INTERFERENCE AND TURNING OF IN-PARALLEL WAKES

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Summary

Multiple in-parallel wakes are studied for ‘internal’ configurations, modelling contained motion periodic about a central axis, and ‘external’ ones set in a stream. The former configurations (but not the latter ones) induce a substantial pressure gradient and wake shapes that are controlled, in an unusual manner, by the local lateral momentum balance. Particular attention is given to the pressure losses and the overall turning of the relatively thick wakes for the internal case, based on slender-flow theory throughout. The links with experiments using external flow to simulate internal conditions for turbomachinery applications are discussed, especially in terms of first- and second-order effects for small wake deficits and the differences between the upstream and the downstream lateral periods for many external wakes.

1. Introduction

This study of multiple wakes is motivated by the application to turbomachinery flows, especially concerning the wakes of a row of blades which are effectively in-parallel, closely contained within the internal casing of an axisymmetric engine; see, for example, Pearson and Arndt (1), Wisler (2). Such wakes occur downstream of a (moving) row of rotor blades for instance, prior to a (static) stator-blade row, where the arrangement is designed with the aim of passing wake fluid with non-uniform velocity periodically in time and space into the gaps between the stator blades. The periods involved are very short, the numbers of blades are large, the gaps can be relatively thin and the wake thicknesses tend to be of the order of the downstream gap thicknesses or not much less. Experiments by Doorly (3), Pfeil et al. (4, 5), Hodson (6), Simon (7), Gostelow (8), Kaszeta et al. (9), Simon and Yuan (10) among others are aimed at simulating the above contained-flow configuration, often by means of moving bars of cylindrical cross-section or squirrel-cage or similar arrangements which, by inducing upward or downward moving wakes in a stream in front of a vertical quasi-planar system of fixed blades, enable relevant measurements and observations to be made more readily without close containment of the flow. This setting can be denoted ‘external’ as opposed to the ‘internal’ setting of contained flow outlined at the start. The interference involved between wakes also has application to flows from sprinklers and fluid-gun sorting devices. The Reynolds numbers are typically large. Computations have been performed by direct simulations in few cases compared with the number of useful engineering models, approximations and inviscid calculations.

The determination of the in-parallel wakes in the above contexts, along with the motion through the successive rows of blades, poses a considerable challenge. The flows in reality tend to be a mixture of turbulent, transitional and laminar flows as well as being three-dimensional and unsteady with numerous time scales present. Yet the large number of rotor and stator blades and their common axis indicate that a two-dimensional local approximation is not unreasonable, at positions...
away from the internal casing and the axis. Again, an assumption of laminar planar wake flows which are steady in a frame moving with the rotor blades seems a sensible starting point. In addition, between the successive blades in a row, the gap thickness may be about 30 to 50 per cent of the gap length in practice, suggesting however remotely the notion of a thin layer. Given such assumptions, we ask here whether a slender-flow approach based on supposing thin layers throughout can help in understanding and predicting some of the real global physical properties. This is examined for the incompressible regime. Particular issues are the role played by the lateral periodicity, around the common axis, that is, globally, in the thin-layer local motions; the physical mechanism controlling the wake shape and consequent turning of the stream; and the determination of wakes with substantial velocity deficits and thicknesses.

It is found below that multiple external wakes can indeed produce laterally periodic wake flow downstream but in general the period there is different from that of the start upstream. The difference depends on the reduced Reynolds number and input velocity profiles. This may be of interest for the external simulation experiments. In contrast, multiple in-parallel internal wakes provoke a response with constant period throughout, due to the action of the induced adverse pressure gradient. Other contrasts between the external and internal wake settings are found to be in the control of wake shape (the external setting invokes interaction with the flow outside the wakes, while the internal setting invokes the lateral momentum force) and in the entry further downstream to a blade row, where pressure discontinuities (Bowles and Smith (11), Smith and Jones (12), Smith (13), Smith et al. (14)) can occur in the former setting but not in the latter, if we exclude period breaking.

Section 2 describes the external-flow configuration and properties, which generally are not periodic, and section 3 the internal case, which is periodic in the local lateral coordinate \( y \) to ensure periodicity on the more global scale, for example, around an axis of symmetry. Section 4 then considers the influences of non-symmetry and Reynolds number, in the internal wakes. Wake shapes are studied in section 5. Section 6 comments on the correction effects present for small wake deficits, apart from the contrasts of the previous paragraph concerning periodicity, wake shapes and overall flow turning, and downstream entry to a blade row.

2. External wakes: constant pressure

The numerous in-parallel wakes of interest here are supposed to occur downstream of a row of numerous identical rotor blades for example, the flow around and between which provides the starting wake velocity profiles. The flow is taken to be steady, if necessary by transformation to an appropriate moving frame. We use non-dimensional variables, the coordinates \((x, y)\) and corresponding velocity components \((u, v)\) (‘streamwise’ and ‘lateral’ respectively) being based on \((L, D)\) and \(U_r(u, Dv/L)\) in turn, where \(L\) is the blade chord or other representative streamwise length, \(D\) denotes the lateral distance between two neighbouring blades and \(U_r\) is a streamwise velocity scale representative of the starting wake profile. Each wake is assumed thin in the sense that \(D/L\) is small. The natural pressure scale is \(\rho U_r^2\), where \(\rho\) is the incompressible fluid density, and we can anticipate corrections of order \((D/L)^2\) so that the pressure here is \(\rho U_r^2[\hat{P} + (D/L)^2 \hat{P}]\) with \(\hat{P}\) of order unity. The number \(N\) of wakes in this external setting is finite and although \(N\) may be large typically we suppose that \((D/L)N\) remains small, leaving the total wake system slender and hence provoking only a small perturbation of the outer stream external to the wakes.

Another consequence of \(D/L\) being small is that the whole system acts as a slender flow,
governed by the thin-layer equations of continuity and momentum
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.1a)
\]
\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -P'(x) + \text{re}^{-1} \frac{\partial^2 u}{\partial y^2}, \quad (2.1b)
\]
subject to the boundary and starting conditions
\[
u \to 1 \quad \text{as} \quad y \to \infty, \quad (2.2a)
\]
\[rac{\partial u}{\partial y} = v = 0 \quad \text{at} \quad y = 0, \quad (2.2b)
\]
\[
u = \nu_0(y) \quad \text{at} \quad x = 0. \quad (2.2c)
\]
The lateral momentum balance establishes that $\frac{\partial P}{\partial y}$ is zero, so that $P'$ appears in the streamwise momentum balance (2.1b). This is along with the dominant viscous term, in which re denotes the reduced Reynolds number $\text{Re} = U_r D^2 / \nu L$, that is $(D/L)^2$ multiplied by the global Reynolds number $\text{Re} \equiv U_r L / \nu$, where $\nu$ is the kinematic viscosity of the fluid. We include the main viscous contribution for the sake of generality; $\text{re}$ may be small, of order unity or large in practice. The condition (2.2a) of matching to a uniform free stream laterally upward at $\infty$ also applies downward at $-\infty$ but for now we assume symmetry, in (2.2b), as that confines the flow problem to $y$ positive or zero without losing the principal feature(s). Again, since the free stream speed is uniform, in (2.1b) the pressure is prescribed, being the free stream pressure
\[
u(x) \equiv \text{constant}, \quad (2.3)
\]
while in (2.2c) $\nu_0$ stands for the starting velocity profile across all the numerous wakes, starting at $x = 0$ say and having $y$-symmetry.

Our concern is with starting conditions that give, with $N$ large, a velocity profile $\nu_0(y)$ which is $y$-periodic in the middle of the total wake although tending to unity at sufficiently large $y$. The profile reflects the presence of a large but finite number $N$ of rotor blades just upstream. Given the parabolic nature of the thin-layer (boundary-layer) equations we consider below where and whether a $y$-periodic flow ensues in the middle part for positive $x$ downstream and, if so, what the lateral period is downstream and its dependence on $N$.

Computational solutions were obtained by use of a semi-implicit finite-difference approach as in Smith and Timoshin (15), Bowles and Smith (16), marching forward in small steps $\Delta x$ from the given profile $\nu_0$ at $x = 0$. (The starting form (2.4a, b) below is also taken into account for $v$ at small $x$ values.) In the approach the undifferentiated terms in (2.1b) at any $x$-station are lagged as the known values at the previous station, leaving (2.1b) as a second-order linear equation in $y$ for $\nu$ subject to two boundary conditions. Centred differencing is applied in $y$, with small steps $\Delta y$, and backward in $x$, to allow solution for $\nu$. Then (2.1a) is solved for $\nu$, given $\nu$ zero at $y = 0$. The approach is then ready to move on to the next station, and so on. A double stepping procedure in $x$ as in (15) is added in order to produce second-order accuracy in $x$, complementing that in $y$.

Numerical results are presented first in Fig. 1a, with two $N$ values, for the case of a starting profile, at $x = 0$,
\[
u_0 = \begin{cases} 
1 - \epsilon \sin^2 \pi y & \text{for } 0 \leq y \leq N, \\
1 & \text{otherwise},
\end{cases}
\]
which is positive and periodic over a substantial $y$ range. The reduced re is 1 here. In the particular
Fig. 1 External wake solutions, for \( \text{re} = 1 \). (a) Velocity profiles at various \( x \) values as shown for input \( u_0 = 1 - \epsilon \sin^2(\pi y), 0 \leq y \leq N, u_0 = 1 \) for \( y > N \), with \( N = 10, 20, \epsilon = 0.5 \). (b) Small-\( x \) form with \( u_0 \) as in (a); note that the \( y \) values where \( u_1 \) is minimal do not have period 1.
Fig. 1 Cont. (c) Velocity profiles at $x = 0.02, 0.04$ for input $u_0 = |\cos(\pi y)|$, $0 \leq y \leq N$, $u_0 = 1$ for $y > N$, $N = 20$; the profile for $x = 0.04$ has the minimum amplitude of oscillation, among those shown examples $\epsilon = 0.5$, $N = 10$ and 20, the results show that by $x = 0.1$ a middle part of the total wake flow does indeed appear to have emerged in a form which is laterally periodic, or nearly so, and independent of $N$, and this persists for some considerable distance downstream. The results for $N = 10, 20$ are virtually identical for $0 \leq y \leq 8$ (for the sake of clarity we show two velocity profiles at non-zero $x$ values for $N = 10$ but only one for $N = 20$). The $y$-period indicated there however is not equal to that of the starting profile, that is unity, since the profile oscillations do not quite align with those of $u_0$. The oscillations in $y$ seen at such positive $x$ values in the middle part eventually decay downstream as a uniform velocity profile (but with $u$ different from unity) seems to be approached. Even further downstream, edge effects enter the middle gradually for large positive $y$, yielding an ultimate approach to the uniform stream value of $u$ equal to unity.

Analytically, we consider (i) the middle part for large $N$, (ii) the solution at small $x$ values, and (iii) the outer part, edge effects and far downstream response for large $N$.

Concerning (i), consider the consequences of supposing the middle part where $y$ is $O(1)$ with $N$ large to be $y$-periodic for $x$ of order unity. Then integration of (2.1b) in $y$ across the period 1 imposed by the $u_0$ profile would give the integral of $u^2$ being conserved for all $x$ and hence equal to $(1 - \epsilon + 3\epsilon^2/8)$ for the case of $u_0(y)$ in Fig. 1a, giving 0.59375 where $\epsilon = 0.5$. Far downstream a uniform state $u \to c$ would emerge in which the constant $c = (0.59375)^{1/2} = 0.77055$ to conserve the $u^2$ integral in $y$ from 0 to 1. Against that, the numerical results for $u$ are nearly $y$-periodic in Fig. 1a but the inferred period is not unity and moreover the uniform state indicated downstream
has $c$ distinct from 0.770.55. This suggests a contradiction, implying that the flow is not $y$-periodic for general $x$ of $O(1)$, although it may be periodic with different periods at $x = 0$ and at large $x$.

The contribution (ii) supports the above. The flow solution expands as

$$(u, \psi) = (u_0, \psi_0) + x(u_1, \psi_1) + \cdots$$

(2.4a)

for small positive $x$, where $\psi$ is the scaled streamfunction satisfying $u = \partial \psi / \partial y$, $V = -\partial \psi / \partial x$ in view of (2.1a) and $\psi = 0$ at $y = 0$, using (2.2b), while $u_0(y) = \psi'_0(y)$ is assumed positive here. Then (2.1b), (2.2b) and (2.3) yield the solution, with $u_1(y) = \psi'_1(y)$,

$$\psi_1(y) = \frac{u_0(y)}{re} \int_0^\gamma \frac{u_0''(\tilde{\gamma})}{u''_0(\gamma)} d\tilde{\gamma}. \quad (2.4b)$$

The example plotted in Fig. 1b is for $re = 1$ with the same $y$-periodic sine-squared $u_0$ profile as in Fig. 1(a). It is clear that $\psi_1$, $u_1$, although oscillatory in $y$, are not periodic. An overall linear growth in $y$ is evident in both Fig. 1b and the result (2.4b). This ties in with the view that the flow solution at $O(1)$ positive values of $x$ is not periodic in $y$, despite the periodic $u_0$ starting profile, within the middle part. Further, (2.4a, b) indicate that the starting profile of $v$ (given by $-\psi_1$) is non-zero and non-periodic.

Similar trends hold for the case in Fig. 1c which has $u_0 = |\cos \pi y|$ being zero at $y = 1/2, 3/2, \ldots, N - 1/2$ but $u_0$ is unity for $y \geq N$. This is included for completeness to check that the reasoning goes through also for non-smooth starting profiles taken immediately aft of the $N$ rotor trailing edges where $u_0 = 0$. The form (2.4a, b) is replaced by Goldstein’s $1/3$ power form, but as $x$ develops the behaviour is basically as in the Fig. 1a case described earlier. The peaks of $u$ at two $x$ stations shown in Fig. 1c do not line up with the periodic section of $u_0$, the profile $u$ is not quite periodic, a downstream $u$ value $c < 1$ is again indicated for the middle part, and, towards the edge, a slower evolution is implied. Henceforth we may focus on smooth $u_0$ profiles.

Concerning (iii), the outer part of the wake for $N > 1$ has the shifted coordinate $y - N$ of order unity for $x$ of $O(1)$ and so the full equations continue to hold there, while the starting profile $u_0$ generally extends from large positive $y - N$ values where $u_0$ is 1 to large negative values where $u_0$ is $1$-periodic. This poses a non-trivial marching problem but downstream at large positive $x$ the $y - N$ scale expands like $x^{1/2}$ due to wake diffusion. Therefore the outer wake or edge effect influences the middle part at distances downstream such that $N \sim x^{1/2}$, that is at distances $x \sim N^2$. On this longer scale, then, $x = N^2 \tilde{x}$ and $y = N \tilde{y}$ with $\tilde{x}, \tilde{y}$ of order unity and $u$ is still of order unity also. So the full equations (2.1a, b) and (2.3) apply again, in terms of $\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y}$ (here $v = N^{-1} \tilde{v}$), subject now to a starting profile as $\tilde{x} \to 0+$ which varies from $c$ (different from 1 usually) relatively near the symmetry line $\tilde{y} = 0$ through a smooth profile to 1 at large $\tilde{y}$. We can expect diffusion of the profile, as $\tilde{x}$ increases, towards $\tilde{u} \equiv 1$ as $\tilde{x} \to \infty$, completing the streamwise evolution of the wake.

The features in (i) to (iii) appear to support the computational trends in Fig. 1a, c overall. Similarly the conflict between the number of lateral conditions required for a solution 1-periodic in $y$ in the middle part (four conditions, namely $\psi$ given at $y = 0$ say, $\psi$ given at $y = 1$ for mass conservation, and periodicity in $u, \partial u / \partial y$) and the lateral order of the governing equations (three, from (2.1a, b) and (2.3)) suggests that a periodic solution is usually impossible. The main exception to that is at large $x$ where an unknown period different from 1 can emerge, on approach to $u = c$. The above contrasts with the properties for internal flow studied below.
3. Internal wakes: lateral periodicity

The difference in setting from that in the previous section is mainly that here, for internal flow, periodicity in \( y \) is required effectively as the upstream (rotor) blades and hence the wakes continue all the way around the axis over a sufficiently large \( y \) range. One distinct consequence can be deduced first through the symmetric case. The same argument as at the start of the previous section then applies but now

\[
P(x) \neq 0, \quad (3.1)
\]
since the presence of periodicity replaces the free stream condition and, however large the lateral period, this can support a significant pressure variation. A similar distinction arises if walls are present in the lateral direction, no matter how far apart.

Further, a wake-flow solution with \( y \)-periodicity of \( O(1) \) dictated by the upstream blade distribution seems called for in this internal setting with (3.1). So only a single wake needs to be integrated now, accompanied by a given periodicity in \( y \); this is another distinction from the external case. The appropriate boundary conditions for each wake are thus

\[
\frac{\partial u}{\partial y} = v = 0 \quad \text{(and \( u \) values are equal)} \quad \text{at} \quad y = 0, 1, \quad (3.2a)
\]
\[
u = u_0(y) \quad \text{(1-periodic in \( y \)) \quad \text{at} \quad x = 0, \quad (3.2b)}
\]
for the internal setting, the \( y \)-period being 1 without loss of generality and the profile \( u_0 \) being taken as symmetric.

The computational approach of section 2 was adapted to allow for (3.1), (3.2) by means of an outer iteration at each \( x \), to determine the pressure ensuring satisfaction of the extra conditions implicit in (3.2a) (compared with the number of conditions in section 2) along with 1-periodicity in \( u \); see also a comment at the end of this section on the number of unknowns. Results are given in Fig. 2a to d. These show, in Fig. 2a, streamwise velocity profiles \( u \) at various \( x \) stations after a Gaussian profile with a 60 per cent deficit at the start, the \( y \)-period again being unity, followed in Fig. 2b, c by \( u \) evaluated along \( y = 0 \) and the pressure \( P \) versus \( x \) for 95, 60 and 10 per cent deficits at the start. Figure 2d gives the \( u \) values at \( y = 0 \) for three fuller starting profiles of sine-squared form. The change in the \( u \), \( P \) variation for different deficits is notable.

Related properties are as follows. For small \( x \), (2.4a) holds again but with unknown pressure

\[
P(x) = \pi_0 + x \pi_1 + \cdots \quad (3.3a)
\]
now, where the constant \( \pi_0 (= P(0)) \) is given but \( \pi_1, \ldots \) are unknown. Consequently (2.4b) is replaced by

\[
\psi_1(y) = \frac{u_0(y)}{\text{re}} \int_0^y \left\{ \frac{u''_0(\tilde{y}) - \pi \text{re}}{u''_0(\tilde{y})} \right\} d\tilde{y}, \quad (3.3b)
\]
provided \( u_0 > 0 \) again, and so the local pressure response \( \pi_1 \) is determined, for satisfaction of (3.2a), as

\[
\pi_1 = \frac{1}{\text{re}} \int_0^1 u''_0(\tilde{y}) u''_0^{-2}(\tilde{y}) d\tilde{y} \int_0^1 u''_0^{-2}(\tilde{y}) d\tilde{y}, \quad (3.3c)
\]
which is finite. The higher-order coefficients implied in (3.3a) are determined similarly, ensuring
Fig. 2 Internal wake, re = 1. (a) Velocity profiles, after a 60 per cent deficit profile at x = 0; 
\( u_0 = 1 - \epsilon \exp[-(\epsilon_1 (y - \frac{1}{2}))^2] \), \( \epsilon = 0.6 \), \( \epsilon_1 = 12 \). (b) \( u(x, 0) \) for \( u_0 \) as in (a) but \( \epsilon = 0.1, 0.6, 0.95 \)
Fig. 2  Cont. (c) $P(x)$ for the three cases in (b). (d) As (b) but for $u_0 = 1 - \epsilon \sin^2(\pi y)$, $\epsilon = 0.1, 0.6, 0.95$.

The lateral period is 1
that the flow solution remains 1-periodic in $y$. The starting profile $v = v_0(y)$ (say) at $x = 0$ is still given by $-\psi_1$, but subject now to (3.3c), and this is incorporated in the computations above.

Moreover, integration in $y$ across the period yields the cross-momentum balance

$$I_2 + P = \text{constant for all } x \left[ I_2 = \int_0^1 u^2 dy \right]$$

(3.4)

from (2.1b) and (3.2a), where the constant is $\int_0^1 u_0^2 dy + \pi_0$ from (3.2b). On the other hand, a uniform state $u \to \hat{c}$ emerges far downstream, with decay into it of the form $\exp(-4\pi^2 x/\hat{c})$ generally, indicating a fast exponential decay which explains the rapid saturation shown in Fig. 2b to d. Here $\hat{c}$ is constant. The above implies a mass flux $\hat{c}$ across a single wake and $\hat{c}$ can be fixed as $\phi_0(1)$ from (3.2a, b), since $\phi_0(0)$ is zero. So $I_2 \to \hat{c}^2$ is known at downstream infinity, and thence (3.4) leads to the relation

$$P(\infty) = \pi_0 + \left\{ \int_0^1 u_0^2 dy - \phi_0^2(1) \right\}.$$  

(3.5)

This indicates, since $\phi_0(1)$ is the integral of $u_0$, that $P(\infty) > \pi_0$ in general, corresponding to a pressure rise along the wake, independently of the form of the $u_0(y)$ profile.

Next, a linearized solution of the form $u = c_0 + \hat{c} u_1$ holds if $u_0$ is $c_0 + O(\hat{c})$ with $\hat{c}$ small and $c_0$ constant, that is, for small wake deficits at the start. Then $u_1$ satisfies $c_0 \partial u_1/\partial x = -P_1' + (\partial^2 u_1/\partial y^2)/\hat{c}$, where $P - \pi_0 = \hat{c} P_1$. Given an input $u_1 = \sin(2\pi y)$ at $x = 0$ for example, for unit re the solution for $x > 0$ is simply $u_1 = \sin(2\pi y) \exp(-4\pi^2 x/c_0)$ and $P_1' \equiv 0$. This feature of zero pressure gradient at $O(\hat{c})$ which is common to the linearized cases ties in with (3.4) since the integral there becomes $c_0^2/2\hat{c} \phi_1(1)$ to order $\hat{c}$ and $\phi_1(1)$ is constant from (3.2a). In fact (3.4) at the next order yields the pressure gradient as

$$P' = 4\pi^2 \hat{c}^2 c_0^{-1} \exp(-8\pi^2 x/c_0)$$

(3.6)

for the above example, of second rather than first order in $\hat{c}$, and positive. A second-order effect is also inferred from the starting form (3.3a to c) in such cases. It is interesting that in retrospect this difference between an external wake (the middle part, in the previous section) and an internal one (this section) appears only as a second-order correction, for the case of small deficits.

Finally, with (3.1) present the number of unknowns now tallies with the number of equations and conditions required to obtain a laterally periodic solution for all $x$, unlike in section 2. The numerical solutions thus obtained all produce adverse pressure gradients, consistent with the analysis, as well as rapid saturation and an increase in the pressure rise for fuller $u_0$ profiles implied by (3.5) which is in line with the computed solutions. We continue with the internal case below.

4. Influences of lateral non-symmetry and reduced Reynolds number

Addressing non-symmetry first, in the $y$-direction, we suppose that a typical streamline dividing one wake from its lateral neighbour in the periodic internal array is given by $y = f(x)$, say, instead of $y$ zero in the previous section. On that streamline $v = u df/dx$, as fluid flows along it. Therefore, use of the Prandtl transposition

$$y - f \equiv \tilde{y}, \quad v - uf' \equiv \tilde{v}$$

(4.1)
not only leaves the controlling equations essentially intact, in the form
\[
\frac{\partial u}{\partial x} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \quad (4.2a)
\]
\[
u \frac{\partial u}{\partial x} + \bar{v} \frac{\partial u}{\partial \bar{y}} = -P'(x) + \text{Re}^{-1} \frac{\partial^2 u}{\partial \bar{y}^2}, \quad (4.2b)
\]
but also yields, with unit periodicity,
\[
\bar{v} = 0 \quad \text{at} \quad \bar{y} = 0, 1, \quad (4.3a)
\]
retrieving the second half of condition (3.2a). The first half, however, is replaced by the lesser requirement
\[
\frac{\partial u}{\partial \bar{y}} \quad \text{(and } u \text{) values at } \bar{y} = 0, 1 \text{ must be equal,} \quad (4.3b)
\]
for periodicity. This same requirement applies to \( u \) in effect as in section 3. The starting condition remains as in (3.2b) with \( \bar{y} \) instead of \( y \), although now \( u_0 \) in general is non-symmetric about \( \bar{y} = \frac{1}{2} \).

The flow solution downstream depends (only) on that \( u_0 \) profile, which in turn provokes a non-zero \( \bar{v} = \bar{v}_0 (\bar{y}) \) profile at \( x = 0 \) given by
\[
\bar{v}_0(\bar{y}) = \frac{u_0(\bar{y})}{\text{Re}} \int_0^\bar{y} \left\{ \frac{\pi_1 \text{Re} - u_0'(\tilde{y})}{u_0(\tilde{y})} \right\} d\tilde{y}, \quad (4.4)
\]
together with \( \pi_1 \) obtained still from (3.3c). The computational approach was again adapted in order to impose (4.3a) as well as equality of \( u, \frac{\partial u}{\partial \bar{y}} \equiv \tau_0 \) at \( \bar{y} = 0, 1 \) for each station. This generalization from section 3 required another iteration loop based on finding \( \tau_0 \) and precise application of (4.4) and (3.3c). Results are given in Fig. 3a to c for various \( u_0(\bar{y}) \) profiles. The flow asymptotes to a symmetric state far downstream. The integral properties (3.4) and (3.5) still hold throughout.

Figure 3a to c is for a non-symmetric \( u_0 \) defined by parameters \( c_1, d_1, d_2 \); the profile \( u_0 \) is also non-smooth, although continuous. Figure 3a presents the effects on the pressure \( P \) from varying \( c_1 \), corresponding to different input velocity deficits. The pressure change is consistent with (3.5). Figure 3b in contrast shows the influence of altering the degree of non-symmetry, via altering \( d_1, d_2 \). This affects the \( u \) profiles more than \( P \). Figure 3c gives the corresponding \( v \) profiles (as well as \( u_0 \)) over a range of \( x \) stations, including \( x_0 \) which, like \( u_0 \), is non-smooth but also discontinuous. The \( v \) solution relaxes rapidly to zero downstream as \( x \) increases.

Secondly, there are mainly two types of effects due to varying \( \text{Re} \), associated with the incident vorticity. One is inviscid, the other viscous. Thus if the input profile \( u_0 \) is fixed in \( y \), at the start of the wake, determining the vorticity there, then use of \( x/\text{Re} \equiv \tilde{x} \) say, \( \text{Re} \equiv \bar{v} \) leaves the controlling equations (4.2a, b) and (4.3a, b) as they are but with \( \tilde{x}, \bar{v}, 1 \) instead of \( x, v, \text{Re} \) respectively. So the earlier results for unit \( \text{Re} \) continue to apply for any non-zero \( \text{Re} \) but with \( x \) scaled on \( \text{Re} \), in line with the diffusion streamwise distance increasing with increasing \( \text{Re} \). It follows that, for a given length of wake, the solution at large \( \text{Re} \) tends to the inviscid limit, implying here that the velocity profiles along the wake remain little altered from the \( u_0 \) form. Conversely, suppose that the input profile varies with \( \text{Re} \), on a characteristic viscous \( y \)-scale of order \( \text{Re}^{-1/2} \), that is, order \( \text{Re}^{-1/2} \). Then the whole wake solutions do not need to be recalculated. The same approach, except for the factor \( \text{Re} \), was used for computing solutions, subject to increased grid resolution for higher-\( \text{Re} \) cases. Solutions are shown in Fig. 4, covering \( \text{Re} \) values of 1, 10^2, 10^4, again for a suitable non-smooth \( u_0 \) profile scaling with \( \text{Re}^{-1/2} \). The marked decrease in pressure variation as \( \text{Re} \) increases is as expected from the properties described earlier, in particular (3.5) and Fig. 2, as the corresponding \( u_0 \) profile becomes less full.
Fig. 3 Non-symmetric internal wakes, $\text{re} = 1$, with $u_0 = 1 - c_1y [0 \leq y \leq d_1], 1 - c_1d_1 + c_2(y - d_1)$ $[d_1 \leq y \leq d_2], 1 - c_1d_1 + c_2(d_2 - d_1) - c_1(y - d_2) [d_2 \leq y \leq 1]$, where $c_2(d_2 - d_1) = c_1(d_1 + 1 - d_2)$.

(a) $P(x)$ when $(d_1, d_2) = (0.6, 0.8)$, for $c_1 = 0.5, 0.9$

5. The wake shapes and overall turning

In external wakes with lateral non-symmetry, the overall wake centreline shape for example (we recall that (4.1) eliminates the shape from the viscous wake problem (4.2a, b) etc.) depends on matching with the slightly disturbed free stream outside. This inner-outer balance is as in Smith and Timoshin (15), involving Cauchy–Hilbert integral relations between the shape and the small pressures induced outside.

In internal wakes with non-symmetry, by contrast, a different mechanism fixes the wake shape $f$. The lateral momentum equation, which usually is of little consequence, actually matters here across the wakes. Its form is

$$u \partial \bar{v}/\partial x + \bar{v} \partial \bar{v}/\partial \bar{y} + f''u^2 - f'P' = -\partial \hat{P}/\partial \bar{y} + \text{re}^{-1} \partial^2 \bar{v}/\partial \bar{y}^2,$$

(5.1)
given that the pressure is

$$\rho U_r^2[P + (D/L)^2 \hat{P}]$$

(5.2)

and (4.1), (4.2a, b) also hold. Although $\psi, \partial \psi/\partial \bar{y}, \partial^2 \psi/\partial \bar{y}^2$ and so on are rendered periodic in $\bar{y}$ by means of (4.3a, b), (4.2b), and $P(x)$ is clearly periodic, the point now is that the pressure correction $\hat{P}$ must also be rendered periodic in the internal configuration, so that

$$\hat{P} \text{ values at } \bar{y} = 0, 1 \text{ must be equal},$$

(5.3)
in particular.
Fig. 3 Cont. (b) $P(x)$ and $u(x, 0)$ when $(d_1, d_2) = (0.6, 0.75), (0.7, 0.85), (0.8, 0.95)$ for $c_1 = 0.5$
Integrating (5.1) over $\bar{y}$ from 0 to 1 and then enforcing (5.3) thus leaves us with

$$\int_0^1 u \frac{\partial \bar{v}}{\partial x} d\bar{y} + f'' I_2 - f' P' = 0,$$

(5.4)
a relation controlling $f$, for all re. The first integral is the same as that for $u \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial u}{\partial x}$, however, by use of (4.2a) with the condition (4.3a), and so integration of (5.4) in $x$ yields

$$I_2 f' = -\int_0^1 u \bar{v} d\bar{y} + c_2,$$

(5.5a)
in view of (3.4) also, that is, since $P' = -I_2$. The constant $c_2$ is determined by the starting velocity profile and initial wake direction, so that

$$c_2 = I_2(0) f'(0) + \int_0^1 u_0 \bar{v}_0 d\bar{y}$$

(5.5b)
is known, with $\bar{v}_0$ given by (4.4). Equation (5.5a) then controls $f(x)$ in $x > 0$, subject to zero $f(0)$ without loss of generality. The right-hand side of (5.5a) can be replaced by an expression independent of $\bar{v}$, namely $c_2 - \frac{1}{2} P' + \bar{v}_0$ minus twice the integral of $\bar{y}u \frac{\partial u}{\partial x}$ across the wake, from (4.2b), but we shall keep to the form (5.5a).

The entire wake shape $f$ depends on the starting profile $u_0$, since that determines the $u, \bar{v}, P$
solutions for \( x > 0 \) in the previous section and hence the integral quantities in (5.5a, b), although \( f'(0) \) also influences \( f \) through the value of \( c_2 \). This value is crucial because far downstream \( \bar{v} \) tends to zero and (5.5a) yields

\[
f' \to f'(\infty) = c_2/\hat{c}^2,
\]

where we recall that \( \hat{c} \equiv \psi_0(1) \). Thus the overall turned angle (slope) of the wake system far downstream is sensitive to \( c_2 \).

Solutions for \( f \) versus \( x \) derived numerically are shown in Fig. 5(a) to (c) for various input profiles \( u_0 \) and starting slopes \( f'(0) \), with reduced Reynolds numbers \( re \) of unity. The approach of section 4 was used to find \( u, \bar{v}, P \) for \( x > 0 \) and then, after \( c_2 \) was determined as in (5.5b), equation (5.5a) was integrated forward in \( x \) for \( f(x) \) with second-order numerical accuracy. The results confirm the sensitive dependence on \( u_0(y) \), \( f'(0) \), especially with regard to the overall turning. Also, in the case of Fig. 5(a) with \( \psi_0(1) = 0.9 \) the computations agree well with the prediction that \( u \to \hat{c} = 0.9 \) rapidly downstream, while (5.5b) yields \( c_2 = -0.159 \) for \( f'(0) \) zero, so that according to (5.6) \( f'(\infty) \) is \( -0.196 \), a value which is in close agreement with the computed slopes sufficiently far downstream. Figure 5a is for one of the non-symmetric cases of Fig. 3a, with \( f'(0) \) zero. Figure 5b includes the effects of varying the non-symmetry in \( u_0 \), for the same cases as in Fig. 3b, again with \( f'(0) \) zero. Far downstream a positive or negative slope \( f'(\infty) \) is found. Figure 5c has the
Fig. 5 Non-symmetric wake shapes $f(x)$, with $\text{re} = 1$, for input $u_0$ as in Fig. 3. (a) Results for $(d_1, d_2) = (0.6, 0.8), c_1 = 0.5$, showing computed solution (upper curve) and asymptote (from (5.6)), when $f'(0) = 0$. (b) Results for the three $u_0$ cases of Fig. 3b, when $f'(0) = 0$
Fig. 5  Cont. (c) Results for the case of (a) but with \( f'(0) = -1, 0, 1 \)

same non-symmetric input profile as in Fig. 5a but now \( f'(0) \) values of \(-1, 0, 1\) are taken. These yield positive or negative slopes \( f'(\infty) \) downstream. The solution here tends to be dominated by \( f' - f'(0) \) since the integral \( I_2 \) varies relatively little.

We note that for small input deficits (see near (3.6)) the induced \( \bar{v} \) is \( \tilde{e} v_1 \), where \( v_1 \) is of order unity, and then (4.4) gives

\[
\bar{v}_1 = \left\{ u'_{10}(0) - u'_{10}(\bar{y}) \right\}/(c_{0re}) \text{ at } x = 0, \quad (5.7a)
\]

with \( u_{10} \) denoting \( u_1 \) evaluated at \( x = 0 \). Hence (5.5b) shows that in this case

\[
c_2 = \tilde{e} \left[ c_{0}^2 \tilde{f}'(0) + \text{re}^{-1} u'_{10}(0) \right] \quad (5.7b)
\]

if \( f = \tilde{e} \tilde{f} \) is supposed small as well. Accordingly the scaled shape \( \tilde{f} \) is governed by

\[
\tilde{f}'(x) - \tilde{f}'(0) = \left\{ u'_{10}(0) - u_{1\bar{x}}(x, 0) \right\}/(c_{0re}^2), \quad (5.7c)
\]

from (5.5a). At most \( x \) positions the scaled shear \( \partial u_1/\partial \bar{y} \) \((x, 0)\) here must be deduced from the \( u_1 \) equation shown prior to (3.6), but far downstream that shear tends to zero, leaving \( \tilde{f}'(\infty) - \tilde{f}'(0) \) equal to \( u'_{10}(0)/(c_{0re}^2) \). In consequence, positive input shear at the dividing streamline at the start of the wake increases the wake slope downstream, whereas negative input shear leads to a decrease, for such small-deficit cases. There is thus a first-order influence on the overall turning.
6. Further comments

Two fundamental differences have been found in sections 2 to 5 between external wakes and internal wakes in multiple parallel formation. One is that the former have a prescribed streamwise pressure gradient (within the limitations of the slender-flow model), which strictly precludes the existence of a flow solution periodic across all the wakes, whereas the internal wakes admit an adjustable pressure gradient that maintains the lateral periodicity, a periodicity which is essential to the present internal cases for geometrical reasons. This is so for laterally symmetric or non-symmetric flows. Although the external wakes, if many, do exhibit periodicity over a middle part of their formation downstream, the lateral period produced there is usually not the same as that of the (input) wakes at their starting position upstream. A second difference arises in the determination of the overall wake shapes, or effective centreline, in laterally non-symmetric motions. The shapes are determined by inner–outer interaction in the external cases but by an explicit balance from the lateral momentum in the internal cases. The latter are interesting and unusual in that both the major pressure response and its small correction (see (5.2)) affect the total wake motion.

The internal wakes also exhibit a sensitivity of the overall turning of the entire wake system to an integral of the input velocity profiles as well as the initial direction of the wake system. This is clear from the computations and analysis in the previous section; moreover, for the small deficit wakes of (5.7a to c) the overall turning or scaled form \( \hat{f}'(\infty) - \hat{f}'(0) \) is shown to depend only on the input shear at the dividing streamline. The argument in the previous two sections holds for any streamline of the wakes, the distinction between different streamlines being due to the different starting conditions (as \( \bar{y} \) is moved up or down different portions of \( u_0, \bar{v}_0 \) for example are taken in order), but all streamlines tend to the same direction far downstream.

Concerning application, for in-parallel wakes in practice aft of rotor blades for instance the ratio \( D/L \) which is taken as small in the theory may have a typical value as large as 0.1 to 0.3. The theoretical error tends to be of the order \( (D/L)^2 \) however, as in (5.2), suggesting errors of only about nine per cent at most then. It is intriguing also that the pressure effect, which distinguishes internal from external flow, is of second order in the input wake deficit when that deficit is sufficiently small, whereas the turning effect is of first order. This feature of the pressure indicates that multiple external wakes should simulate the behaviour of internal wakes reasonably well in such circumstances, if not in more general settings; we know as yet of no experimental results indicating difficulty in simulation. Again, in reality the reduced Reynolds number is often very large, compressibility is significant and the wake flows are turbulent. Wake turbulence modelling, however, may leave the main findings of the present study intact or little disturbed, especially as integral properties are found to play a substantial role.

There is also the question of how the current work fits into an overall plan for predicting the flow through successive rows of blades. We consider the internal case mostly. Here the pressure is an unknown in each gap laterally, between successive blades, as well as in each wake, and for slender flows the pressure depends predominantly on the streamwise position only. As part of the gap problem the possibility of pressure discontinuities occurring streamwise across the leading-edge stations must be addressed (11 to 14). Such a discontinuity requires a lateral shift of the velocity profile incident at the leading edge and the magnitude of the shift depends on the pressure adjustment required at the corresponding trailing edge for the Kutta condition there. Because of the lateral periodicity present, however, no pressure adjustment or profile shift is necessary for the internal case. The major difficulty instead is that each gap flow is unsteady in general, due to the relative downward or upward movement of the blade row upstream and the consequent relative movement
of the wakes incident upon that gap. Strictly the wakes themselves follow from a calculation for the entire blade row upstream, and so on. Nevertheless it seems clear that the whole internal-flow solution for successive rows of slender blades, wakes, blades and so on can be determined consistently by a slender-flow approach as here, in principle, subject to the omission of short-scale phenomena. The same applies for much of the external setting also, although there the pressure discontinuities at the leading edges of blade rows cannot be ruled out, because of the external pressure imposition, in addition to the different nature of the wake flows (sections 2 and 3).

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References