A Microscopic Theory of Rotational Motion in Deformed Odd-Mass Nucleus

--- An Additional Term to the Cranking Moment of Inertia ---

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A microscopic theory of rotational motion in odd-mass nucleus is given. The basic idea is the use of certain constraints of the pair operators devised by one of the present authors (M. Y.). One of the main results is the appearance of additional term to the cranking moment of inertia, which cannot be expected in the Hartree-Bogoliubov approach.

§ 1. Introduction

Nuclear rotational motion has been regarded as one of the most important problems in the theory of nuclear collective excitations. One of the reasons is that a consistent description of the rotation is of fundamental significance in quantum theory of many-body system. In addition to the above-mentioned reason, we can find the importance in the recent experimental studies of medium and heavy nuclei. Through the experimental information, we have already known that there exist many nuclei which show, more or less, the rotation-like excitations. Therefore, it is an important task to develop a powerful theory for analysing the structure of such nuclei.

A conventional description of the rotation is to take into account the change of the self-consistent Hartree-Bogoliubov field induced by an external field $-\omega J_r$. By applying this idea to the case of even-mass nucleus, we can obtain a set of inhomogeneous linear equations, the homogeneous parts of which are very similar to the equations for conditions of the stability of the Hartree-Bogoliubov field.\) The interaction parts contained in these equations generally give rise to additional term to the standard cranking moment of inertia. However, in the case of the quadrupole force, such a term does not appear. Therefore, for the sake of investigating the deviation from the simple cranking formula, the pairing type or $T$-odd particle-hole type interaction* has been inevitably introduced. The additional

*) An example of the interaction is $\sigma \cdot \sigma$-force. In this case, $\sigma$ is the $T$-odd particle-hole type operator. The other kind of the interaction is of $T$-even particle-hole type, for example, the quadrupole force.
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A term induced by such an interaction is usually called the Migdal term. Further, by making a formal extension of this idea to the case of odd-mass nucleus, correction terms to the moment of inertia of the even system are derived. These come from the blocking and the scattering effects of the last odd quasi-particle on the even-mass core. Therefore, if the moment of inertia of the even system has not the Migdal term, then, the odd system does not possess this term, either. However, let us remember the conditions for that even nucleus does not possess the Migdal term. The first condition is, of course, the following: The interaction is of \( T \)-even particle-hole type in contrast to \( T \)-odd property of the angular momentum. In addition to this condition, it is also necessary that the occupation probabilities of any single-particle state and its time-reversed one are the same. On the other hand, such occupation probabilities are different from each other in the case of the odd system. Therefore, it can be expected that the odd-mass system has additional term to the simple cranking moment of inertia even if the interaction is of \( T \)-even particle-hole type. In this sense, we have to throw doubt on the formal application of the conventional approach to the rotational motion of the odd-mass system.

Recently, the present authors, together with Matsuzaki, proposed a microscopic theory of rotational motion in deformed odd-mass nuclei. This theory is called the algebraic approach and the basic standpoint is different from that of the conventional approach. According to this theory, pair operators composed of the bilinear forms of fermions are regarded as fundamental. By solving certain constraints governing the pair operators, we can understand the structure of the system under consideration. On the basis of this method, we have succeeded in clarifying many phenomenological aspects of rotational model from many-body theory. It is characteristic that this theory does not contain the violation of spherical symmetry at any stage. However, we could not derive the moment of inertia with the additional term expected above. Although it is one of the reasons that the model adopted there does not include the pairing correlation, the main reason seems to be attributed to rough estimate of various matrix elements of the pair operators. Therefore, the problem should be reexamined by making a refined estimation of the matrix elements.

The main purpose of the present paper is to propose a microscopic theory of rotational motion in odd-mass nucleus on the basis of the algebraic approach. Our starting equations are, as already mentioned, certain constraints of the pair operators, which consist of equations of motion and operator identities. The basic idea is, in some sense, the extension of the theory for the even-mass system proposed by Belyaev and Zelevinsky in 1970 and supplemented by one of the present authors (M. Y.) in 1974. In 1973, Belyaev and Zelevinsky reformulated their basic idea in terms of the method of generalized density matrix elements in the theory of collective motion. However, they could not realize our expectation. We can obtain the rotational solution in the framework of our approximation. Our main result is the appearance of the additional term to the conventional
cranking moment of inertia, which is just our expectation. Further, what is interesting is that the additional term has a close connection with the deviations from the rotor rules of E2-moments and -transitions.

In the next section, we will recapitulate our basic formalism. For simplicity, we adopt the single-\(j\) shell model with the pairing plus quadrupole force. Section 3 will be devoted to giving solution procedure of our equations after setting up our starting assumptions. In §§4 and 5 we will mention the zeroth and first order approximations, respectively. In §6, the final solution, especially, the additional term to the cranking moment of inertia will be given. In §7, after making a comparison with the Belyaev-Zelevinsky theory, we will give the physical interpretation of our results.

§ 2. Preliminaries

In order to illustrate the essential point of our idea, we adopt a simple model: The system consists of \(n\) identical nucleons \((n\) : odd integer) moving in the single-\(j\) orbit with the pairing plus quadrupole force. Our present theoretical framework is based on a theory developed by one of the present authors (M. Y.), which will be, hereafter, referred to as I.\(^9\) In this section, we will give a recapitulation of the theory I.

First, we introduce the following pair operators:

\[
A_{JM}^{±} = \frac{1}{2} \left( A_{JM} \pm (-)^{J+M} A_{-J-M} \right), \quad B_{JM}^{±} = \frac{1}{2} \left( B_{JM} \pm (-)^{J+M} B_{J-M} \right). \tag{2.1}
\]

The operators \(A_{JM}^{±}\) and \(B_{JM}^{±}\) are defined by nucleon operators \(c_{jm}^{±}\) and \(c_{jm}\) as follows:

\[
A_{JM} = \sum_{mm'} \langle j_m j_{m'} | J M \rangle c_{jm'} c_{jm'}, \quad B_{JM} = \sum_{mm'} \langle j_m j_{m'} | J M \rangle c_{jm'} (-)^{j+m'} c_{j-m'}. \tag{2.2}
\]

Our pair operators have the properties

\[
A_{JM}^{±} = (-)^{J} A_{J-M}^{±}, \quad B_{JM}^{±} = (-)^{J} B_{J-M}^{±}. \tag{2.3}
\]

Therefore, \(A_{JM}^{±}\) and \(B_{JM}^{±}\) with odd \(J\) and \(B_{JM}^{±}\) with even \(J\) are equal to zero.

Our model Hamiltonian is expressed as

\[
H = (\epsilon - \lambda) N_0 - 1/G \cdot A_0 - 1/2 \cdot \sum_M Q_M (-)^M Q_{-M}, \tag{2.4}^* \]

where

\[
N_0 = \sqrt{2j+1} B_{00}^{(+)}, \quad A_0 = \frac{1}{2} G \sqrt{2j+1} A_{00}^{(+)}, \quad Q_M = q \cdot B_{0M}^{(+)}. \tag{2.5}
\]

It has been mentioned in I that the foundation of our theory exists in certain constraints which govern the pair operators. The first one is a set of equation of motion as dynamical constraints:

\[^*\] We adopt the number non-conserving treatment and neglect the effect of pairing vibration. Therefore, the effect of \(A_{00}^{(+)}\) on the pairing correlation can be neglected and the Hamiltonian contains chemical potential \(\lambda\).
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\[ [H, A_{ij}^\pm] = P_{ij}^\pm, \quad [H, B_{ij}^\pm] = Q_{ij}^\pm, \]

\[ P_{ij}^\pm = 2(\epsilon - \lambda) A_{ij}^\pm - \delta_{ij} \delta_{\alpha \mu} (1 \pm 1) / 2 \cdot \sqrt{j + 1} \cdot [J, B_{ij}^\pm]. \]

\[ Q_{ij}^\pm = [J, A_{ij}^\pm], - \gamma q^2 \sum_j Z (2j) \sum_{k, \alpha \beta} \langle k2a | jM \rangle [Q_{ij}^\pm, A_{ij}^\pm], \]

Here, \( Z (ILJ) \) denotes the Racah coefficients defined in the form

\[ Z (ILJ) = \sqrt{(2I + 1)(2L + 1)} W (j;j;L;J). \]

The second one is a set of operator identities as kinematical constraints:

\[ B_{ij}^\pm = R_{ij}^\pm, \quad 0 = S_{ij}^\pm, \]

\[ R_{ij}^\pm = -1/2 \cdot 2 \delta_{ij} \delta_{\alpha \mu} (2j + 1) \cdot (1 + (-1)^\epsilon) \]

\[ + 1/2 \cdot \sum_j Z (ILJ) \sum_{k, \alpha \beta} \langle k2a | jM \rangle \]

\[ \times \left[ \left[ B_{ij}^\alpha, B_{ik}^\beta \right], \pm \left[ A_{ij}^\alpha, A_{ik}^\beta \right], + \left[ B_{ik}^\alpha, B_{ij}^\beta \right], \pm \left[ A_{ik}^\alpha, A_{ij}^\beta \right] \right]. \]

\[ S_{ij}^\pm = 1/2 \cdot \sum_j Z (ILJ) \sum_{k, \alpha \beta} \langle k2a | jM \rangle \]

\[ \times \left[ \left[ B_{ij}^\alpha, A_{ik}^\beta \right], \pm \left[ A_{ij}^\alpha, B_{ik}^\beta \right], + \left[ B_{ik}^\alpha, A_{ij}^\beta \right], \pm \left[ A_{ik}^\alpha, B_{ij}^\beta \right] \right]. \]

These are the basic relations of our theory. By investigating these relations under certain assumptions, we will give a description of rotational motion of odd-mass system.

\[ \textbf{§ 3. Solution procedure} \]

Let us start from setting up our basic assumptions. We seek solution of Eqs. (2-6) and (2-7) under the following assumptions:

(1) The system makes a rotational band having no coupling with any state which does not belong to the band (the isolated band). We denote each member of the rotational band as \( | \tilde{\gamma}_\alpha; RR \rangle (R = R_\alpha, R_\alpha + 1, R_\alpha + 2, \ldots) \), the energy of which is \( E(| \tilde{\gamma}_\alpha; R \rangle) \). The quantities \( R \) and \( R_\alpha \) are the angular momentum and its projection, respectively, and \( \tilde{\gamma}_\alpha \) denotes a set of additional quantum numbers specifying the band. Of course, \( R_\alpha \) is a half-integer.

(2) The energy interval between any states belonging to the band, \( E(| \tilde{\gamma}_\alpha; R \rangle) - E(| \tilde{\gamma}_\alpha; R \rangle) \) (\( \equiv W (RR) \)), are sufficiently small, for example, compared with the characteristic energies of the single-particle excitations.

The main interest in the present paper is to calculate the matrix elements

\[ \langle \tilde{\gamma}_\alpha; R | A_{ij}^\pm | \tilde{\gamma}_\alpha; R \rangle \text{ and } \langle \tilde{\gamma}_\alpha; R | B_{ij}^\pm | \tilde{\gamma}_\alpha; R \rangle \text{ and the excitation energy } W (RR) \text{ on the basis of the constraints (2-6) and (2-7) combined with the above-mentioned assumptions. Hereafter, we will use the following abbreviation for the matrix element of any tensor operator } \mathcal{Q}_{ij}^\pm: \]

\[ \langle \tilde{\gamma}_\alpha; R | \mathcal{Q}_{ij}^\pm | \tilde{\gamma}_\alpha; R \rangle \equiv \mathcal{Q}_{ij} (RR'). \]
The quantities $A^{(\pm)}_{J}(RR')$ and $B^{(\pm)}_{J}(RR')$ satisfy the relations

$$A^{(\pm)}_{J}(RR') = \pm (-)^{R-R'} \sqrt{\frac{2R'+1}{2R+1}} A^{(\pm)}_{J}(R'R),$$
$$B^{(\pm)}_{J}(RR') = (-)^{R-R'} \sqrt{\frac{2R'+1}{2R+1}} B^{(\pm)}_{J}(R'R).$$

(3.2)

Now, let us rewrite our basic relations (2.6) and (2.7) in terms of the matrix elements:

$$W(RR') A^{(\pm)}_{J}(RR') = P^{(\pm)}_{J}(RR'), \quad W(RR') B^{(\pm)}_{J}(RR') = Q^{(\pm)}_{J}(RR'),$$

(3.3)

$$B^{(\pm)}_{J}(RR') = R^{(\pm)}_{J}(RR'), \quad 0 = S^{(\pm)}_{J}(RR'),$$

(3.4)

$$P^{(\pm)}_{J}(RR') = 2(\epsilon - \lambda) A^{(\pm)}_{J}(RR') - \delta_{J0}^{(RR)} (1 \pm 1) / 2 \sqrt{2j+1} A^{(RR)}$$
$$+ (\delta_{J1}^{(RR)} + \delta_{J0}^{(R'R)}) B^{(\pm)}_{J}(RR')$$
$$- \chi q \sum_{I} Z(2J) \sqrt{(2J+1)(2R''+1)} [W(RR'2I; JR'')$$
$$\times Q(RR'') A^{(\pm)}_{J}(R''R') + W(R'R2I; JR'') A^{(\pm)}_{J}(RR'')Q(R''R')],$$

(3.5a)

$$Q^{(\pm)}_{J}(RR') = (\delta_{J1}^{(RR)} + \delta_{J0}^{(R'R)}) A^{(\pm)}_{J}(RR') - \chi q \sum_{I} Z(2J) \sqrt{(2J+1)(2R''+1)}$$
$$\times [W(RR'2I; JR'')Q(RR'')B^{(\pm)}_{J}(R''R')$$
$$- W(R'R2I; JR'')B^{(\pm)}_{J}(RR'')Q(R''R')],$$

(3.5b)

$$R^{(\pm)}_{J}(RR') = 1/2 \sum_{ILJ} Z(1LJ) \sqrt{(2J+1)(2R''+1)}$$
$$\times [W(RR'LI; JR'') (B^{(+)}_{L}(RR'') B^{(-)}_{J}(R''R') \mp A^{(+)}_{L}(RR'') A^{(-)}_{J}(R''R')$$
$$+ B^{(-)}_{L}(RR'') B^{(+)}_{J}(R''R') \mp A^{(-)}_{L}(RR'') A^{(+)}_{J}(R''R'))$$
$$+ W(R'RLL; JR'') (-)^{I+L+J} (B^{(+)}_{L}(RR'') B^{(-)}_{J}(R''R') \mp A^{(+)}_{L}(RR'') A^{(-)}_{J}(R''R')$$
$$\times A^{(+)}_{L}(R''R') + B^{(+)}_{L}(RR'') B^{(-)}_{L}(R''R') \mp A^{(-)}_{L}(RR''R') A^{(+)}_{L}(R''R'))],$$

(3.6a)

$$S^{(\pm)}_{J}(RR') = 1/2 \sum_{ILJ} Z(1LJ) \sqrt{(2J+1)(2R''+1)}$$
$$\times [W(RR'LI; JR'') (B^{(+)}_{L}(RR'') A^{(+)}_{J}(R''R') \mp A^{(+)}_{L}(RR'') B^{(+)}_{J}(R''R')$$
$$+ B^{(-)}_{L}(RR'') A^{(-)}_{J}(R''R') \mp A^{(-)}_{L}(RR'') B^{(+)}_{J}(R''R'))$$
$$+ W(R'RLL; JR'') (-)^{I+L+J} (A^{(+)}_{L}(RR'') B^{(+)}_{J}(R''R') \mp B^{(+)}_{L}(RR'') A^{(+)}_{J}(R''R')$$
$$\times A^{(+)}_{L}(R''R') + A^{(+)}_{L}(RR'') B^{(-)}_{L}(R''R') \mp B^{(-)}_{L}(RR'') A^{(+)}_{L}(R''R'))],$$

(3.6b)

*) Here we have neglected the constant term in Eqs. (2.7b) in a sense similar to the case of even system given in I.
These relations have been derived with the use of the spectral decomposition method combined with the assumption (1). The operators given in Eq. (2·5) can be expressed in the following matrix forms:

\[ N(RR) = \sqrt{2j+1}B_0(=n), \]
\[ \mathcal{A}(RR) = 1/2 \cdot G \sqrt{2j+1}A_0(=\mathcal{A}(RR)), \quad Q(RR') = qB_1(=\mathcal{A}(RR')). \]  

(3·7)

The set of Eqs. (3·3), (3·4) and (3·7) is difficult for us to solve exactly. However, with the help of a kind of self-consistent method, it is possible to solve them successively from lower to higher order, starting from an initial condition mentioned below. We initially regard certain parts of \( \mathcal{A}(RR) \) and \( Q(RR') \) as the parameters to be determined by connecting with the final solutions \( A_0(=\mathcal{A}(RR)) \) and \( B_1(=\mathcal{A}(RR')) \) through Eq. (3·7). The assumption (2) permits us to put \( W(RR') \) as zero firstly, which is the initial condition. Under this condition, we can solve Eqs. (3·3) and (3·4). We call this stage the zeroth order approximation. The next stage (the first order approximation) can be obtained by a certain linearization of any quadratic term combined with the results of the zeroth order approximation. The higher stages are also obtained by a similar method. In the present paper, we will show the solution up to the first order approximation.

§ 4. Zeroth order approximation

Following the solution procedure given in § 3, we will investigate the structure of the zeroth order approximation. The basic equations at this stage can be obtained by putting \( W(RR') \) as zero:

\[ 0 = P_{j}^{(\pm)}(RR')^{(0)}, \quad 0 = Q_{j}^{(\pm)}(RR')^{(0)}, \]  
\[ B_{j}^{(\pm)}(RR')^{(0)} = R_{j}^{(\pm)}(RR')^{(0)}, \quad 0 = S_{j}^{(\pm)}(RR')^{(0)}, \]  

where the superscript (0) denotes the zeroth order approximation. Equations (4·1) and (4·2) have the following types of the solutions:

\[ \begin{pmatrix} A_{j}^{(\pm)}(RR')^{(0)} \\ B_{j}^{(\pm)}(RR')^{(0)} \end{pmatrix} = \begin{pmatrix} a_{j}^{(\pm)} \\ b_{j}^{(\pm)} \end{pmatrix} (-)^{j}\langle R-R_0|0|R'-R_b \rangle. \]  

(4·3a)

It can be derived that the quantities \( \mathcal{A}(RR) \) and \( Q(RR') \) should have the following \((R, R')\)-dependence at the zeroth order approximation:

\[ \mathcal{A}(RR) = \mathcal{A}(=\mathcal{A}^{(0)}(RR)), \]
\[ Q(RR') = Q \cdot \langle R-R_0 | 0 | R'-R_b \rangle (=Q^{(0)}(RR')). \]  

(4·3b)

where \( \mathcal{A} \) and \( Q \) are constants to be initially regarded as the parameters. The quantities \( a_{j}^{(\pm)} \) and \( b_{j}^{(\pm)} \) have to satisfy the following equations:
In order to determine $a_j^{(z)}$ and $b_j^{(z)}$, it is convenient to introduce new variables $a_m^{(z)}$ and $b_m^{(z)}$ defined as

$$
\begin{align*}
0 &= 2(\epsilon - \lambda) a_j^{(z)} - \lambda(\theta_0 - 2b_j^{(z)}) - 2\gamma q \sum_T Z(T2J)\langle J00|J0\rangle Q a_i^{(z)}, \\
0 &= a_j^{(z)}, \\
b_j^{(z)} &= \sum_T Z(TILJ)\langle J00|J0\rangle (a_i^{(z)}a_{L}^{(z)} + b_i^{(z)}b_{L}^{(z)} + b_i^{(-z)}b_{L}^{(-z)}), \\
b_j^{(z)} &= 2\sum_T Z(TILJ)\langle J00|J0\rangle b_i^{(-z)}b_{L}^{(-z)}, \\
0 &= \sum_T Z(TILJ)\langle J00|J0\rangle b_i^{(-z)}a_{L}^{(-z)}.
\end{align*}
$$

(4.4)

We can transform Eqs. (4.4) into the following forms in terms of the new variables:

$$
\begin{align*}
2\varepsilon_m a_m^{(+)} &= \Delta (1 - 2b_m^{(-)}), \\
\varepsilon_m &= \frac{\langle jm|T0|20\rangle (-)^J_m}{\langle jm|T0|20\rangle (-)^J_m}, \\
b_m^{(+) - b_m^{(-)}} &= a_m^{(z)} + b_m^{(z)}, \\
b_m^{(-)} &= 2b_m^{(-)}b_m^{(-)}, \\
0 &= b_m^{(-)}a_m^{(z)},
\end{align*}
$$

(4.6)

where $\varepsilon_m$ is defined as

$$
\varepsilon_m = (\epsilon - \lambda) - \gamma q \langle jm|T0|20\rangle (-)^J_m Q.
$$

(4.7)

Now, let us search for the solutions of Eqs. (4.6). They can be classified into the following two cases:

(i) The case $b_m^{(-)} = 0$:

$$
\begin{align*}
a_m^{(+)} &= \frac{1}{2}, \\
b_m^{(+) - b_m^{(-)}} &= \frac{1}{2}, \\
0 &= b_m^{(-)}a_m^{(z)},
\end{align*}
$$

(4.8)

(ii) The case $b_m^{(-)} \neq 0$:

$$
\begin{align*}
a_m^{(+)} &= 0, \\
b_m^{(+) - b_m^{(-)}} &= 1/2, \\
0 &= b_m^{(-)}a_m^{(z)},
\end{align*}
$$

(4.9)

It should be noted that all quantities are given in terms of functions of $\lambda$, $\Delta$, and $Q$. We can see that case (i) is of familiar form in the even-mass system. In fact, case (i) is completely identical with the solution of the Hartree-Bogoliubov theory applied to the Hamiltonian (2.4) if we determine $\lambda$, $\Delta$, and $Q$ through Eq. (3.7) at this stage. Therefore, case (ii) is conjectured to reflect the influence of the last odd particle. Finally, it should be noted that $m$ belonging to case (ii) cannot be specified and remains undetermined, up to the present stage.

§ 6. First order approximation

Now we proceed to give the first order equations. Equations (3.3) and (3.4)
are composed of linear and quadratic terms with respect to $A_{i}^{(±)}(RR')$, $B_{i}^{(±)}(RR')$ and $W(RR')$. We replace the quadratic parts with the products of the zeroth and first order quantities such as, for example,

$$[B_{i}^{(±)}(RR'')B_{i}^{(±)}(R''R')]^{(1)}$$

$$= B_{i}^{(±)}(RR'')^{(0)}B_{i}^{(±)}(R''R')^{(1)} + B_{i}^{(±)}(RR'')^{(1)}B_{i}^{(±)}(R''R')^{(0)}, \quad (5\cdot1)$$

where (1) denotes the first order approximation. Then, we obtain the first order equations:

$$W(RR')^{(1)} A_{i}^{(±)}(RR')^{(0)} = P_{i}^{(±)}(RR')^{(1)},$$

$$W(RR')^{(1)} B_{i}^{(±)}(RR')^{(0)} = Q_{i}^{(±)}(RR')^{(1)},$$

$$B_{i}^{(±)}(RR')^{(1)} = R_{j}^{(±)}(RR')^{(1)}, \quad 0 = S_{i}^{(±)}(RR')^{(1)}. \quad (5\cdot3)$$

Here we notice that a new term, which we will denote as $Q_{i}^{(±)}(RR')$, appears at this stage (Eq. (4·3)). Equations (5·2) and (5·3) have the following types of the solutions:

$$W(RR')^{(1)} = [R(R+1) - R'(R'+1)]/2\beta,$$

$$\begin{align*}
\frac{A_{i}^{(±)}(RR')^{(1)}}{B_{i}^{(±)}(RR')^{(1)}} &= \frac{1}{\sqrt{2}} \left( \frac{\alpha_{i}^{(±)}}{\beta_{i}^{(±)}} \right) \frac{1}{2\beta} \left[ (-y') \sqrt{(R-R_{0}) (R+R_{0}+1)} 
\times \langle R-R_{0}-1|J| R'-R_{0} \rangle + (-y') \sqrt{(R'+R_{0}) (R'-R_{0}+1)} 
\times \langle R-R_{0}+1|J| R'-R_{0} \rangle \pm \sqrt{(R'-R_{0}) (R'+R_{0}+1)} \langle R-R_{0}+1|J| R'-R_{0} \rangle, \right. \\
&\left. = \sqrt{(R'-R_{0}) (R'-R_{0}+1)} \langle R-R_{0}+1|J| R'-R_{0} \rangle \right]. \quad (5\cdot5a) \\
Q_{i}^{(±)}(RR') \text{ is given by} \quad Q_{i}^{(±)}(RR') &= \left( \tilde{Q}/\sqrt{2} \right) \left[ \sqrt{(R'-R_{0}) (R'-R_{0}+1)} \langle R-R_{0}+2|J| R'-R_{0}+1 \rangle 
- \sqrt{(R'-R_{0}) (R'+R_{0}+1)} \langle R-R_{0}+1|J| R'-R_{0}+1 \rangle \right], \quad (5\cdot5b)
\end{align*}$$

where $\tilde{Q}$ is a constant. Each coefficient in Eq. (5·5a) must be determined from the following equations:

$$\begin{align*}
- \sqrt{J(J+1)}/2a_{j}^{(±)} = 2(\epsilon-\lambda)\tilde{\alpha}_{j}^{(±)} + (1 \pm 1)J\tilde{\beta}_{j}^{(±)} \\
-2\chi \sum_{I} Z(I2J) \langle I20|J1\rangle Q\tilde{\alpha}_{i}^{(±)} + \langle I021|J1\rangle a_{i}^{(±)} \tilde{Q}, \\
- \sqrt{J(J+1)}/2b_{j}^{(±)} = (1 \mp 1)\tilde{\alpha}_{j}^{(±)} \\
+ 2\chi \sum_{I} Z(I2J) \langle I20|J1\rangle Q\tilde{\beta}_{i}^{(±)} + \langle I021|J1\rangle b_{i}^{(±)} \tilde{Q}, \\
\tilde{\beta}_{j}^{(±)} = 2\sum_{I} Z(I2J) \langle I01|J1\rangle [b_{j}^{(±)} \tilde{\alpha}_{j}^{(±)} + a_{j}^{(±)} \tilde{\alpha}_{j}^{(±)} + b_{j}^{(±)} \tilde{\beta}_{j}^{(±)}], \\
O = (1 - (-y')^{2}) \sum_{I} Z(I2J) \langle I01|J1\rangle [b_{j}^{(±)} \tilde{\alpha}_{j}^{(±)} - a_{j}^{(±)} \tilde{\beta}_{j}^{(±)} - b_{j}^{(±)} \tilde{\alpha}_{j}^{(±)}]. \quad (5\cdot7)
\end{align*}$$

Let us determine $\tilde{\alpha}_{j}^{(±)}$ and $\tilde{\beta}_{j}^{(±)}$ from Eqs. (5·6) and (5·7) by introducing the
following new variables:

\[
\begin{align*}
\tilde{\alpha}_m^{(\pm)} &= \sum_j j m + 1 j - m | J I \rangle \langle -J -m | \tilde{\alpha}_m^{(\pm)}, \\
\tilde{\beta}_m^{(\pm)} &= -\tilde{\alpha}_m^{(-m-1)}, \quad \tilde{\beta}_m^{(\pm)} = \mp \tilde{\beta}_m^{(-m-1)}.
\end{align*}
\]

Then, Eqs. (5.6) and (5.7) can be transformed into the following forms:

\[
\begin{align*}
\sqrt{(j - m) (j + m + 1) / 2} \cdot (a_{m+1}^{(\pm)} - a_m^{(\pm)}) &= (\varepsilon_{m+1} + \varepsilon_m) \tilde{\alpha}_m^{(\pm)} \\
&+ (1 \pm 1) \delta \tilde{\beta}_m^{(\pm)} + \tilde{\varepsilon}_m (a_{m+1}^{(+)} + a_m^{(\pm)}), \\
\sqrt{(j - m) (j + m + 1) / 2} \cdot (b_{m+1}^{(\pm)} - b_m^{(\pm)}) &= (\varepsilon_{m+1} - \varepsilon_m) \tilde{\beta}_m^{(\pm)} \\
&- (1 \mp 1) \delta \tilde{\alpha}_m^{(\pm)} - \tilde{\varepsilon}_m (b_{m+1}^{(+)} - b_m^{(\pm)}),
\end{align*}
\]

where \( \tilde{\varepsilon}_m \) denotes

\[
\tilde{\varepsilon}_m = -\chi g \langle jm + 1 j - m | 21 \rangle (\varepsilon_J - m) \tilde{Q}.
\]

The solutions of the above equations are given by

\[
\begin{align*}
\tilde{\alpha}_m^{(\pm)} &= -(\varepsilon_{m+1} - \varepsilon_m)^{-1} \left[ 2 \delta \sqrt{(j - m) (j + m + 1) / 2} (a_{m+1}^{(\pm)} + a_m^{(\pm)}) (b_{m+1}^{(-)} - b_m^{(-)}) - \tilde{\varepsilon}_m (a_{m+1}^{(\pm)} - a_m^{(\pm)}) \right], \\
\tilde{\beta}_m^{(\pm)} &= (\varepsilon_{m+1} - \varepsilon_m)^{-1} \left[ \sqrt{(j - m) (j + m + 1) / 2} (b_{m+1}^{(\pm)} - b_m^{(\pm)}) + \tilde{\varepsilon}_m (b_{m+1}^{(\pm)} - b_m^{(-)}) \right], \\
\tilde{\alpha}_m^{(\pm)} &= (\varepsilon_{m+1} - \varepsilon_m)^{-1} \left[ \sqrt{(j - m) (j + m + 1) / 2} (a_{m+1}^{(\pm)} - a_m^{(-)}) \right], \\
\tilde{\beta}_m^{(\pm)} &= (\varepsilon_{m+1} - \varepsilon_m)^{-1} \left[ \sqrt{(j - m) (j + m + 1) / 2} (b_{m+1}^{(\pm)} - b_m^{(\pm)}) - 2 \delta (\varepsilon_{m+1} + \varepsilon_m)^{-1} \right] \\
&\times (a_{m+1}^{(\pm)} - a_m^{(\pm)}) + \tilde{\varepsilon}_m (b_{m+1}^{(-)} - b_m^{(-)})].
\end{align*}
\]

It is clear that all quantities given at this stage are functions of \( \mathcal{J} \) and \( \tilde{Q} \) in addition to \( \mathcal{J} \) and \( \tilde{Q} \).

**§ 6. Determination of excitation energies and transition matrix elements**

We are now in a stage to give a complete determination of the transition matrix elements and the excitation energies. The quantities \( a_m^{(\pm)} \), \( b_m^{(\pm)} \), \( \tilde{\alpha}_m^{(\pm)} \) and \( \tilde{\beta}_m^{(\pm)} \) can be given as functions of \( \mathcal{J} \) and \( \tilde{Q} \) if \( m \) belonging to case (ii) is specified and \( \tilde{Q} \) and \( \mathcal{J} \) are given.

To this end, we notice that quadrupole moment \( Q_{JM} \) and angular momentum \( J_J \) are given in terms of \( q \cdot B_{3J}^{(\pm)} \) and \( \sqrt{1/3 \cdot j (j+1) (2j+1) \cdot B_{3J}^{(\pm)} \cdot B_{3J}^{(\pm)}} \) respectively. Within the framework of our approximation, these relations lead us to the following:
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\[ \sqrt{R(R+1)} \delta_{RR} = \sqrt{1/3} \right j (j+1) (2j+1) (B_l^{(-1)} (RR')^{(0)} + B_l^{(-1)} (RR')^{(1)}), \quad (6.1a) \]

\[ \tilde{Q}(\sqrt{\mathcal{Z}} \mathcal{G})^{-1} \left[ \sqrt{(R' + R_0)} (R' - R_0 + 1) \langle R - R_0 2\rangle | R' - R_0 - 1 \rangle \right] = qB_t^{(2)} (RR')^{(1)} \cdot (6.1b) \]

From the above relations, we can obtain

\[ R_0 = \sum_m m \cdot b_m^{(-)} (= 2 \sum_{m > 0} mb_m^{(-)}), \quad (6.2a) \]

\[ \mathcal{J} = - \sum_m \sqrt{(j-m)(j+m+1)/2} \tilde{\mathcal{J}}_m^{(-)}, \quad (6.2b) \]

\[ \tilde{Q} = q \sum_m \langle jm + 1 | j - m | 21 \rangle (-)^{j-m} \tilde{\mathcal{J}}_m^{(+)}. \quad (6.2c) \]

Let us start from specifying \( m \) which belongs to case (ii). Equation (6.2a) gives a constraint to many solutions (4.8) and (4.9) for \( b_m^{(-)} \). We consider the simple case where only one pair \( b_m^{(-)} \) has non-zero value and others vanish. Then, from Eq. (6.2a), we get

\[ R_0 = 2mb_m^{(-)}. \quad (m > 0) \quad (6.3) \]

Since \( b_m^{(-)} = \pm 1/2 \) (Eq. (4.9)), we have the conclusion

\[ m = R_0, \quad b_m^{(-)} = \pm 1/2. \quad (6.4) \]

Next we will determine \( \mathcal{J} \) by substituting \( \tilde{\mathcal{J}}_m^{(-)} \) given in Eqs. (5.11) into Eq. (6.2b). \( \mathcal{J} \) consists of two terms as follows:

\[ \mathcal{J} = \mathcal{J}_o + \mathcal{J}_1, \quad (6.5a) \]

\[ \mathcal{J}_o = \frac{1}{2} \sum_m \frac{(j-m)(j+m+1)}{\mathcal{E}_{m+1} - \mathcal{E}_m} \left[ \frac{2}{\mathcal{E}_{m+1}} (a_m^{(+)} - a_m^{(-)}) - (b_m^{(+)} - b_m^{(-)}) \right], \quad (6.5b) \]

\[ \mathcal{J}_1 = - \sum_m \sqrt{(j-m)(j+m+1)} \cdot \mathcal{E}_m \left( b_m^{(+)} - b_m^{(-)} \right). \quad (6.5c) \]

The first term \( \mathcal{J}_o \) can be rewritten as

\[ \mathcal{J}_o = \frac{1}{2} \sum_m \frac{(j-m)(j+m+1)}{E_{m+1} + E_m} \cdot \frac{1}{2} \left( 1 - \mathcal{E}_m + \mathcal{J}^2 \right) \]

\[ - \frac{1}{2} \sum_{m \neq R_0} \frac{(j-m)(j+m+1)}{E_{m+1} + E_m} \cdot \frac{1}{2} \left( 1 - \mathcal{E}_m + \mathcal{J}^2 \right) \]

\[ + \frac{1}{2} \sum_{m \neq R_0} \frac{(j-m)(j+m+1)}{E_{m+1} - E_m} \cdot \frac{1}{2} \left( 1 + \mathcal{E}_m + \mathcal{J}^2 \right). \quad (6.6) \]

The above \( \mathcal{J}_o \) has a form analogous to the cranking moment of inertia in odd-mass nuclei. The first term corresponds to the moment of inertia of even-mass core and the others describe the influence of last odd quasi-particle. To be interesting is the existence of the additional term \( \mathcal{J}_1 \), which will be discussed later. \( \tilde{Q} \) can be
determined from Eq. (6.2c) as follows:

$$\tilde{Q} = q \sum_{m=0}^{\infty} \sqrt{(j-m)(j+m+1)/2} \langle jm + 1 - m | 21 \rangle (-)^{j-m} (\varepsilon_{m+1} - \varepsilon_m)^{-1} (b_{m+1}^{(-)} - b_m^{(-)})$$

Thus, in terms of $\lambda$, $J$ and $Q$ we could express all quantities appearing in our theory.

Finally, we mention how to determine values of $\lambda$, $J$ and $Q$. Within the framework of the zeroth and first order approximation, these values cannot be determined. The reason is the following: If we determine these values by the use of Eqs. (3·7) at this stage of approximation, the value of $\tilde{Q}$ expressed in Eq. (6.7) becomes infinite. Therefore, inevitably, we have to proceed to the stage of the second order approximation at which we should exploit Eq. (3·7). In fact the terms with the same dependence of $R$ as the zeroth order approximation appear at this approximation. With the help of these terms, we can determine the values of $\lambda$, $J$ and $Q$ by Eqs. (3·7):

$$n = \sum_{m=0}^{\infty} \left[ \frac{1}{2} \left[ 1 - (\varepsilon_m/E_m) \right] + \frac{1}{2} \left( a_m^{(-)} - b_m^{(-)} \right) \right],$$

$$4/G = \sum_{m=0}^{\infty} \left[ \frac{1}{2} \left( a_m^{(-)} + b_m^{(-)} \right) \right],$$

$$Q = q \sum_{m=0}^{\infty} \left[ jm + 1 - m | 20 \rangle (-)^{j-m} 1/2 \left[ 1 - (\varepsilon_m/E_m) \right] \right]$$

$$+ q \langle jRm | jRm | 20 \rangle (-)^{j-Rm} + \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \left[ jm + 1 - m | 20 \rangle (-)^{j-Rm} \right]$$

where $a_m^{(-)}$ and $b_m^{(-)}$ appear as the second order correction terms to $a_m^{(+)}$ and $b_m^{(+)}$ respectively. These can be regarded as the relations extended from those in the conventional Hartree-Bogoliubov theory. Of course second order quantities can be determined by the procedure similar to the first order approximation. Within our qualitative discussion, it may be unnecessary to give the detailed results.

§ 7. Discussion

7·1) Physical interpretation of the additional term to the cranking moment of inertia

One of the most important results of our theory is the appearance of the additional term $\tilde{J}_1$ to the cranking moment of inertia. In this subsection, we will give physical interpretation of this term from various aspects.

First, let us rewrite $\tilde{J}_1$ given in Eq. (6.5c) more concretely. With the aid of Eqs. (5·10), the relation (6.5c) is rewritten as

$$\tilde{J}_1 = \tilde{r} \cdot \tilde{Q},$$

$$\tilde{r} = \frac{zq}{2} \sum_{m=0}^{\infty} \sqrt{(j-m)(j+m+1)} \langle jm + 1 - m | 21 \rangle (-)^{j-m} (b_{m+1}^{(-)} - b_m^{(-)}).$$
\( \mathcal{Q} \) is given in Eq. (6·7). The above relation tells us that \( \mathcal{J}_1 \) vanishes if all \( b^{(-)}_{m} \) are equal to zero. This means that all \( m \) belong to the solutions of the case (i), i.e., in the sense of the conventional Hartree-Bogoliubov approach, the occupation probabilities of any single-particle state and its time-reversed one are the same. However, \( \mathcal{J}_1 \) does not vanish in our present system, because \( b^{(+)0}_2 \) is equal to \( \pm \frac{1}{2} \). Our results \( b^{(+)0}_2 = 1/2 \) and \( b^{(-)0}_2 = \pm \frac{1}{2} \) can be translated into the situation that the occupation probabilities of the state \( R_0 \) and \( -R_0 \) are equal to unity and zero, respectively, in the sense of the Hartree-Bogoliubov theory. Therefore, this is just our expectation mentioned in § 1.

Next we will investigate the physical meaning of \( \mathcal{J}_1 \) in the language of rotational model. For this purpose, let us express our pair operators in terms of \( D \)-function and angular momentum operators as was done in I for the case of even-mass system:

\[
\begin{pmatrix}
A^{(+)0}_J \\
B^{(+)0}_J
\end{pmatrix} = \begin{pmatrix}
\alpha^{(+)0}_J \\
\beta^{(+)0}_J
\end{pmatrix} D^{(0)}_{J=0}(\theta) + \frac{1}{\mathcal{J}} \begin{pmatrix}
\tilde{\alpha}^{(+)0}_J \\
\tilde{\beta}^{(+)0}_J
\end{pmatrix} \frac{1}{2} \left[ [R_1, D^{(0)}_{J=1}(\theta)], + [R_{-1}, D^{(0)}_{J=1}(\theta)] \right].
\]

\[
\begin{pmatrix}
A^{(-)0}_J \\
B^{(-)0}_J
\end{pmatrix} = \begin{pmatrix}
0 \\
\beta^{(-)0}_J
\end{pmatrix} D^{(0)}_{J=0}(\theta) + \frac{1}{\mathcal{J}} \begin{pmatrix}
\tilde{\alpha}^{(-)0}_J \\
\tilde{\beta}^{(-)0}_J
\end{pmatrix} \frac{1}{2} \left[ [R_1, D^{(0)}_{J=1}(\theta)], - \gamma \gamma' [R_{-1}, D^{(0)}_{J=1}(\theta)] \right].
\]

(7·2a)

(7·2b)

where \( D^{(0)}_{J}(\theta) \) and \( R_K \) satisfy

\[
[R_K, D^{(0)}_{J}(\theta)] = \sqrt{J(J+1)} \sum_{\nu} \langle J\nu 1 K | J\nu \rangle D^{(0)}_{J}(\theta),
\]

\[
[R_K, R_\nu] = -\sqrt{2} \sum_{\nu} \langle J\nu 1 K | J\nu \rangle R_\nu.
\]

(7·3)

All the coefficients on the right-hand sides should be the same as those given in the previous sections in order to insure that these pair operators satisfy dynamical and kinematical constraints under our assumptions. In contrast to the representations (7·2), we obtained the following relations for the \( K=0 \) band in I:

\[
\begin{pmatrix}
A^{(+)0}_J \\
B^{(+)0}_J
\end{pmatrix} = \begin{pmatrix}
\alpha^{(+)0}_J \gamma \\
\beta^{(+)0}_J \gamma
\end{pmatrix} D^{(0)}_{J}(\theta),
\]

\[
\begin{pmatrix}
A^{(-)0}_J \\
B^{(-)0}_J
\end{pmatrix} = \frac{1}{\mathcal{J}} \begin{pmatrix}
\tilde{\alpha}^{(-)0}_J \gamma \\
\tilde{\beta}^{(-)0}_J \gamma
\end{pmatrix} \frac{1}{2} \left[ [R_1, D^{(0)}_{J=1}(\theta)], - \gamma \gamma' [R_{-1}, D^{(0)}_{J=1}(\theta)] \right].
\]

(7·4a)

(7·4b)

The relations (7·4) give the so-called rigid rotor rules for transition matrix elements. Therefore, the differences between Eqs. (7·2) and (7·4) can be regarded as the effects due to the addition of one particle to the rigid rotor. In general, it may be expected that the total system is influenced in the following two manners: i) The geometrical shape of nucleus may be different from that of the even system and ii) the direction of the principal axes may be changed. Of course, these two effects cannot be distinguished clearly. The second effect, however, is not taken into account usually when we apply the cranking model to the odd-mass system. In
our present case, $A_{ij}^{(x)}$ and $B_{ij}^{(y)}$ in Eq. (7·2a) have the terms $D^{(i)}_{ij}(\theta)$ with $\mu = \pm 1$. On the other hand, the relation (7·4a) does not contain such terms. It may be interpreted that our principal axes are different from those of the Hartree-Bogoliubov field and rather fluctuate around them by quantum effect. Thus we are led to the conclusion that our theory contains the above-mentioned two effects on the equal footing.

On the basis of the above arguments, let us consider the physical meaning of $\mathcal{J}_1$. Since $\mathcal{J}_1$ is proportional to $\tilde{Q}$ (Eq. (7·1a)) and $\tilde{Q}$ is related to the fluctuation of principal axes (Eq. (7·2a)), we may mention that it reflects mainly the second effect. Therefore, $\mathcal{J}_1$ comes from the new effect, which has not been considered in the conventional approach, i.e., the cranking theory.

7-2) The comparison with Belyaev-Zelevinsky description of rotational motion

As was mentioned in § 1, Belyaev and Zelevinsky proposed a microscopic theory of rotation in 1970, and in 1973 they reformulated their basic idea in terms of the method of generalized density matrix in the theory of collective excitations. Our theory is, in some sense, the extension of their former theory to the case of the odd-mass system. In this section, we will make a comparison between their theory and ours.

Belyaev and Zelevinsky described the ground-state rotational band of an even-mass nucleus ($K=0$), starting from the same assumptions as those mentioned in § 3. They gave a solution procedure after stressing that the assumption (2) is consistent with the following: $A_{ij}^{(x)}$ and $B_{ij}^{(y)}$ are large, while $A_{ij}^{(x)}$ and $B_{ij}^{(y)}$ are small. However, this estimation is contradictory to our kinematical constraints in the case of the odd-mass system. In fact, our solutions show that $B_{ij}^{(y)}$ is of the same order as $A_{ij}^{(x)}$ and $B_{ij}^{(y)}$. It is noted, however, that our solutions reduce to the ones given by Belyaev and Zelevinsky in the formal limit $R_0 \to 0$. Therefore, we can conclude that assumption (2) is completely independent of the estimation of $A_{ij}^{(x)}$ and $B_{ij}^{(y)}$ in the general case. The consistency of the case $K=0$ band is no more than accidental.

7-3) Concluding remark

We developed a microscopic theory of rotational motion from the algebraic approach. Characteristics of our theory are the following:

i) The spherical symmetry is not violated at any stage, ii) the notion of principal axes is not employed in the explicit form and iii) the information of the adjacent even-mass nuclei is unnecessary. These are in contrast to the Hartree-Bogoliubov theory applied to the cranking Hamiltonian. Owing to these merits of our theory, we found the additional term of the moment of inertia in the odd-mass nuclei which did not appear in the cranking calculation. Further, the interesting term is of the same order as the usual ones. Therefore, it may be important to investigate this term in the realistic model.
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