omega_i \) thus obtained is inaccurate, since this approximation ignores completely \( v \)-dependence of \( F_v(x) \). On the other hand, Suchkov investigated the Jeans branch under the approximation of \( \omega_r = 0 \), and concluded that the Jeans instability is not effective for the formation of the spiral arms. However as is shown in § 3, modified Jeans modes with \( \omega_r \neq 0 \) appear in case of cold halo.

The number of solutions of dispersion relation (8) is infinite. Considering
the usual condition for the Landau growing mode, i.e.,
\[
\frac{\omega_r - m\Omega}{k} \frac{df}{dv} \bigg|_{\nu = (\omega_r/k)} > 0,
\]
we expect that the growing mode exists in the following restricted region of \( \omega_r \):
\[
0 < \gamma + R = \frac{\omega_r}{\kappa} < R.
\]

§ 3. Numerical results of growing modes

We have solved Eq. (8) numerically under various physical situations and found the growing modes in the region of \( \omega_r \) determined by Eq. (15). The calculated results are shown in Figs. 1~4, and the assumed values of parameters are summarized in Table. In Fig. 1 the growing modes for varying \( \beta \) are shown and this represents the dependence of \( \omega \) on the velocity dispersion of disk component.

![Fig. 1](https://example.com/fig1.png)

Fig. 1. The growing modes for varying \( \beta \). \( \nu_r \) and \( \nu_i \) are plotted versus \( K \) in (a) and (b), respectively. See Table for the taken parameters.

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Fig. 2. The growing modes for varying $S$.

$\alpha$. The case $\beta=0.5$ corresponds to a little smaller velocity dispersion than Toomre's minimum dispersion. Then, for $D=0$ (with no halo component), this mode ought to be purely growing ($\nu_r=0$) by the Jeans instability, but in our case $\nu_r$ is slightly changed from zero by virtue of halo component. In the cases $\beta=0.75$ and $\beta=1$ where the velocity dispersions are larger than Toomre's minimum dispersion, the disk component is stabilized and for $D=0$ this mode ought to be purely oscillating. In our case $D=1$, however, the Landau growing mode occurs by the resonant wave-particle (halo stars) interaction. The growth rate $\nu_t$ is smaller than $10^{-2}$ in general.

In Fig. 2 the solutions for varying $S$ are shown, i.e., varying ratios of velocity dispersions of halo component to disk one. In the cases $S=5$ and 10, the Landau growing occurs in the same manner as in Fig. 1. The larger is $S$, the smaller is $\nu_t$ for the fixed $K$. For $S=1$, the rapidly growing mode appears and this mode has been considerably modified in $\nu_r$ in contrast to the case $\beta=0.5$ in Fig. 1.

The reason of this modification is considered as follows: If we neglect the contribution of disk component, the dispersion relation (8) reduces to

$$\frac{k_{ch}^2}{2\pi G \sigma_h} = -\frac{1}{2} Z'(\left\langle \frac{\omega}{\sqrt{2k_{ch}}} \right\rangle).$$

(16)
We obtain from Eq. (16) a purely growing mode and a purely damping mode corresponding to $k < (2\pi G\sigma_h/c_s^2)$, $(K<1/\beta)$ and $k > (2\pi G\sigma_h/c_s^2)$, $(K>1/\beta)$, respectively. This is exactly the criterion of Jeans instability as in a previous paper, though dominant terms of $Z'(\zeta)$ are $1+i\sqrt{\pi}e^{-\zeta^2}$ when $k \approx (2\pi G\sigma_h/c_s^2)$, i.e., the resonant interaction between wave and halo stars is large. When we take the disk component into account, this mode is modified to have non-zero $\omega_r$. In this sense we may consider that the growing mode of $K \approx 1$ in our case $S=1$, $D=1$ and $\beta=1$ is the Jeans one of the halo component modified by the disk component.

In Fig. 3 we have shown the dependence of solutions on $D$. As we have assumed $S=10$, the growth rates are nearly proportional to $D$ as expected from Eq. (8). Thus, the larger mass the halo component has, the larger is the growth rate of the Landau growing mode.

In Fig. 4, the case $R=1.6$ is compared with the case $R=1.0$. The former
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Corresponds to the solar neighborhood and the latter to the uniform rotation if \( m = 2 \). The growth rate for \( R = 1.6 \) is about three times as high as that for \( R = 1.0 \). This is due to the difference of the values \( \alpha'(v+R)/KS \).

Summarizing the above results, large growth rates can be obtained in either case of the low velocity dispersion in the disk or in the massive halo. In other cases slow growth of waves occur due to the Landau mechanism.

§ 4. Discussion

First, we discuss the possible relevance of the previous results to spiral arms. The density wave theory proposed by Lin and Shu\(^9\) has encountered some theoretical difficulties, i.e., the damping due to the radial transfer\(^9\) of wave energy in several galactic rotation periods and due to the shock dissipation\(^9\) in the interstellar gas. From the viewpoint of the wave theory, however, the growth of waves may overcome these difficulties.

Since the waves with \(-1 < \nu_r < 0\) grow in disk-halo system as is shown in § 3, it is interesting to apply their growth to the excitation and maintenance of spiral arms in disk galaxies.

In Fig. 5 we have plotted \( \nu_v \) lines for varying \( D \) and \( S \), i.e., for varying halo situations. For the wave with \( \nu_r \sim -0.5 \) to grow fast enough, say, \( \nu_v \geq 0.015 \), massive halo with rather low velocity dispersion is required. Then, it is important to clarify the real halo situation.

Here we remark that the growth rate obtained by Marochnik and Suchkov was overestimated because of inaccurate estimate of \( \nu_r \) for their adopted parameters. (They estimated \( \nu_r = -0.02 \).

Lastly, we add a brief comment on our work.\(^1, 2\)

We have investigated the collective instabilities of self-gravitating systems on the following fundamental assumptions, i.e., the local analysis, the homogeneity of the equilibrium state and the linearization of the perturbations. These assumptions may have essential defects for the wave phenomena and instabilities of self-gravitating systems though the physical mechanisms of the instabilities are well clarified. Therefore we must investigate the effects of non-locality, inhomogeneity and non-linearity in the next step.

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References