WAVE REFLECTION AND TRANSMISSION FROM ANISOTROPIC LAYERS THROUGH RICCATI EQUATIONS

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Summary

Reflection and transmission matrices, associated with obliquely-incident plane harmonic waves, are investigated for a planarly-stratified inhomogeneous layer. The material in the layer and in the half-spaces is a dissipative anisotropic solid. Upon establishing a Cauchy problem for the local impedance matrix in the layer, the condition is found for the determination of the initial value. Next reflection and transmission matrices of the layer are related to the final value of the impedance. Moreover the dependence of the reflection matrix on the thickness of the layer is found to satisfy a Riccati differential equation. This shows in turn that the local reflection and the reflection of the layer are the same in the sense of a principle of localization. As a result, the principle is generalized to anisotropic dissipative layers.

1. Introduction

Wave propagation in stratified media has been widely investigated because of the applications to seismology, geophysical prospecting, ocean acoustics, non-destructive evaluation and electromagnetic remote sensing (1 to 6). Here we consider a time-harmonic (inhomogeneous) plane wave obliquely incident on a solid layer of thickness \(d\), sandwiched between two homogeneous solid half-spaces, and investigate the problem of finding the reflection and transmission matrices of the layer. For the sake of generality, the material is allowed to be dissipative and anisotropic.

Under the assumption that the material parameters depend on only one Cartesian coordinate, say \(z\), the equation of motion and the stress–strain relation can be reduced to a system of six first-order ordinary differential equations of the form

\[ \frac{d\mathbf{w}}{dz} = \mathbf{A} \mathbf{w}. \]  

Up to a wave-like factor, the unknown vector \(\mathbf{w}\) is the pair \([\mathbf{u}, \mathbf{t}]\) of the displacement \(\mathbf{u}\) and traction \(\mathbf{t}\), at planes \(z = \text{constant}\). The matrix \(\mathbf{A}\) is constant for \(z < 0\) or \(z > d\). An equation of the form (1) is found in (2) for stratified elastic solids. The formulation of the governing equations in the form (1) traces back to a paper by Stroh (7) where uniform elastic solids are considered.

The calculation of the reflection and transmission matrices of the layer, \(\mathbf{R}\) and \(\mathbf{T}\), results in a two-point boundary-value problem for (1) in \([0, d]\) (6). By means of the impedance matrix \(\mathbf{Y}\), such
that \( t = Y u \), the calculation of \( R \) and \( T \) is turned into a Cauchy problem. Alternatively, the matrices \( R \) and \( T \) can be computed through the propagator (or fundamental) matrix \( \Omega \) of the system (1) in \([0, d]\). Algorithms for the numerical calculation of \( R \) and \( T \), in terms of \( \Omega \), in stratified isotropic solids can be found in (2,8).

It may be convenient to regard as unknowns the amplitudes \( v \) of forward- and backward-propagating waves. This is accomplished through the eigenvectors of a \( A(2,6,8 \to 13) \). Meanwhile, the reflectivity or local reflection matrix function \( R(z) \) is defined which relates the amplitudes of forward- and backward-propagating waves \( (11) \). In terms of the variables \( v \), the matrices \( R \) and \( T \) are also established.

The purpose of this paper is threefold. The first is to examine the main properties of the matrix function \( Y(z) \) which is found to obey a Riccati equation. The associated problem requires that the initial value is given. We find that the initial value of \( Y(z) \) is determined when an appropriate matrix is non-singular. To be specific, examples are given for isotropic solids. Next, subject to the existence of the initial value for \( Y(z) \), the impedance \( Y(z) \) is related to the propagation matrix \( \Omega(z) \). We find that \( Y(z), z \in [0, d] \), exists if and only if an appropriate matrix involving \( \Omega(z) \) is non-singular. Moreover, the differential equation for the propagator matrix \( K(z) \), relative to the displacement, is also derived in terms of \( Y(z) \).

The second purpose is to relate \( R \) and \( T \) to \( Y \) or \( \Omega \) for the layer. In both cases \( (Y \) or \( \Omega) \) we find that the non-singularity of an appropriate matrix, \( B' \), allows for the existence of \( R \) and \( T \). Meanwhile, the singularity of \( B' \) implies non-uniqueness of the solution or incompatibility depending on the range of \( B' \).

The third purpose is to examine how \( R \) changes as the thickness of the layer changes. We find that \( R(z) \), relative to the layer \([z, d]\), satisfies a Riccati differential equation and show that \( R \) and \( \Omega \) are the same in that they satisfy the same differential equation and the same initial condition. This aspect relates seemingly different definitions of reflection matrix in planarly-stratified materials. Indeed, as a result a generalization is given of the Principle of Localization \( (14) \) to anisotropic dissipative layers.

2. First-order governing equations

Consider a three-dimensional anisotropic viscoelastic solid with one-dimensional inhomogeneity, in the \( z \)-direction. The material properties are homogeneous in the half-spaces \( z < 0 \) and \( z > d \). For \( z \in (0, d) \) the material properties are continuous, bounded functions of \( z \). We let the properties suffer jump discontinuities only at \( z = 0 \) and \( z = d \). Also with regard to previous works, it is a routine matter to account for a finite number of jump discontinuities in \((0, d)\).

The equation of motion is taken in the form

\[
\rho \ddot{u} = \nabla \cdot T,
\]

where \( \rho \) is the mass density, \( u \) is the displacement, \( T \) is the stress tensor and \( \nabla \) is the gradient with respect to the position vector \( x \). The constitutive equation is such that, for time-harmonic displacements,

\[
T(x, t) = G(z, \omega)\nabla u(x, t),
\]

where \( \omega \) is the angular frequency while \( G \) is a complex-valued fourth-order tensor; in indicial notation \( T_{ijh} = G_{jkhl} \partial u_l / \partial x_k \).
We look for solutions of the form
\[ u(x, t) = \hat{u}(z) \exp[i(k \cdot x - \omega t)], \tag{2} \]
where \( k \) is a possibly-complex vector perpendicular to the z-axis. In practice, \( k \) is induced by the incident wave. Also let
\[ t(x, t) = \hat{t}(z) \exp[i(k \cdot x - \omega t)], \tag{3} \]
and \( t = T e_1 \) being the traction at the pertinent plane with \( z = \text{constant}. \)

As shown in (15), subject to the restriction \( \Re G < 0 \) of thermodynamic character, the column vector \( w = [\hat{u}, \hat{t}]^T \) satisfies the system of first-order differential equations (1) where \( A = A(z) \), parametrized by \( \omega \) and \( k \), is a matrix in \( \mathbb{C}^{6 \times 6} \). Henceforth we find it convenient to write \( 6 \times 6 \) matrices through \( 3 \times 3 \) blocks in the form
\[ A = \begin{bmatrix} A^I & A^{II} \\ A^{III} & A^{IV} \end{bmatrix}. \]

For the matrix \( A \) (of (1)), the blocks are given by
\[ A^I = -i \left( e_3 G e_3 \right)^{-1} (e_3 G k), \quad A^{II} = (e_3 G e_3)^{-1}, \]
\[ A^{III} = -\rho \omega^2 I + (k_1 G k_1) - (k_1 G e_3) (e_3 G e_3)^{-1} (e_3 G k_1) \]
where, for any pair of vectors \( a, b, aGb \in \mathbb{C}^{6 \times 6} \) is the matrix such that \( (aGb)_{ij} = a_p G_{ip} b_k \). The symbol \( I \) denotes the \( 3 \times 3 \) or \( 6 \times 6 \) identity matrix as appropriate. Incidentally, the blocks \( A^{II} \) and \( A^{III} \) are symmetric matrices.

For formal convenience, let a prime stand for \( d/dz \). Also, let the subscript \( -, + \) denote the values at \( z = 0, d \). Though discontinuities of \( A \) are allowed (at \( z = 0, d \)), the vectors \( u \) and \( t \), and hence \( w \), are supposed to be continuous. The system (1) is linear and the matrix \( A \) is taken to be bounded. Hence (16, Chapter 4; 17, Chapters 1, 2) there exists the propagator matrix function
\[ \Omega : [0, d) \rightarrow \mathbb{C}^{6 \times 6}, \] such that
\[ w(z) = \Omega(z) w(0). \tag{4} \]

Hence \( \Omega \) is the solution to the Cauchy problem (2, 8, 9, 16)
\[ \Omega' = A \Omega, \quad \Omega(0) = I. \tag{5} \]

The matrix \( \Omega \) turns out to be non-singular for every value of \( z \in [0, d] \) and determines \( w \) by (4).

3. Impedance matrix and propagator matrix

Let \( Y \) be the impedance matrix such that
\[ \hat{t} = Y \hat{u}. \tag{6} \]

Hence the upper part of (1), \( \hat{u}' = A^I \hat{u} + A^{II} \hat{t} \), implies that \( \hat{u} \) satisfies the first-order differential equation
\[ \hat{u}' = (A^I + A^{II} Y) \hat{u}. \tag{7} \]
By means of (1), (6) and (7) we find that the matrix $Y$ satisfies the Riccati differential equation

$$Y' = A^{III} + A^{IV} Y - Y A^I - Y A^{II} Y.$$  \hfill (8)

Conversely, equation (6) and the observation that $\dot{Y} = Y' \dot{u} + Y \ddot{u}$ show that (7) and (8) imply (1). This is the proof of the following statement.

**Theorem 1.** If $\dot{u}$ and $Y$ satisfy (7) and (8) in $[0, d]$ then the vector $w = [\dot{u}, Y \dot{u}]^T$ satisfies the system (1).

The impedance matrix $Y$ may be determined through the propagator matrix $\Omega$ for $w$. Also, $\Omega$ determines the propagator matrix $K : (0, d) \rightarrow C^{3 \times 3}$, such that

$$\dot{u}(z) = K(z)\dot{u}(0), \quad K(0) = 1.$$  \hfill (9)

In this regard replace $\dot{u}$ with $Y \dot{u}$ and observe that, by (4),

$$\dot{u}(z) = [\Omega^I(z) + \Omega^{II}(z) Y(0)] \dot{u}(0), \quad Y(z) \dot{u}(z) = [\Omega^{III}(z) + \Omega^{IV}(z) Y(0)] \dot{u}(0).$$  \hfill (10)

Hence we have

$$K(z) = \Omega^I(z) + \Omega^{II}(z) Y(0).$$  \hfill (11)

Moreover, if $\Omega^I(z) + \Omega^{II}(z) Y(0)$ is non-singular we have

$$Y(z) = [\Omega^{III}(z) + \Omega^{IV}(z) Y(0)] [\Omega^I(z) + \Omega^{II}(z) Y(0)]^{-1}. \hfill (12)$$

**Theorem 2.** Let $\Omega(z)$ be the propagator matrix of (1). If the Riccati equation (8) allows for the solution $Y(z)$, $z \in [0, d]$, with a prescribed initial value $Y(0)$, then the matrix $K$ given by (10) is the propagator matrix for (7) and hence satisfies the differential equation

$$K' = (A^I + A^{II} \dot{Y}) K.$$  \hfill (13)

**Proof.** Evaluate (10) at $z = 0$ and compare with (5) to see that $K(0) = 1$. Left multiply the first equation in (9) by $Y$ and compare with the second one. The arbitrariness of $\dot{u}(0)$ produces

$$Y [\Omega^I + \Omega^{II} Y(0)] = \Omega^{III} + \Omega^{IV} Y(0).$$

Hence, differentiation of (10), use of (5) and the definition of $K$ provide (12).

**Theorem 3.** The Riccati equation (8) allows for the solution $Y(z)$, $z \in [0, d]$, satisfying the initial condition for $Y(0) = Y_0$ if and only if the matrix $K(z) = \Omega^I(z) + \Omega^{II}(z) Y_0$ is non-singular in $[0, d]$.

**Proof.** Suppose that $\Omega^I(z) + \Omega^{II}(z) Y_0$ is non-singular and hence define $Y$ through (11). Differentiation with respect to $z$ of (11), $z \in (0, d)$, substitution of $\dot{\Omega}$ with the expressions arising from (5), some rearrangement and application of (11) show that $Y$ satisfies the Riccati equation (8). Also, letting $\Omega(0) = \Omega$ in (11) we find that $Y$ takes the prescribed value $Y_0$ at $z = 0$.

Conversely, if $Y(z)$ is a solution of (8) in $[0, d]$ then $K(z) = \Omega^I(z) + \Omega^{II}(z) Y_0$ is the fundamental matrix for (7). Such a matrix is non-singular (16).
In words, Theorem 3 shows that \( \det K \neq 0 \), for \( z \in [0, d] \), is necessary and sufficient for the existence of the solution \( Y \) to the Cauchy problem for the Riccati equation (8) with initial value \( Y_0 \).

The equivalence between the solutions of a linear system and those of the associated Riccati matrix equation is examined in (17, 18). In both cases a condition is established which is the counterpart of the non-singularity of \( K \). Specifically, in (18) a linear system is considered with constant coefficients, which corresponds to the particular case of a uniform layer. Next the solution of the Riccati equation is determined explicitly by means of the propagator matrix. By means of such a representation, proof is given that the singularities of the solution correspond to those of the matrix \( K \).

Quite generally, we observe that the solution to (5) for \( \Omega \) holds provided only that \( A \) is bounded. Instead, the solution to (8), as for any Riccati equation (19), need not exist as is shown in (18) for complex constant coefficients.

4. Propagation modes

Wave propagation may be described in terms of propagation modes, in which case it is essential to partition them in forward- and backward-propagating components (2, 9, 11). Within the layer the usefulness of this approach is questionable. Rather, it is essential within the homogeneous half-spaces to describe the reflection–transmission process.

Assume that the eigenvectors of \( A(z) \), for \( z \in (-\infty, 0) \cup (0, d) \cup (d, \infty) \), form a basis in \( \mathbb{C}^6 \). If the six roots of the secular equation are distinct the existence of six independent eigenvectors for \( A \) is guaranteed and \( A \) is said to be simple. If, instead, multiple eigenvalues occur then our assumption corresponds to letting \( A \) be semisimple (20). As shown in (21, Chapter 7), simplicity is a generic property and non-simple matrices occur only for particular values of \( k_\| \) (22).

Let \( P(z) : \mathbb{R} \to \mathbb{C}^{6 \times 6} \) be the matrix whose columns are the six chosen independent eigenvectors of \( A \), namely

\[
P = [p_1, \ldots, p_6].
\]  

Also let

\[
A(z) = \text{diag} [\lambda_1, \ldots, \lambda_6]
\]

be the diagonal matrix whose entries are \(-i\) times the eigenvalues of \( A \). Let

\[
E = \exp \left[ i \int_{z_0}^z A(\tau) \, d\tau \right]
\]

\[
= \text{diag} \left[ \exp \left( i \int_{z_0}^z \lambda_1(\tau) \, d\tau \right), \ldots, \exp \left( i \int_{z_0}^z \lambda_6(\tau) \, d\tau \right) \right] = \begin{bmatrix} E' & 0 \\ 0 & E'' \end{bmatrix},
\]

where \( z_0 \) is a chosen value of \( z \). The matrices \( P \) and \( E \) are non-singular. Hence, by (15) we have

\[
E' = iE \Lambda = iA E.
\]  

Let \( v \) be the 6-tuple \( v = (PE)^{-1}w \). Substitution of \( w = PEv \) in (1) and comparison with (16) shows that \( v \) satisfies the differential equation

\[
v' = Zv, \quad Z := -E^{-1}P^{-1}P'E.
\]
To fix ideas let \( z_0 = 0 \). In the half-space \(-\infty < z < 0\) the matrices \( P \) and \( \Lambda \) are constant. Hence (17) implies that \( Z \) vanishes and \( v(z) \) is constant. Denote by a subscript minus sign the values at \( z < 0 \), or at \( z = 0 \) if they are constant. Hence we let \( v = v_- \) for \( z < 0 \). This in turn allows \( w \) to be written as

\[
\begin{bmatrix}
u(x, t) \\
v(t, x, t)
\end{bmatrix} = \sum_{j=1}^{6} v_{-j} \exp[i(k_1 \cdot x + \lambda_{-j} z - \omega t)] p_{-j}, \quad z \in (-\infty, 0). \tag{18}
\]

The representation (18) indicates that \( w \) is the superposition of six inhomogeneous waves (12, 13). We regard \( v_{-j} \exp[i(k_1 \cdot x + \lambda_{-j} z - \omega t)] p_{-j} \) as the \( j \)th propagation mode, or wave, with wave vector \( k_1 + \lambda_{-j} e_3 \) and wave amplitude \( v_{-j} \).

It is reasonable to assume that three waves are forward-propagating and the other three are backward-propagating. This partition holds for small values of \( |k_1| \) or for any real value of \( k_1 \) in elastic solids (13). We use the superscripts \( f \) and \( b \) to denote forward and backward. Hence we let

\[
v = [v^f, v^b]^T
\]

and, to fix ideas, \( v^f = [v_{-1}, v_{-2}, v_{-3}]^T \) and \( v^b = [v_{-4}, v_{-5}, v_{-6}]^T \).

Let \( d < z < \infty \). The value of \( E \) is written as

\[
E(z) = E \exp(i \Lambda_+(z - d)) \quad \text{with} \quad E = \exp \left[ i \int_0^d \Lambda(z) \, dz \right].
\]

Hence

\[
\begin{bmatrix}
u(x, t) \\
v(t, x, t)
\end{bmatrix} = \sum_{j=1}^{6} E_j v_{+j} \exp[i(k_1 \cdot x + \lambda_{+j} (z - d) - \omega t)] p_{+j}, \quad z \in (d, \infty). \tag{19}
\]

Here too we assume that the partition \( v = [v^f, v^b]^T \), where \( v^f, v^b \in \mathbb{C}^3 \), holds. The solution for \( z \geq d \) involves \( E v_+ \). We might redefine \( v_+ \) so as to avoid the occurrence of \( E \) but we prefer to keep \( v_+ \) as it stands.

Within the layer, we have

\[
\begin{bmatrix}
u(x, t) \\
v(t, x, t)
\end{bmatrix} = \sum_{j=1}^{6} E_j(z) v_j(z) \exp[i(k_1 \cdot x - \omega t)] p_j(z), \quad z \in (0, d). \tag{20}
\]

The quantities \( v_j(z) \) may be viewed as the local amplitudes of the pertinent waves. However, \( Z \) is usually a non-diagonal matrix and hence \( v(z) \) is the solution of a coupled system. The picture of superposition of (local) waves is still possible provided that we interpret the consequences of the coupled system as the occurring of a mode conversion within the layer.

By way of comment we say that in general the eigenvalues and the eigenvectors of \( A \) cannot be evaluated explicitly (23) and recourse is made to numerical evaluations. In addition, equation (17) involves the derivatives of the material parameters through \( P' \). On the one hand this shows that stronger assumptions are required on \( A \). Moreover, an approach based on the wave amplitudes requires that the partition of \( v \) into \( v^f \) and \( v^b \) is true at each value of \( z \). We find it more reasonable, in addition to being more general, that the partition holds only at the homogeneous half-spaces where this is required to interpret and to face any reflection–transmission problem and hence any scattering problem. With this observation we now investigate reflection and transmission due to a layer by assuming only that (18) and (19) hold along with the partition of waves.
5. Reflection and transmission via the propagator matrix

Consider the reflection–transmission process that is generated by an incident homogeneous wave which is coming from \( z = -\infty \) and hits the surface \( z = 0 \). The reflected and transmitted waves propagate in the half-spaces \( z < 0 \) and \( z > d \). The incident wave is characterized by \( v^f_− \), the reflected wave by \( v^b_− \), the transmitted wave by \( v^f_+ \). Accordingly we define the reflection and transmission matrices of the layer \( R^f \), \( T^f \) such that

\[
 v^b_− = R^f v^f_-, \quad v^f_+ = T^f v^f_-
\]

the superscript \( f \) in \( R^f \) and \( T^f \) being a reminder that the incident wave is forward-propagating (comes from \(-\infty\) ). The matrices \( R^f \) and \( T^f \) are evaluated by determining \( v^b_− \) and \( v^f_+ \) for any incident vector \( v^f_- \).

At any discontinuity surface the jump of \( v \) is evaluated by requiring the continuity of \( w \). Let \( z = \text{constant} \) be any discontinuity surface for \( A \). Hence

\[
 P(z_-)E(z)v(z_-) = w(z_-) = P(z_+)E(z)v(z_+)
\]

whence we have

\[
 v(z_+) = E^{-1}(z)(P(z_+))^{-1}P(z_-)E(z)v(z_-). \quad (21)
\]

Let \( \Omega_+ = \Omega(d) \) be the propagation matrix of the layer in that \( w(d) = \Omega_+w(0) \). Since \( w(d) = P_+E v_+ = \Omega_+P_-v_- \) we can write \( P_+E v_+ = \Omega_+P_-v_- \) whence

\[
 P_+E \begin{bmatrix} v^f_- \\ 0 \end{bmatrix} = \Omega_+P_- \begin{bmatrix} v^f_-\\ v^-_+ \end{bmatrix} \quad (22)
\]

and then

\[
 B \begin{bmatrix} \mathcal{E}^l v_+ \\ 0 \end{bmatrix} = \begin{bmatrix} v^f_- \\ R^f v^-_+ \end{bmatrix},
\]

where \( B = P_-^{-1} \Omega_+^{-1} P_+ \) is a property of the whole layer. Hence we have

\[
 B^l E^l v^f_- = v^f_- \quad \text{and} \quad B^{II} E^l v^f_- = R^f v^-_+. \quad (23)
\]

If \( B^l \) is non-singular then

\[
 \mathcal{T}^f = (B^l E^l)^{-1} \quad \text{and} \quad R^f = B^{II}(B^l)^{-1}. \quad (24)
\]

If, instead, the incident wave is coming from \( z = \infty \) then we have

\[
 v_+ = \begin{bmatrix} v^f_+ \\ v^b_+ \end{bmatrix}, \quad v_- = \begin{bmatrix} 0 \\ v^b_- \end{bmatrix},
\]

where \( v^b_+ \) represents the incident wave. Accordingly, we can write

\[
 v^b_- = \mathcal{T}^b v^b_+ \quad \text{and} \quad v^f_+ = \mathcal{R}^b v^b_+.
\]
The same matrix \( B \) holds, as before, such that
\[
B \mathcal{E} \begin{bmatrix} v_f^+ \\ v_b^+ \end{bmatrix} = \begin{bmatrix} 0 \\ v_b^- \end{bmatrix}.
\]

Hence we find that
\[
B^I \mathcal{E}^I v_f^+ + B^{II} \mathcal{E}^{IV} v_b^+ = 0, \quad B^{III} \mathcal{E}^I v_f^+ + B^{IV} \mathcal{E}^{IV} v_b^+ = 0.
\] (25)

We solve this system for \( v_b^- \) and \( v_f^+ \) and hence we find that
\[
T_b = \begin{bmatrix} -B^{III} \left( B^I \right)^{-1} B^{IV} \mathcal{E}^{IV} & \mathcal{E}^{IV} \end{bmatrix},
\]
\[
R_b = \left( \mathcal{E}^{IV} \right)^{-1} (B^I)^{-1} B^{III} \mathcal{E}^{IV}.
\] (26)

If incident waves are coming from both sides, \( z = \pm \infty \), then we regard \( v_b^+ \) and \( v_f^+ \) as known while \( v_b^- \) and \( v_f^- \) are still unknown. By linearity, the effects of the two incident waves superpose to give
\[
\begin{bmatrix} v_f^+ \\ v_b^- \end{bmatrix} = S \begin{bmatrix} v_f^- \\ v_b^+ \end{bmatrix},
\] (27)

where \( S \in \mathbb{C}^{6 \times 6} \), given by
\[
S = \begin{bmatrix} T_f & R_b \\ R_f & T_b \end{bmatrix},
\] (28)

is associated with the whole layer. The matrix \( S \) is called the scattering matrix. Collecting the results about the matrices \( T^f, R^f, T^b, R_b \) we find that \( B \) can be written as
\[
B = \begin{bmatrix} \left( T^f \right)^{-1} \left( \mathcal{E}^I \right)^{-1} & -\left( T^f \right)^{-1} \mathcal{E}^{IV} \left( \mathcal{E}^I \right)^{-1} \\ \mathcal{R}^I \left( \mathcal{E}^I T^f \right)^{-1} \left( \mathcal{E}^I \right)^{-1} & \left( T^b - \mathcal{R}^I \left( T^f \right)^{-1} \mathcal{R}^b \right) \left( \mathcal{E}^{IV} \right)^{-1} \end{bmatrix}.
\] (29)

Owing to the definition of \( B \) we can write
\[
\mathbf{\Omega} = \mathbf{P} \cdot B \cdot \mathbf{P}^{-1}.
\] (30)

Incidentally, for any reflection–transmission problem we can regard the solution as the pair of vectors \( v_f^+, v_b^- \). As shown by (24) and (26), the matrix \( S \) can be determined provided \( B^I \) (and \( T^f \)) is non-singular. Moreover, the following theorem shows that the non-singularity of \( B^I \) is crucial for existence and uniqueness of the solution \( v_f^+, v_b^- \).

**Theorem 4.** If \( B^I \) is non-singular then the solution to the reflection–transmission problem is unique, the matrices \( R^f, R_b, T^f, T^b \) and \( S \) are uniquely determined and the relation between \( S \) and \( \mathbf{\Omega} \) is invertible. If, instead, \( B^I \) is singular then the solution to the reflection–transmission problem is non-unique or does not exist according as \( \left( B^{II} \mathcal{E}^{IV} v_b^+ \right) \) belongs to the range of \( B^I \) or does not.
Proof. If $B'$ is non-singular then $\mathcal{R}^f$, $\mathcal{T}^f = (B'\mathcal{E}^f)^{-1}$, $\mathcal{R}^b$, $\mathcal{T}^b$ and $S$ are determined by (24), (26) and (28). Further, by (30), $S$ determines $\Omega$, while, by (29), the knowledge of $\Omega$, and hence of $B$ determines $S$ in that $\mathcal{T}^f = (B'\mathcal{E}^f)^{-1}$, $\mathcal{R}^f = (B'\mathcal{E}^b)^{-1}$, and so on. By means of $S$, the solution is determined uniquely through (27).

Now let $B'$ be singular. Hence the kernel of $B'$, $\mathcal{N}(B')$, is non-empty. Let $v \in \mathcal{N}(B')$. By (23) we have $B'v = 0$ and $B'^{III}v = \hat{v}^b$. Let $\hat{v}^f$, $\hat{v}^b$ be a solution associated with the incident wave $v^f$, which means that $v^f$ is in the range of $B'$. The pair of waves $v^f + \alpha(\mathcal{E}^f)^{-1}v$, $v^b + \alpha v^b$ for every $\alpha \in \mathbb{C}$ is also a solution associated with the same incident wave, thus showing non-uniqueness. If, instead, $v^f$ is not in the range of $B'$ then no solution exists while $v^f$ is the incident wave.

By the same token, let $v^b$ be the incident wave and consider $\hat{v}^f$ such that $\mathcal{E}^f\hat{v}^f = \hat{v} \in \mathcal{N}(B')$. The relations (25) are satisfied by

$$\hat{v}^f, \quad \hat{v}^b = B'^{III}\mathcal{E}^f\hat{v}^f$$

with a zero incident wave $v^b$. Hence, if $v^f$, $v^b$ is a solution while $\hat{v}^f$ is the incident wave then so is $v^f + \alpha(\mathcal{E}^f)^{-1}\hat{v}^f, v^b + \alpha \hat{v}^b, \alpha \in \mathbb{C}$, which shows non-uniqueness. Existence is ensured if (25) can be solved for $\hat{v}^f$ and $\hat{v}^b$, which is the case if $r[B'] = r[B', B'^{II}E^f\hat{v}^b]$, where $r$ is rank. If, instead, $r[B'] \neq r[B', B'^{II}E^f\hat{v}^b]$ then no solution exists.

The existence of non-zero solutions while no incident wave occurs is naturally associated with the occurrence of surface waves (or guided waves). In particular, Theorem 4 shows that such a circumstance is related to the singularity of $B'$ in analogy with (24) for scattering by an infinite layer. Non-uniqueness or incompatibility occur also at the plane interface between solid half-spaces (13).

Since $\mathcal{E}^f$ is non-singular, experimentally the non-singularity of $B'$ may be ascertained by measuring $\mathcal{T}^f$ the transmission matrix, for forward-propagating waves. Indeed, $\mathcal{T}^f$ is singular if and only if $B'$ is singular. Mathematically, by means of $A$ we determine $\Omega$, and then we evaluate $B' = (P^{-1}_{−}\Omega P_{−})^f$.

It is worth remarking that $A$ is parametrized by the frequency $\omega$ and the transverse wave vector $k_t$. Now, finding $B'$, when the material properties and the parameters $\omega, k_t$ are given, is a matter of calculation. The converse, namely finding $k_t$ such that $B'$ is a given matrix, possibly singular, is a very awkward problem.

6. Reflection–transmission and impedance matrices

The impedance matrix $Y$ satisfies the Riccati differential equation (8). To find $Y$ we have to integrate (8) and to know the value of $Y$ at a point. The reflection–transmission problem indicates that the natural value to know is $Y(d)$. We set aside the case where $u(d)$ vanishes (fixed boundary) and then we assume that

$$t(d) = Y(d)u(d)$$

can be written for any admissible $u(d)$. Moreover, to fix ideas we let $v^b = 0$, namely the incident wave is coming from $z = -\infty$.

Since $w = PEv$ we let $z = d$ and write

$$\begin{bmatrix} u(d) \\ Y(d)u(d) \end{bmatrix} = P_+E \begin{bmatrix} v^f \\ 0 \end{bmatrix} = \begin{bmatrix} P'_{+}\mathcal{E}^fv^f \\ P'^{III}_{+}\mathcal{E}^f\hat{v}^f \end{bmatrix} = \begin{bmatrix} B^f \hat{v}^f \\ S \end{bmatrix},$$

(31)
namely
\[ u(d) = P^I_+ E^I v^f_+, \quad Y(d) u(d) = P^{III}_+ E^I v^f_+. \] (32)

It is understood that \( Y(z), z \in [0, d], \) is the solution to the Cauchy problem for (8).

**Theorem 5.** The Cauchy problem for the Riccati equation (8) is defined if and only if \( P^I_+ \) is non-singular in which case \( Y(d) = P^{III}_+ (P^I_+)^{-1}, \) unless the half-space \( z > d \) is empty in which case \( Y(d) = 0. \) The matrices \( R^I \) and \( T^I \) are uniquely defined, and only if, the matrix \( Y(0)P^{II}_- - P^{IV}_- \) is non-singular. If, instead, \( Y(0)P^{II}_- - P^{IV}_- \) is singular then the reflection–transmission problem allows for infinitely many solutions or is incompatible according as \( (Y(0)P^{II}_- - P^{III}_-)v^f_+ \) belongs to the range of \( Y(0)P^{II}_- - P^{IV}_- \) or does not.

**Proof.** Let \( P^I_+ \) be non-singular. Hence we can solve (32) for \( v^f_+ \) to find that
\[ Y(d) u(d) = P^{III}_+ (P^I_+)^{-1} u(d). \]
The arbitrariness of \( u(d) \) gives \( Y(d) = P^{III}_+ (P^I_+)^{-1}. \)

If \( P^I_+ \) is singular then \( Y(d) \) is undetermined and the Cauchy problem is undefined as we show by contradiction. Apply \( Y(d) \) to the first equation in (32) to get
\[ Y(d)P^I_+ E^I v^f_+ = P^{III}_+ E^I v^f_+. \]

Since \( P^I_+ \) is singular, there is a vector \( \tilde{v} \neq 0 \) such that \( P^I_+ E^I \tilde{v} = 0. \) Hence, letting \( v^f_+ = \tilde{v} \) we have \( P^{III}_+ E^I \tilde{v} = 0. \) It then follows from (31) that
\[ P^I_+ E \begin{bmatrix} \tilde{v} \\ 0 \end{bmatrix} = 0 \]
in contradiction to the assumption that \( P^I_+ \) is non-singular. Hence \( Y(d) \) is undetermined.

If, instead, the half-space \( z > d \) is empty then \( t(d) = 0 \) for every \( u(d) \) whence it follows that \( Y(d) = 0. \)

We first examine the reflection. Let now \( P^I_+ \) be non-singular, in which case \( Y(d) \) is determined, and suppose that, by integration, (8) yields \( Y(z), z \in [0, d]. \) The relation \( w = PEv \) at \( z = 0 \) gives
\[ u(0) = P^I_+ v^f_+ + P^{III}_+ v^b_+, \quad Y(0) u(0) = P^{III}_+ v^f_+ + P^{IV}_+ v^b_. \]

Application of \( Y(0) \) to the first relation and comparison with the second gives
\[ Y(0)(P^I_+ v^f_+ + P^{III}_+ v^b_+) = P^{III}_+ v^f_+ + P^{IV}_+ v^b_. \]

Hence
\[ (Y(0)P^I_- - P^{IV}_-) v^f_+ = -(Y(0)P^I_- - P^{III}_-) v^b_. \] (33)

whence it follows that
\[ R^I = -(Y(0)P^I_- - P^{IV}_-)^{-1} (Y(0)P^I_- - P^{III}_-). \] (34)
This result relates the reflection matrix of the layer, \( R^f \), with \( Y(0) \) subject to the assumption that \( Y(0)P^f - P^IV \) is non-singular.

We now examine the transmission. Still by \( w = PEv \) and \( \bar{v}^b = R^f v^f \) we have

\[
u(0) = [P^f + P^f R^f]v^f.
\]

Given the incident wave \( v^f \) we determine \( u(0) \). Hence, by integration of (7) we obtain \( u(d) \). Since \( u(d) = P^f E^f v^f \), it follows that \( v^f \) is determined by \( v^f = (E^f)^{-1}(P^f)^{-1}u(d) \) whence \( T^f \) follows.

If \( Y(0)P^f - P^IV \) is singular then (33) cannot be solved for \( v^b \) when the left-hand side is not in the range of \( Y(0)P^f - P^IV \), in which case incompatibility occurs. To prove non-uniqueness, we observe that there is a non-zero pair solution \((\bar{v}^b, \bar{v}^f)\) while \( v^f = 0 \). For, let \( \bar{v}^b \) be a non-zero element of \( Y(0)P^f - P^IV \). Since \( \bar{u}(0) = P^f \bar{v}^b \), by integrating (7) we find the value \( \bar{u}(d) \).

Hence the transmitted wave \( \bar{v}^f \) is given by

\[
\bar{v}^f = (E^f)^{-1}(P^f)^{-1} \bar{u}(d).
\]

The occurrence of the solution \( \bar{v}^f, \bar{v}^b \) while the incident wave vanishes yields the non-uniqueness of the solution to the reflection–transmission problem.

A further question may arise about the possibility of a direct check of the consistency of the results (24) and (34) for \( R^f \) through \( \Omega \) and \( Y \). Observe that (22) can be written as

\[
P^f E^f v^f = (\Omega^f P^f + \Omega^f P^f P^f) v^f + (\Omega^f P^f + \Omega^f P^f P^f) v^b,
\]

\[
P^f E^f v^f = (\Omega^f P^f + \Omega^f P^f P^f) v^f + (\Omega^f P^f + \Omega^f P^f P^f) v^b.
\]

Since \( P^f E^f \) is non-singular, left multiplication of (35) by \((P^f E^f)^{-1}\) yields

\[
v^f = (P^f E^f)^{-1}(\Omega^f P^f + \Omega^f P^f P^f) v^f + (P^f E^f)^{-1}(\Omega^f P^f + \Omega^f P^f P^f) v^b.
\]

Substitution of (37) in (36), the observation that \( P^f P^f P^f = Y(d) =: Y_+ \) and some rearrangement yield

\[
(\Omega^f P^f + \Omega^f P^f P^f) (v^f P^f + P^f) + (\Omega^f P^f + \Omega^f P^f P^f) (P^f I^f v^f + P^f v^b) = 0.
\]

Hence, in view of (11) at \( z = d \), we can write

\[
(\Omega^f P^f + \Omega^f P^f P^f) (Y_+ P^f v^f + Y_+ P^f P^f v^b - P^f I^f v^f + P^f v^b) = 0,
\]

where \( Y_- = Y(0) \). If the matrix \( Y_+ \Omega^f - \Omega^f P^f \) is non-singular this equation implies that

\[
Y_- P^f I^f v^f + Y_- P^f P^f v^b - P^f I^f v^f - P^f P^f v^b = 0.
\]

The system (37), (38) is equivalent to (23), in that both of them are equivalent to (22). Accordingly, the corresponding representations of the reflection matrix \( R^f \) are equivalent as well. Since (38) is equivalent to (33) we conclude that the representations (24) and (34) of \( R^f \) are equivalent.

If the matrix \( Y_+ \Omega^f - \Omega^f P^f \) is singular then (33) implies (38). Hence, the representation of \( R^f \), obtained through the impedance, holds also within the propagator matrix approach.
7. Principle of localization

In (14), a principle of localization is asserted through the following statement: a plane wave travelling through an inhomogeneous medium proceeds as if there were an instantaneous reflection and transmission at each interface of a stratum \([z, z + \Delta z]\). The statement is proved by considering a scalar wave equation, which is the case for longitudinal and transverse waves in isotropic solids with normal incidence. Here we show that an analogous statement holds in the general case of anisotropic stratified layers.

The principle of localization provides a procedure for the determination of the local reflection and transmission matrices. Such a procedure is named invariant imbedding (25).

Look at a layer in \((z, d)\), where \(z < d\), say \(z \in (0, d)\) and regard the half-spaces \((\text{\(-\infty, z\)}\) and \((d, \infty]\) as homogeneous. As in the previous sections, the material properties may be discontinuous at the boundaries. In a moment we examine the dependence of reflection and transmission of the layer on the thickness \(d - z\).

The relations (27) and (28) hold in the present case by observing that \(v^f\) and \(v^b\) occur at \(z_+\). Hence we write the analogue of (27) in the form

\[
\begin{bmatrix}
  v^f_z(d) \\
  v^b_z(z)
\end{bmatrix}
= S(z)
\begin{bmatrix}
  v^f_z(z) \\
  v^b_z(d)
\end{bmatrix},
\]

(39)

where \(S(z)\) is a reminder that the value of \(S\) is associated with the layer in \((z, d)\). It is understood that the solution \(v^f_z(d), v^b_z(z)\) exists and is unique. With this in mind we write the analogue of (28) as

\[
S(z) = \begin{bmatrix}
  T^f_z(z) & R^b_z(z) \\
  R^f_z(z) & T^b_z(z)
\end{bmatrix}.
\]

(40)

Now we keep the quantities \(v^f_z(d)\) and \(v^b_z(d)\) fixed, which means that we regard the waves in \((d, \infty]\) as fixed. If the position \(z\) of the first interface varies then the scattering matrix \(S(z)\) and the waves \(v^b_z(z), v^f_z(z)\) change so that the the waves \(v^f_z(d), v^b_z(d)\) are kept unchanged.

Henceforth we regard \(v^f(z), v^f_z(z)\) and \(S(z)\) as differentiable, which in turn means that we let \(z\) run over subintervals of \((0, d)\) and set aside possible values of \(z\) where \(A\), and hence also \(v^f, v^b, S\), is discontinuous. That is why here we let \(v^f_z, v^b_z\) denote the fixed values at the right of the layer while \(v^f, v^b, S\) are differentiable, and hence continuous, functions of \(z\).

**Theorem 6.** If \(v^f_z\) and \(v^b_z\) are fixed and \(v^f(z), v^b(z), S(z)\) are differentiable then the blocks \(T^f, R^f, R^b, T^b\) of the scattering matrix of the layer \((z, d)\) change with the position \(z\) of the first interface according to the differential equations

\[
T^f = -T^f Z^f - T^f Z^{1f} R^f,
\]

(41)

\[
R^f = -R^f Z^f + Z^{1f} R^f - R^f Z^{1f} R^f,
\]

(42)

\[
R^b = -T^f Z^{1f} T^b,
\]

(43)

\[
T^b = (Z^{1f} - R^f Z^{1f}) T^b.
\]

(44)

**Proof.** Since \(v^f(z), v^f_z(z)\) and \(S(z)\) are differentiable, the derivative of (39) gives

\[
\begin{bmatrix}
  0 \\
  (v^b)'
\end{bmatrix} = S' \begin{bmatrix}
  v^f \\
  v^b
\end{bmatrix} + S \begin{bmatrix}
  (v^f)' \\
  0
\end{bmatrix}.
\]
In view of (17) we have
\[
\begin{bmatrix}
0 & Z^{II}v^f + Z^{IV}v^b
\end{bmatrix} = S' \begin{bmatrix} v^f \\ v^b \end{bmatrix} + S \begin{bmatrix} Z^I v^f + Z^{II} v^b \\ 0 \end{bmatrix}.
\]
Now we make use of (39) to express \( v^b \) and \( v^f \) in terms of the incident waves \( v^f, v^b \) to obtain
\[
\begin{bmatrix}
0 & Z^{II}v^f + Z^{IV} (R^f v^f + T^b v^b)
\end{bmatrix} = S' \begin{bmatrix} v^f \\ v^b \end{bmatrix} + S \begin{bmatrix} Z^I v^f + Z^{II} (R^f v^f + T^b v^b) \\ 0 \end{bmatrix}.
\]
This condition can be given the form
\[
\begin{bmatrix} M^I & M^{II} \\ M^{II} & M^{IV} \end{bmatrix} \begin{bmatrix} v^f \\ v^b \end{bmatrix} = 0
\]
which must hold for every pair of vectors \( v^f, v^b \). Such is the case if and only if the blocks \( M^I, \ldots, M^{IV} \) vanish. By means of (40), the vanishing of \( M^I, \ldots, M^{IV} \) results in the differential equations (41) to (44).

The local reflection matrix \( R \) and the local transmission matrix \( T \) such that
\[
v^b (z) = R(z) v^f (z), \quad T(z) v^f (z) = v^f (d_z),
\]
while \( v \) is governed by (17), satisfy (41) and (42) as \( R^f \) and \( T^f \) do. This is shown in (15) by following a procedure strictly similar to that for \( Y \). Moreover,
\[
T^f (d_z) = T (d_z) = 1, \quad R^f (d_z) = R (d_z) = 0,
\]
while
\[
R^f (0) = R (0), \quad T^f (0) = T (0)
\]
are the reflection and transmission matrices of the layer \([0, d] \). Hence it follows that
\[
R^f (z) = R (z), \quad T^f (z) = T (z)
\]
for any \( z \in [0, d] \). This result can be phrased through the following principle of localization: a wave \( w (z) \exp [i (\mathbf{k} \cdot \mathbf{x} - \omega t)] \) travelling through a planarly-stratified anisotropic dissipative solid, at \( z \), is associated with local reflection and transmission matrices \( R, T \) which are equal to the reflection and transmission matrices \( R^f, T^f \) of the layer \([z, d] \).

Incidentally, the result holds at every point \( z \) where \( v \) is differentiable and hence continuous. In such a case the value of \( A \) in the half \(( -\infty, z] \) equals \( A (z) \). Discontinuities of \( A \) are allowed to occur at the interfaces \( z = 0, d \) as well as at any internal interface \( z = z_0 \in (0, d) \). A discontinuity of \( A \) simply means that \( v^f, v^b, S \) are discontinuous at \( z = 0, \ldots, z_0, \ldots, d \). Indeed, \( v^f (z_0), v^b (z_0) \) are related to \( v^f (z), v^b (z) \) by (21) and hence the jumps of \( v^f, v^b, T, R \) follow. The possibility of discontinuities is a further generalization with respect to (14, 22).

Often a procedure is applied for the derivation of differential equations for \( R \) and \( T \) which is named invariant imbedding. The procedure consists in deriving the derivative of \( R^f (z) \) and \( T^f (z) \) inasmuch as the position of the first interface changes. It is the proof of the principle of localization which justifies the recourse to the invariant imbedding.
8. Remarks
The reflection and transmission matrices of an inhomogeneous layer, $z \in [0, d]$, are shown to be determined via two approaches which are based on the impedance matrix $Y$ and the propagator matrix $\Omega$. To find the matrix function $Y(z)$, a Cauchy problem for a system of nine Riccati equations has to be solved. Also, the initial condition for $\Omega(z)$ has to be solved. Also, the initial condition for $\Omega$ matrix be determined via two approaches which are based on the impedance matrix $Y$.

The non-singularity of $B^I$ guarantees existence and uniqueness of the matrices $T^f$, $R^f$, $T^b$ and $R^b$. Conversely, the singularity of $B^I$ may result in the occurrence of surface waves or in the non-existence of the solution. Now, $B^I$ is given by the material properties through the eigenvector matrices $P_\sigma$, $P_\lambda$ and the propagator matrix $\Omega_{\lambda}$. Since $B^I$ is singular if and only if $T^f$ is singular it follows that a measure of $T^f$ determines whether or not $B^I$ is singular.

Concerning $Y$, the initial condition can be given if and only if $P^I_{\lambda}$ is non-singular. It is of interest to consider examples and inspect whether the singularity of $P^I_{\lambda}$ is likely to occur.

Look at isotropic solids with Lamé parameters $\mu$, $\lambda$. Let $k_\parallel$ be directed along the x-axis. Hence the two sets of components $[\hat{a}_x, \hat{t}_x]$ and $[\hat{u}_x, \hat{u}_z, \hat{t}_z]$ decouple. The first case represents horizontally-polarized waves and

$$P = \begin{bmatrix} 1 & 1 \\ i\phi & -i\phi \end{bmatrix},$$

where $\phi = \mu \sigma_T$ and $\sigma_T = \sqrt{(\rho \omega^2/\mu) - k_x^2}$. Hence $P^I = 1$ is non-singular and $Y(d)$ is well defined. The second case represents vertically-polarized waves. To save writing we consider only $P^I$ which reads

$$P^I = \begin{bmatrix} k_x & -\sigma_T \\ \sigma_L & k_x \end{bmatrix},$$

where $\sigma_L = \sqrt{\rho \omega^2/(\lambda + 2\mu)} - k_x^2$. It follows that $P^I$ is singular when $k_x^2 = \rho \omega^2/(\lambda + 3\mu)$.

For elastic materials, the positive definiteness of the elasticity tensor requires that $\mu > 0$, $3\lambda + 2\mu \geq 0$. It then follows that the denominator $\lambda + 3\mu$ does not vanish and there are two values of $k_x$ such that $Y(d)$ is undetermined. Except for these two values of $k_x$, $P^I_{\lambda}$ is non-singular.

9. Conclusions
This paper emphasizes that the reflection–transmission matrices of an inhomogeneous anisotropic layer can be determined through the propagator matrix $\Omega$ or the impedance matrix $Y$ which satisfies a matrix Riccati equation. In both approaches, no assumption is made about wave splitting which in fact would be a restrictive condition on the balanced partition of the wave modes at any point of the inhomogeneous layer. Also, no invariant imbedding is required. Indeed, we have shown that the result obtained often in the literature by having recourse to invariant imbedding is found to be a proved consequence even for anisotropic layers.

The solution for $\Omega$ exists provided only that the matrix $A$ is bounded. The solution for $Y$, instead, may not exist both because of the indeterminacy of the initial value $Y(d)$ or because of the properties of Riccati equations. That is why the recourse to $\Omega$ looks preferable. Also, numerically, the determination of $\Omega$ proves easier than $Y$ as examined in (26) in connection with isotropic dissipative solids with a linear dependence of $\mu$ on $z$. 
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