Proton Electromagnetic Form Factors at Very High Momentum Transfers

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(Received November 15, 1975)

The electromagnetic form factors of the proton at asymptotically large space-like momentum transfers are studied on the basis of usual relativistic field theory. On the assumption of convergence of the Low equation, we obtain the asymptotic sum rules which relate the form factors to the real part of the on-mass-shell proton-proton elastic scattering amplitudes. The rough evaluation of the sum rules for the 4-momentum transfer squared $t$ up to $\sim -700$ GeV$^2$ is made with the information of high energy scattering experiments at the ISR. The usual "dipole" behavior of the form factors has not been expected. Some interesting features of the asymptotic behavior of form factors have emerged.

§ 1. Introduction

The electromagnetic structure of the proton has been investigated by the use of the electron as a probe. From the one photon exchange cross section, the form factors which provide a quantitative description of the internal structure of the proton are extracted. The behavior of them at asymptotically large space-like momentum transfers is related to the behavior of charge and current distribution near the origin in configuration space.

At present we have experimental information about them for the 4-momentum transfer squared $t$ up to about $-25$ GeV$^2$. And there has been a large number of theoretical models fitting the data, of widely differing functional forms in the high momentum transfer region. Each model has been based on particular dynamical assumptions and it has not been clearly discussed on the region of $t$ where the model works. The more definite and rather model independent investigation is needed for understanding the inner structure of proton at a fundamental level.

In this paper, the form factors in the limit $t \to -\infty$ are studied on the basis of usual relativistic local-field theory but without any particular dynamical assumption. That is, the basic assumption is that the electromagnetic form factors of bare or free field particles, which are the constituents of the physical proton, are independent of the 4-momentum transfer. All known strongly interacting particles will be the candidates for the constituents and we exclude the unknown particles, such as quarks.

On the assumption of the convergence of the Low equation, we derive the asymptotic sum rules which relate the form factors to the real part of the physical
proton-proton elastic scattering amplitudes. By using the recent high energy experimental data at the ISR, the rough evaluation of the sum rules becomes practicable for \( t \) up to \( \sim -700 \text{ GeV}^2 \). It is found that the completely empirical approximations at low momentum transfers, which are usually referred to as the form factor "scaling" and the "dipole" behavior, are not expected. The magnetic form factor decreases more slowly than the "dipole" formula, and the behavior of the electric form factor is not monotonic but has a minimum near \( t \sim -200 \text{ GeV}^2 \).

In § 2, the structure of the form factors is analyzed in the pion-nucleon system. In § 3, our simple mathematical procedure is presented. After renormalizing the charge, we can relate the asymptotic form factors apart from a constant to the on-mass-shell scattering amplitudes. The dominance of the proton current contribution is revealed. The rough evaluation of the sum rules is carried out in § 4. In the final section, some concluding remarks are given.

§ 2. Structure of the form factors

Using relativistic invariance and gauge invariance, the matrix element of the electromagnetic current operator between the physical proton states is written in terms of two form factors \( G_E(t) \) and \( G_M(t) \), which are the so-called electric and magnetic form factors respectively. In the particular coordinate system, the Breit system, it can be written in the form

\[
\left\langle \frac{\Delta}{2} \left| J^\rho(0) \right| -\frac{\Delta}{2} \right\rangle = \frac{m}{E} G_E(t) \vec{u} \left( \frac{\Delta}{2} \right) \gamma_\rho u \left( -\frac{\Delta}{2} \right),
\]

\[
\left\langle \frac{\Delta}{2} \left| J^I(0) \right| -\frac{\Delta}{2} \right\rangle = \frac{m}{E} G_M(t) \vec{u} \left( \frac{\Delta}{2} \right) \gamma_5 u \left( -\frac{\Delta}{2} \right),
\]

where \( J^\rho(0) \) is the electromagnetic current operator in the Heisenberg representation at the space time point \( x_\mu = 0 \). And \( u(\Delta/2) \) is the positive energy Dirac spinor of the proton of mass \( m \), momentum \( \Delta/2 \) normalized \( uu = -1 \), \( E = (\Delta^2/4 + m^2)^{1/2} \) and \( t = -\Delta^2/4 \). If the proton is "pointlike", \( G_E \) and \( G_M \) are independent of \( t \) and \( G_E = G_M = 1 \).

In order to find the general structure of the matrix element, the left-hand sides of (2.1), in the limit \( \Delta^2 \rightarrow \infty \), we consider the system of the nucleon and pion, for simplicity. Through taking the interaction representation, we can see, to the first order of the electromagnetic interaction, how the bare or free nucleon and pion currents contribute to them. The free current operator is

\[
j^\rho_\nu(x) = j^\rho_\nu^N(x) + j^\rho_\nu^\pi(x)
\]

\[
= \phi_\rho(x) \bar{\phi}_\nu \phi_\mu(x) + \bar{\phi}_\rho(x) \gamma_5 \tau_\rho \psi_\mu(x),
\]

where \( \phi \) and \( \bar{\phi} \) are the nucleon field operators, \( \phi_\rho(x) \) the pion field operator in the interaction representation, and \( \tau_\rho \) are the conventional isotopic spin matrices. Then, the matrix element in the Heisenberg picture can be rewritten formally in the
interaction picture:
\[
\langle \frac{\mathbf{A}}{2} | J_\mu^e(0) | -\frac{\mathbf{A}}{2} \rangle = \left( \frac{\mathbf{A}}{2} \right) T \left( S j_\mu^e(0) \right) \frac{\mathbf{A}}{2} ,
\]
where \( T \) stands for the time-ordered product of operators. \( S = U(\infty, -\infty) \) is the S-matrix which describes the strong interaction of the system and is given by the interaction Hamiltonian \( H_{\text{int}} \):
\[
U(t, -\infty) = T \exp\left[ -i \int_{-\infty}^{t} d\tau H_{\text{int}}(\tau) \right],
\]
\[
H_{\text{int}}(t) = g \int d^3x \bar{\psi}(x) \gamma_5 \tau_\alpha \psi(x) \varphi_\alpha(x).
\]
And,
\[
\frac{-\mathbf{A}}{2} = U(0, -\infty) \frac{\mathbf{A}}{2}.
\]
It is understood that the mass renormalization counter terms for the nucleon and the pion are implicitly included in \( H_{\text{int}} \).

Making use of the Dyson-Wick contraction theorem and after some manipulations, one can go back to the Heisenberg picture and get the contribution of the pion current (Fig. 1(a)):
\[
\bar{M}_\mu^e(\mathbf{A}) = \langle \frac{\mathbf{A}}{2} | J_\mu^e(0) | -\frac{\mathbf{A}}{2} \rangle
\]
\[
= \frac{\left( \partial_{\alpha\beta} \partial_{\beta\gamma} - \partial_{\alpha\gamma} \partial_{\beta\beta} \right)}{(2\pi)^4} \int \frac{d^4k}{i} \frac{k_\mu}{k_+^2 - \left( k + \frac{\mathbf{A}}{2} \right)^2 - \mu^2 + i\epsilon} \frac{k_\gamma}{k_+^2 - \left( k - \frac{\mathbf{A}}{2} \right)^2 - \mu^2 + i\epsilon}
\]
\[
\times \int dx e^{ikx} \left( \frac{\mathbf{A}}{2} \right) \frac{T}{2} \left( j_\mu(x), j_\mu\left( -\frac{x}{2} \right) \right) \frac{\mathbf{A}}{2}.
\]
\[
\frac{\mathbf{A}}{2}
\]

\[
\frac{-\mathbf{A}}{2}
\]
Fig. 1. Diagrams for the pion current contribution (a) and for the proton current contribution (b). The solid line is a nucleon and dashed line a pion.

where \( j_\mu = (\Box + \mu^2) \varphi_\alpha \) is the source operator in the Heisenberg representation and \( \mu \) is the mass of the pion.

Excluding the lowest order diagram (\( S=1 \), in (2.3)), the contribution of the nucleon current (Fig. 1(b)) is expressed as follows:
\( \tilde{\mathcal{M}}^N (\mathcal{A}) = \left\langle \frac{\mathcal{A}}{2} \ ; \ J_x^{(n)} (0) \ ; \ -\frac{\mathcal{A}}{2} \right\rangle \)

\[
\frac{i}{(2\pi)^4} \int dk \frac{h(k) e^{ikx}}{[k_0^2 - (k + (\mathcal{A}/2))^2 - \mu^2 + i\epsilon][k_0^2 - (k - (\mathcal{A}/2))^2 - \mu^2 + i\epsilon]} \times \int dx \ e^{ikx} \left\langle \frac{\mathcal{A}}{2} \ ; \ T \left( f_{\mathcal{A}} \left( \frac{x}{2} \right) , f_{\mathcal{A}} \left( -\frac{x}{2} \right) \right) \ ; \ -\frac{\mathcal{A}}{2} \right\rangle, \tag{2.8}
\]

where \( k = (k_0, k + \mathcal{A}/2), k' = (k_0, k - \mathcal{A}/2) \) and \( \mathcal{P} = p_0 \gamma_0 - p \cdot \gamma \) and \( f = (i\gamma^a \partial_a - m) \phi \) is the source operator of the proton in the Heisenberg representation. We will take account of the contribution of the lowest order diagram with the problem of the renormalization in § 3.C.

§ 3. Derivation of the sum rules

A. Feynman integral in the limit \( |\mathcal{A}| \to \infty \)

With the choice of the Breit system, the 4-momentum transfer between the proton states has only spatial components and this situation simplifies our study of the properties of the integral over the internal momentum in the limit \( \mathcal{A} \to \infty \).

Let us consider the integral in (2.7) and (2.8)

\[
I(\mathcal{A}, x) = \int dk \frac{h(k) e^{ikx}}{[k_0^2 - (k + (\mathcal{A}/2))^2 - \mu^2 + i\epsilon][k_0^2 - (k - (\mathcal{A}/2))^2 - \mu^2 + i\epsilon]}, \tag{3.1}
\]

where \( h(k_0) = 1, k_0 \) or \( k_0^2 \). Equivalently, it can be rewritten in the form

\[
I(\mathcal{A}, x) = -\frac{1}{2} \int dk \frac{h(k) e^{ikx}}{k \cdot \mathcal{A}} \left( \frac{1}{k_0^2 - (k - (\mathcal{A}/2))^2 - \mu^2 + i\epsilon} - \frac{1}{k_0^2 - (k + (\mathcal{A}/2))^2 - \mu^2 + i\epsilon} \right) \]

\[
-\int_0^1 dt e^{ikx} \int dk \frac{h(k) e^{ikx} \exp(ikx)}{k \cdot \mathcal{A}} \sin \frac{\mathcal{A} \cdot k}{k \cdot \mathcal{A}}. \tag{3.2}
\]

Consider this integral in a frame in which \( \mathcal{A} \) lies in the third direction, then the integrand has the infinitely oscillating factor \( \sin (\mathcal{A} \cdot k_0) / k_0 \) as \( \mathcal{A} \to \infty \). When the integration with respect to \( k_0 \) is performed, we can now use Fourier's single-integral formula

\[
\frac{1}{2} \left\{ f(x + 0) + f(x - 0) \right\} = \lim_{\mathcal{A} \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} dt f(t) \frac{\sin \frac{\mathcal{A} \cdot (x - t)}{\mathcal{A}}}{x - t}. \tag{3.3}
\]

Thus, in the limit \( \mathcal{A} \to \infty \), we have

\[
\lim_{\mathcal{A} \to \infty} I(\mathcal{A}, x) = \frac{\pi}{i\mathcal{A}} \int dk_0 dk_1 \frac{h(k_1) e^{ikx}}{k_0^2 - k_1^2 - (\mathcal{A}^2/4) - \mu^2 + i\epsilon}, \tag{3.4}
\]

where \( k_1 = (k_0, k_1, 0) \).

B. Pion current contribution

With the aid of the last result, becomes obvious the relation between the form
factors in the limit $\mathcal{A} \rightarrow \infty$ and the scattering matrix with the 4-momentum transfer squared $t = -\mathcal{A}$.

In this subsection, we show how the contribution of the pion current to the form factors relates to the on-mass-shell pion-nucleon charge exchange scattering amplitudes. According to (3.4), we obtain

$$
\lim_{\alpha \rightarrow \infty} \tilde{M}^\alpha_{\mu}(\mathcal{A}) = \frac{-\pi}{(2\pi)^4} \int d\mathbf{k}_1 \int d\mathbf{k}_0 \frac{k^\mu_0 T(k_0, \mathbf{k}_1, \mathcal{A})}{k_0^2 - \omega^2 + i\epsilon}, \quad (\mu = 0, 1, 2)
$$

where $\omega = (k_1^2 + \mathcal{A}/4 + \mu^2)^{1/2}$ and

$$
T(k_0, \mathbf{k}_1, \mathcal{A}) = i(\partial_{\alpha} \partial_{\mu} - \partial_{\alpha} \partial_{\mu}) \int dx e^{i x^\mu} \left( \frac{\mathcal{A}}{2} \right)^{\frac{1}{2}} T\left( j_\alpha \left( \mathbf{x} \right), j_\beta \left( -\mathbf{x} \right) \right) \frac{-\mathcal{A}}{2}. \quad (3.6)
$$

$T(k_0, \mathbf{k}_1, \mathcal{A})$ is the pion-nucleon charge exchange scattering matrix apart from a pion wave function renormalization factor $Z$, which is to be removed by renormalization and is neglected here. From the invariance considerations it is written in terms of two invariant amplitudes

$$
T(k_0, \mathbf{k}_1, \mathcal{A}) = i \left( \frac{m}{E} \right) \bar{u} \left( \frac{\mathcal{A}}{2} \right) \left\{ -A^{-\left( -\right)}(k_0, \mathbf{k}_1^2, \mathcal{A}) + \frac{k_1^2 B^{-\left( -\right)}(k_0, \mathbf{k}_1^2, \mathcal{A})}{k_0^2 - k_0 + i\epsilon} \right\} \tau_z u \left( \frac{-\mathcal{A}}{2} \right), \quad (3.7)
$$

where $k_1^2 = k_1^2 - k_1^2$, $\bar{k}_1 = k_0 - k_1 \cdot \gamma$ and $A^{-\left( -\right)}$, $B^{-\left( -\right)}$ are free from the kinematical factors which come from the spin of the nucleon. And it has the expression at fixed $k_1$, referred to as the Low equation:

$$
T(k_0, k_1, \mathcal{A}) = \frac{1}{2\pi} \int dk_0 \left\{ \frac{g(k_0', k_1^2, \mathcal{A})}{k_0' - k_0 + i\epsilon} + \frac{g(k_0', k_1^2, \mathcal{A})}{k_0' - k_0 + i\epsilon} \right\}, \quad (3.8)
$$

where

$$
g(k_0, k_1, \mathcal{A}) = g_{12}(k_0, k_1, \mathcal{A}) - g_{21}(k_0, k_1, \mathcal{A}),
$$

$$
g_{\alpha\beta}(k_0, k_1, \mathcal{A}) = (2\pi)^4 \sum_n \delta(k_0 + E - p_n) \delta(k_1 - p_{n1}) \delta(p_{n2})
$$

$$
\times \left\{ \frac{\mathcal{A}}{2} \right\} j_\alpha \left( 0 \right) \langle n | j_\beta \left( 0 \right) \left| -\mathcal{A} \right\rangle. \quad (3.9)
$$

We assume the convergence of the Low equation (3.8) in the limit $\mathcal{A} \rightarrow \infty$. By introducing (3.8) into (3.5), the integration with respect to $k_0$ leads to

$$
\lim_{\alpha \rightarrow \infty} \tilde{M}^\alpha_{\mu}(\mathcal{A}) = \frac{-i\pi^2}{(2\pi)^4} \int d\mathbf{k}_1 \eta(\mu) L_\mu(\omega, \mathbf{k}_1, \mathcal{A}), \quad (3.10)
$$

where

$$
\eta(\mu) = \begin{cases} 1 & \text{for } \mu = 0, \\ \frac{k_i}{\omega} & \text{for } \mu = i = 1, 2, \end{cases}
$$

$$
L_\mu(\omega, \mathbf{k}_1, \mathcal{A}) = \frac{1}{4\pi} \int_{-\mathcal{A} + \epsilon(k)}^{\infty} dk_0' \frac{g(k_0', -\mathbf{k}_1^2, \mathcal{A}) + \epsilon(\mu) g(k_0', \mathbf{k}_1^2, \mathcal{A})}{k_0' + \omega}, \quad (3.11)
$$

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where

\[ \epsilon(\mu) = \begin{cases} 
1 & \text{for } \mu = 0, \\
-1 & \text{for } \mu = 1, 2, 
\end{cases} \]

\[ c(k) = (k_+^2 + (m + \mu)^2)^{\frac{1}{2}} \]

and it is understood that the pole terms are implicitly included in (3.11). \( L_\mu(\omega, k_\perp, \mathcal{D}) \) is related to the real part of the on-mass-shell scattering amplitudes.

We express the real part of the scattering matrix as follows, understanding that the pole terms are implicitly included,

\[ \text{Re } T(\omega, k_\perp, \mathcal{D}) = \frac{P}{2\pi} \int_{-\mu+c(k)}^{\infty} dk'_0 \frac{g(k'_0, k_\perp, \mathcal{D})}{k'_0 - \omega} - \frac{1}{2\pi} \int_{-\mu+c(k)}^{\infty} dk'_0 \frac{g(k'_0, -k_\perp, \mathcal{D})}{k'_0 + \omega}, \]

(3.12)

where \( P \) stands for the principal value integral. Changing the integration variable \( k'_0 \) to \( k'_0 = E \) and taking into account that \( E + \omega = \mathcal{D} + O(\mathcal{D}^{-1}), E - \omega = O(\mathcal{D}^{-1}) \) as \( \mathcal{D} \to \infty \), we can conclude

\[ \lim_{\mathcal{D} \to \infty} \text{Re } T(\omega, k_\perp, \mathcal{D}) = -\frac{1}{2\pi} \int_{-\mu+c(k)}^{\infty} dk'_0 \frac{g(k'_0, -k_\perp, \mathcal{D})}{k'_0 + \omega}, \]

(3.13)

provided that (3.12) is convergent. We cannot say explicitly about the order of magnitude of the principal value integral in (3.12). It should be noticed, however, that as \( \mathcal{D} \to \infty \) only at the point \( k'_0 = \omega \) in \( g(k'_0, k_\perp, \mathcal{D}) \) in (3.12), the mass of the pion \( \lambda^2 = k_0^2 - k_\perp^2 - \mathcal{D}/4 \) is finite and takes the value of the physical one \( \mu^2 \). And it is not inconsistent with the assumption of convergence of the Low equation even if the imaginary part of the physical scattering matrix \( g(\omega, k_\perp, \mathcal{D}) \) has a finite value.

We can now replace \( L_\mu(\omega, k_\perp, \mathcal{D}) \) in (3.10) by the minus of \( \text{Re } T(\omega, k_\perp, \mathcal{D}) \). Using (3.7), we obtain

\[ \lim_{\mathcal{D} \to \infty} \tilde{M}_E^i(\mathcal{D}) \equiv \frac{1}{16\pi^2 \mathcal{D}} \left[ \frac{m}{E} \right] \left[ \bar{u} \frac{(\mathcal{D})}{2} \frac{\overline{u}(-\mathcal{D})}{2} \right] \int dk_\perp \text{Re } A^i(\omega, k_\perp^2, \mathcal{D}) 
- \bar{u} \left[ \frac{(\mathcal{D})}{2} \frac{\overline{u}(-\mathcal{D})}{2} \right] \int dk_\perp \text{Re } B^i(\omega, k_\perp^2, \mathcal{D}), \]

(3.14)

\[ \lim_{\mathcal{D} \to \infty} \tilde{M}_i^i(\mathcal{D}) \equiv \frac{1}{16\pi^2 \mathcal{D}} \left[ \frac{m}{E} \right] \bar{u} \left[ \frac{\mathcal{D}}{2} \right] \frac{\overline{u}(-\mathcal{D})}{2} \int dk_\perp k_\perp^2 \text{Re } B^i(\omega, k_\perp^2, \mathcal{D}). \]

(3.15)

(\( i = 1, 2 \))

Finally, we are led to the asymptotic sum rules for the pion current part of the form factors \( \tilde{G}_E^i \) and \( \tilde{G}_i^i \),

\[ \lim_{\mathcal{D} \to \infty} \tilde{G}_E^i(\mathcal{D}) \equiv \frac{1}{16\pi^2 \mathcal{D}} \int dk_\perp \left[ \frac{E}{m} \text{Re } A^i(\omega, k_\perp^2, \mathcal{D}) - \text{Re } B^i(\omega, k_\perp^2, \mathcal{D}) \right] \]

(3.16)
where we have used the identity \( \bar{u}(\frac{1}{2}) u(-\frac{1}{2}) = (E/m) \bar{u}(\frac{1}{2}) \gamma_\mu u(-\frac{1}{2}) \).

C. Asymptotic sum rules

Although the contribution of the nucleon current \( \tilde{M}_\mu^N(\mathcal{A}) \), excluding the lowest order diagram, is more complicated due to the spin kinematics, our procedure parallels that of the p'ion current part and is given in the Appendix.

In the above derivation, the problem of renormalization and the effect of the contribution of the lowest order diagram have been omitted. In order to take account of them, let us introduce usual two form factors \( F_1 \) and \( F_2 \):

\[
\langle p' | J_\mu(0) | p \rangle = \left( \frac{m^3}{p_\mu p_\mu} \right)^{1/2} \bar{u}(p') \left\{ \gamma_\mu F_1(q^2) + \frac{i \kappa}{2m} q_\mu q_\nu F_2(q^2) \right\} u(p), \tag{3.18}
\]

where \( q_\mu = p_\mu - p_\mu \), \( t = q^2 \) and \( \kappa \) is the anomalous magnetic moment of the proton. Taking account of the lowest order diagram and after the charge renormalization is carried out in the usual way, \( \tilde{F}_1 \) is normalized as \( \tilde{F}_1(0) = 1 \) and it may be written as

\[
\tilde{F}_1(q^2) = 1 + \tilde{F}_1(q^2) - \tilde{F}_1(0), \tag{3.19}
\]

where \( \tilde{F}_1 \) is to be calculated by the contribution of \( \tilde{M}_\mu^N \) and \( \tilde{M}_\mu^S \), in which it is understood that the strong interaction renormalization has been carried out. The form factors \( G_E \) and \( G_M \) are defined by the linear combination of \( F_1 \) and \( F_2 \):

\[
G_E(\mathcal{A}) = F_1(\mathcal{A}) - \frac{\kappa}{4m^2} \mathcal{A} \tilde{F}_2(\mathcal{A}),
\]

\[
G_M(\mathcal{A}) = F_1(\mathcal{A}) + \kappa \tilde{F}_2(\mathcal{A}). \tag{3.20}
\]

Then, introducing (3.19) into (3.20) we obtain the form factors \( G_E \) and \( G_M \) as follows:

\[
G_E(\mathcal{A}) = c + \bar{G}_E^N(\mathcal{A}) + \bar{G}_E^S(\mathcal{A}),
\]

\[
G_M(\mathcal{A}) = c + \bar{G}_M^N(\mathcal{A}) + \bar{G}_M^S(\mathcal{A}), \tag{3.21}
\]

where \( c = 1 - \tilde{F}_1^N(0) - \tilde{F}_1^S(0) \) and \( \bar{G}_E^N \) and \( \bar{G}_M^N \) are derived from \( \tilde{M}_\mu^N \).

Our results of the calculations of \( \bar{G}_E^N \) and \( \bar{G}_M^N \) in the limit \( \mathcal{A} \to \infty \) are

\[
\lim_{\mathcal{A} \to \infty} \bar{G}_E^N(\mathcal{A}) = -\frac{1}{32\pi^2} \int dk_1 \left[ \mathcal{A} \Re \left\{ F_S(E_k, \mathcal{A}) - F_S(-E_k, \mathcal{A}) \right\} 
\right.
\]

\[
- \frac{2\mathcal{A}}{E_k} \Re \left\{ F_T(E_k, \mathcal{A}) + F_T(-E_k, \mathcal{A}) \right\}
\]

\[
+ \frac{8(k^2 + m^2)}{E_k} \Re \left\{ F_V(E_k, \mathcal{A}) + F_V(-E_k, \mathcal{A}) \right\}, \tag{3.22}
\]
where $E_k = (k_{1z}^2 + \lambda^2/4 + m^2)^{1/2}$ and the $F$'s are the invariant amplitudes for the on-mass-shell proton-proton scattering, $p(k_{1z} + \lambda/2) + p(-\lambda/2) \rightarrow p(k_{1z} - \lambda/2) + p(\lambda/2)$, which is written in the form\(^6\)

$$T_{pp} = F_S(E_k, \lambda^2)S + F_T(E_k, \lambda^2)T + F_A(E_k, \lambda^2)A + F_V(E_k, \lambda^2)V + F_P(E_k, \lambda^2)P,$$  \hspace{1cm} (3.24)

where

\begin{align*}
S &= \bar{u}\left(\frac{\lambda}{2}\right)\gamma_5\gamma^\mu u\left(-\frac{\lambda}{2}\right)\bar{u}\left(k_{1z} - \frac{\lambda}{2}\right)\gamma^\mu u\left(k_{1z} + \frac{\lambda}{2}\right), \\
T &= \frac{1}{2} \bar{u}\left(\frac{\lambda}{2}\right)\gamma_5\gamma^\mu u\left(-\frac{\lambda}{2}\right)\bar{u}\left(k_{1z} - \frac{\lambda}{2}\right)\gamma^\mu u\left(k_{1z} + \frac{\lambda}{2}\right), \\
A &= \bar{u}\left(\frac{\lambda}{2}\right)\gamma_5 u\left(-\frac{\lambda}{2}\right)\bar{u}\left(k_{1z} - \frac{\lambda}{2}\right)\gamma_5 u\left(k_{1z} + \frac{\lambda}{2}\right), \\
V &= \bar{u}\left(\frac{\lambda}{2}\right)\gamma_5 u\left(-\frac{\lambda}{2}\right)\bar{u}\left(k_{1z} - \frac{\lambda}{2}\right)\gamma_5 u\left(k_{1z} + \frac{\lambda}{2}\right), \\
P &= \bar{u}\left(\frac{\lambda}{2}\right)\gamma_5 u\left(-\frac{\lambda}{2}\right)\bar{u}\left(k_{1z} - \frac{\lambda}{2}\right)\gamma_5 u\left(k_{1z} + \frac{\lambda}{2}\right).
\end{align*}

The $F(-E_k, \lambda^2)$'s are the corresponding amplitudes for the proton-antiproton scattering and the pairs $F(E_k, \lambda^2) \pm F(-E_k, \lambda^2)$ in (3.22), (3.23) correspond to the difference of amplitudes $T_{pp} - T_{pp}$.

As $\lambda^2 \rightarrow \infty$, the usual three scalars for the proton-proton scattering read

\begin{align*}
s &= \lambda^2 + k_{1z}^2 + 4m^2 + O(\lambda^{-1}), \\
t &= -\lambda^2, \\
u &= -k_{1z}^2 + O(\lambda^{-1}).
\end{align*}

Thus, (3.21) in the limit $\lambda^2 \rightarrow \infty$ relates the form factors to the on-mass-shell scattering amplitudes, which are integrated with respect to the center-of-mass energy squared $s$ in the physical region from the threshold of backward scattering with fixed $t = -\lambda^2$.

Since the high energy differential cross section, in general, has a steep backward peak, the integrals in the sum rules are dominated by the contribution of the amplitudes near the backward direction. However, the elastic proton-proton scattering cross section alone shows the forward-backward symmetry due to the Pauli principle and the backward peak of other processes decreases negligibly small with $s$. Thus the sum rules are dominated by the contribution of the real part.
of the proton-proton scattering amplitudes. So far, we have considered only the pion-nucleon system. According to the above considerations, the dominance of the proton current contribution to the asymptotic sum rules, however, will be valid even if we take all known hadrons into considerations, provided the fields and interactions are of renormalized type.

After all, we are led to the asymptotic sum rules:

\[
\lim_{d \to \infty} G_E(d^2) = C - \frac{1}{32\pi^2} \int d^4k \left\{ 8E \Re F_S(E_k, d^2) - \frac{2d^2}{E_k} \Re F_T(E_k, d^2) \right\},
\]

\[
\lim_{d \to \infty} G_M(d^2) = C - \frac{1}{32\pi^2} \int d^4k \left\{ 8m^2 \Re F_T(E_k, d^2) - \frac{2(d^2/2)}{E_k} \Re F_V(E_k, d^2) \right\},
\]

where \( C = 1 - F_1(0) \) contains the contribution from all hadron currents and we have not the method of evaluating it. Regarding the form factor \( F_2 \), we can derive an asymptotic sum rule without unknown constant \( C \):

\[
\lim_{d \to \infty} F_2(d^2) = \frac{4m^2}{\kappa d^2} \left( \lim_{d \to \infty} \left\{ \bar{G}_M^M(d^2) - \bar{G}_M^E(d^2) \right\} \right). \tag{3.29}
\]

§ 4. Rough evaluation

We now carry out the evaluation of the sum rules (3.27) and (3.28) apart from the constant \( C \). At present, however, our knowledge of the real part of the scattering amplitudes is very limited and it is inevitable that our numerical evaluations are very rough.

By leaving the leading terms in the integrands, the sum rules read

\[
\lim_{d \to \infty} G_E(d^2) \simeq C - \frac{1}{8\pi^2} \int d^4k \left\{ \Re F_S(E_k, d^2) - \Re F_T(E_k, d^2) \right\}, \tag{4.1}
\]

\[
\lim_{d \to \infty} G_M(d^2) \simeq C + \frac{1}{8\pi^2} \int d^4k \Re F_V(E_k, d^2). \tag{4.2}
\]

Here, we first assume that near the forward and the backward direction the leading amplitude among the five independent ones is the spin independent one. In terms of \( s \)-channel helicity amplitudes \( \phi_i, \phi_i^0 \) it means only the combinations \( \phi_1 + \phi_4 \) near forward and \( \phi_1 - \phi_4 \) near backward \( (\phi_3(\pi - \theta) = -\phi_3(\theta)) \) contribute. We neglect all other helicity (flip) amplitudes. This assumption with kinematical analysis leads to near forward direction at high energy

\[
\phi_4 \to \phi = \left( \frac{m^2}{\sqrt{s}} F_S + \frac{\sqrt{s}}{2} F_V \right),
\]

\[
F_T = 0 \tag{4.3}
\]
Introducing the parameter $a$, we can write

$$F_s = -a \frac{\sqrt{s}}{m^2} \phi,$$

$$F_v = -(1-a) \frac{2}{\sqrt{s}} \phi.$$  \hfill (4.5)

The second assumption is that we deal with the parameter $a$ as a constant. Finally, we assume the behavior of the non-forward amplitude as

$$\text{Re } \phi = \text{Re } \phi(t=0) e^{B N t},$$

where $\text{Re } \phi(t=0)$ is to be evaluated using the optical theorem $\text{Im } \phi(t=0) = (\sqrt{s}/8\pi) \sigma_t$, in which $\sigma_t$ is the total cross section, and $\rho$ the real to imaginary parts of forward scattering. $B$ is the slope parameter of the diffraction cross section and we regard it to be constant.

With these three assumptions and using the $t\leftrightarrow u$ symmetry of the amplitude, we obtain

$$\lim_{t \to \infty} G_E(d^2) = C + \frac{a}{8\pi^2 m^2} \int d\mathbf{k}_\perp \sqrt{s} \text{ Re } \phi(s, -d^2, u)$$

$$= C + \frac{a \rho \sigma_t(d^2)}{8\pi} (\pi) \int_0^{-\infty} du e^{B N u}$$

$$= C + \frac{a \rho \sigma_t(d^2)}{32\pi^2 m^2 B}.$$

and in the same way,

$$\lim_{t \to \infty} G_M(d^2) = C - \frac{(1-a) \rho \sigma_t(d^2)}{16\pi^2 B}.$$  \hfill (4.8)

The recent high energy scattering experiments at the ISR have measured $\rho(d^2)$, $\sigma_t(d^2)$ and the slope parameter $B$. As to $\rho(d^2)$ we have data up to $s=d^2 \sim 700$ GeV$^2$.

When extrapolated smoothly to the low momentum transfer region, the result of $F_2(\sim -a \rho \sigma_t/B)$ (3.29) implies that the parameter $a$ should be taken to be positive and that of $G_M$ (4.8) implies $a<1$. The numerical results of $G_E-C$ and $G_M-C$ are illustrated in Fig. 2, in which we have used the value of $B=10$ GeV$^{-2}$ and chosen the parameter $a=0.5$. The form factor $F_2$ is shown in Fig. 3. Since our evaluation has been very crude, the numerical values in Figs. 2 and 3 are not to be taken seriously. We think, however, that the result obtained gives an outline of the high energy form factors.
§ 6. Discussion

a. We point out some interesting features of the asymptotic behavior of the proton form factors in our results:
(1) The magnetic form factor $G_M(J^2)$ decreases more slowly than the “dipole” formula.
(2) The electric form factor $G_E(J^2)$ does not decrease monotonously but has a minimum near $J^2 \sim 200$ GeV$^2$.
(3) It is possible that the sign of $G_E(J^2)$ is negative near $J^2 \sim 200$ GeV$^2$.

All these features come from the behavior of the real part for the proton-proton forward (and so backward) scattering amplitudes. We want, here, to point out that the real part of the invariant amplitude $\text{Re} F(s, t=0) \sim -s \sigma(s) \sigma_t(s)$ reveals a maximum near $s \sim 200$ GeV$^2$. This behavior accompanies the growing up of the total cross section. It is likely that the maximum indicates “a change” of the structure of the proton. In our point of view the change manifests in the minimum of $G_E$.

b. The dominant contribution of proton-proton scattering in the sum rules has originated in the Pauli principle. Therefore, it is inferred that the statistics plays an essential role in the structure of hadrons near the origin in configuration space.

c. We have not been able to state clearly the region where our asymptotic sum rules work. If, however, the future measurements of the form factors for $t$ up to several ten GeV$^2$ reveal a quite different behavior from our predictions, our basic assumption that the electromagnetic interaction of bare field hadrons is
“pointlike” is probably to be discarded. We feel that more definite consideration on the precise meaning of $\mathcal{F} \to \infty$ is needed in connection with the problem of the applicability of field theory.

Acknowledgements

The author would like to express his sincere thanks to Professors K. Yamamoto and T. Kanki for their frequent helpful discussions.

Appendix

—Nucleon Current Contribution—

According to the formula (3.4), the matrix element of the nucleon current, excluding the lowest order diagram, is written as follows:

$$
\lim_{\tau \to \infty} M_{\pi}^{\pi} (\Delta^2) = \frac{\pi}{(2\pi)^4 \Delta} \int dk_0 dk_1 \left[ \frac{\left( \vec{k}_1 + m \right) \gamma_\mu (\vec{k}_1 + m) \gamma_\nu \gamma_5 O_{\mu \nu}}{k_0^2 - E_k^2 + i\epsilon} \right] \int dx \, e^{ik_1 \cdot x} \left[ \frac{4}{2} T \left( f_{\rho} \left( \frac{x}{2} \right), f_{\sigma} \left( -\frac{x}{2} \right) \right) \frac{-\Delta}{2} \right],
$$

where $E_k = (k_0^2 + \Delta^2/4 + m^2)^{1/2}$, and $k_1 = (k_0, k_1 + \Delta/2), k_1' = (k_0, k_1 - \Delta/2)$. We shall introduce five invariant amplitudes for proton-proton scattering as follows:

$$
i \int dx \, e^{ik_1 \cdot x} \left[ \frac{4}{2} T \left( f_{\rho} \left( \frac{x}{2} \right), f_{\sigma} \left( -\frac{x}{2} \right) \right) \frac{-\Delta}{2} \right] = \frac{m}{E} \sum_{i=1}^{5} (O_{i})_{\rho \sigma} \bar{u}(\Delta/2) O_{i} u \left( -\frac{\Delta}{2} \right) F_{i}(k_0, k_1^2, \Delta^2)
$$

$$
= \frac{m}{E} \left\{ \left[ \delta_{\rho \sigma} \bar{u} u F_{S} + \frac{1}{2} (\sigma_{\rho \sigma})_{\rho \sigma} \bar{u} u F_{T} + (\gamma_{\mu})_{\rho \sigma} \bar{u} \gamma_{\sigma} u F_{A}
\right.
\right.

$$
+ \left. (\gamma_{\rho})_{\rho \sigma} \bar{u} \gamma_{\sigma} u F_{V} + (\gamma_{\nu})_{\rho \sigma} \bar{u} \gamma_{\sigma} u F_{P} \right\}.
$$

Introducing (A·2) into (A·1), we have

$$
\lim_{\tau \to \infty} \hat{M}_{\pi}^{\pi} (\Delta^2) = \frac{-i}{2(2\pi)^4 \Delta} \frac{m}{E} \int dk_0 dk_1 \left[ \frac{1}{k_0^2 - E_k^2 + i\epsilon} \right] \sum_{i=1}^{5} \left\{ \text{Tr} \left[ \left( \vec{k}_1 + m \right) \gamma_\mu (\vec{k}_1 + m) O_{i} \right] \bar{u} O_{i} u F_{i}(k_0, k_1^2, \Delta^2)
\right. \right.

$$
+ \left. \text{Tr} \left[ \left( \vec{k}_1 + m \right) \gamma_\mu (-\vec{k}_1' + m) O_{i} \right] \bar{u} O_{i} u F_{i}(-k_0, k_1^2, \Delta^2) \right\},
$$

where the $F(-k_0)$’s correspond to the amplitudes of proton-antiproton scattering. Calculation of the traces in (A·3) yields

$$
\lim_{\tau \to \infty} \hat{M}_{\pi}^{\pi} (\Delta^2) = \frac{-i\pi}{2(2\pi)^4 \Delta} \frac{m}{E} \int dk_0 dk_1 \left[ \frac{1}{k_0^2 - E_k^2 + i\epsilon} \right]
$$

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\[
\times \left[ (\bar{u}u) 8m^2 \left\{ F_S(k_0, k_1^*, \Delta^2) - F_S(-k_0, k_1^*, \Delta^2) \right\} + (i\bar{u}\sigma_\mu u) 4m \mathcal{A} \int \frac{1}{k_0^2 - E_k^2 + i\epsilon} \left\{ F_T(k_0, k_1^*, \Delta^2) + F_T(-k_0, k_1^*, \Delta^2) \right\} \\
+ (\bar{u}\gamma_\mu u) 8(k_0^2 + m^2) \left\{ F_V(k_0, k_1^*, \Delta^2) + F_V(-k_0, k_1^*, \Delta^2) \right\} \right], \quad (A\cdot4)
\]

\[
\lim_{\Delta \to \infty} \tilde{M}_s^S(\Delta^2) = \frac{-i\pi}{2(2\pi)^4} \left( \frac{m}{E} \right) \int d\mathbf{k}_0 \ d\mathbf{k}_1 \frac{1}{k_0^2 - E_k^2 + i\epsilon} \left[ (\bar{u}\sigma_\mu u) 4m \mathcal{A} \int \frac{1}{k_0^2 - E_k^2 + i\epsilon} \left\{ F_T(k_0, k_1^*, \Delta^2) + F_T(-k_0, k_1^*, \Delta^2) \right\} \\
- (\bar{u}\gamma_\mu u) (k_0^2 + \frac{4\Delta^2}{3}) \left\{ F_V(k_0, k_1^*, \Delta^2) + F_V(-k_0, k_1^*, \Delta^2) \right\} \right], \quad (i=1,2) \quad (A\cdot5)
\]

and \( \lim_{\Delta \to \infty} \tilde{M}_s^S(\Delta^2) = 0. \)

Now, we write the Low equation for \( F_S(k_0) - F_S(-k_0), \) \( F_T(k_0) + F_T(-k_0) \) and \( F_V(k_0) + F_V(-k_0), \) which correspond to the amplitudes for \( (pp\to pp) - (p\bar{p}\to p\bar{p}). \)
Performing the integration with respect to \( k_0, \) and after the same considerations in \$3.B, we have

\[
\lim_{\Delta \to \infty} \tilde{M}_s^S(\Delta^2) = \frac{-i\pi}{2(2\pi)^4} \left( \frac{m}{E} \right) \left[ \bar{u} \int d\mathbf{k}_1 \ 8m \Re \left\{ F_S(E_k, \Delta^2) - F_S(-E_k, \Delta^2) \right\} \\
+ i\Delta \bar{u}\sigma_\mu u \int d\mathbf{k}_1 \frac{4m}{E_k} \Re \left\{ F_T(E_k, \Delta^2) + F_T(-E_k, \Delta^2) \right\} \\
+ \bar{u}\gamma_\mu u \int d\mathbf{k}_1 \frac{8(k_0^2 + m^2)}{E_k} \Re \left\{ F_V(E_k, \Delta^2) + F_V(-E_k, \Delta^2) \right\} \right], \quad (A\cdot6)
\]

\[
\lim_{\Delta \to \infty} \tilde{M}_s^S(\Delta^2) = \frac{-i\pi}{2(2\pi)^4} \left( \frac{m}{E} \right) \left[ i\Delta \bar{u}\sigma_\mu u \int d\mathbf{k}_1 \frac{4m}{E_k} \Re \left\{ F_T(E_k, \Delta^2) + F_T(-E_k, \Delta^2) \right\} \\
- \bar{u}\gamma_\mu u \int d\mathbf{k}_1 \frac{8(k_0^2 + \Delta^2/3)}{E_k} \Re \left\{ F_V(E_k, \Delta^2) + F_V(-E_k, \Delta^2) \right\} \right]. \quad (i=1,2) \quad (A\cdot7)
\]

Using the identities for \( q_\mu = (0, \Delta): \)

\[
\bar{u} \left( \frac{\Delta}{2} \right) \sigma_\mu q_\mu u \left( -\frac{\Delta}{2} \right) = i \frac{\Delta^2}{2m} \bar{u} \left( \frac{\Delta}{2} \right) \gamma_\mu u \left( -\frac{\Delta}{2} \right),
\]

\[
\bar{u} \left( \frac{\Delta}{2} \right) \sigma_\mu q_\mu u \left( -\frac{\Delta}{2} \right) = -2im \bar{u} \left( \frac{\Delta}{2} \right) \gamma_\mu u \left( -\frac{\Delta}{2} \right),
\]

we obtain the final results \((3\cdot22)\) and \((3\cdot23)\).
References

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