Author's Closure[9]

For three-dimensional impacts where the initial speed of sliding is small enough that slip vanishes before separation, when slip vanishes the points of contact subsequently either stick or sliding resumes in a new direction, θ_*. As Professor Mac Sithigh correctly points out, in general, θ_* is not directly opposite to the direction $\bar{\phi} - \pi$ of the tangential constraint force for stick. In a second phase of slip, the direction θ_* depends on the inertia properties and the coefficient of friction μ. For an asymmetrical impact configuration with a limiting coefficient of friction for stick $\bar{f}\,(= \mu_*)$, it is only in the limit as $\mu \to f$ that $\theta_* \to \bar{\phi}$.

Professor Mac Sithigh mentions that for some impact configurations, this analysis yields the correct sign for normal acceleration only if the coefficient of friction is less than a limiting or critical value. In my 1990 paper (*Proc. Roy. Soc. London*, Vol. A431, p. 169) I used the term "jamb" to describe impacts where the coefficient of friction was larger than the critical value. For these impacts, an initial change in velocity is not a continuous function of the normal impulse τ. The phenomenon of jamb is associated with Painleve's paradox as explained by Lötstedt (1981); it is likely to be important for surface damage due to abrasive wear during impact.

Reference

Lötstedt, P., 1981, *Angew. Math. Mech.*, Vol. 61, pp. 605–616.

On the Geometry of Nonholonomic Dynamics[10]

J. G. Papastavridis[11]. We would like to point out the following:

1 This is a clearly and carefully written paper; but, unfortunately, it does not contain anything new either mechanically or mathematically; see for e.g., Kondo (1955–1968), Ferrarese (1963), Gugino (1936), Dobronravov (1970, 1976), Hamel (1949), Prange (1935), Synge (1936), Schouten (1954), Vagner (1941), and Vranceanu (1936). Even its example (ball rolling on spinning turntable) can be found in Dobronravov (1976, pp. 201–209) in greater detail.

2 Further, its statement (p. 689) that Lagrange's equations are nothing but a *mathematical* rearrangement of Newton's laws (projections of the latter on certain "tangent directions" in configuration space) and its related implication that, therefore "virtual concepts" are pointless and/or irrelevant, are fundamentally flawed. To go from Newton to Lagrange, Hamel et al., one needs physical, or constitutive, postulates for the constraint reactions, in addition to the well-known differential-geometric transformation of the inertia terms. Schematically: Lagrange = Newton/Euler + Constitutive Postulate. As in continuum mechanics, one cannot do elasticity (fluid mechanics) without Hooke's (Navier-Stokes') "law;" because without them the problem is, in general, indeterminate, i.e., has more unknowns than equations. Here is why: the equations of motion of a typical system particle P: $m_p \mathbf{a}_p = \mathbf{F}_p + \mathbf{R}_p$ (where $m_p/\mathbf{a}_p/\mathbf{F}_p/\mathbf{R}_p$ = mass/acceleration/impressed force (usually known)/constraint reaction (unknown), on P; $p = 1, \ldots, N \equiv$ # particles) contain $6N$ scalar unknowns: $3N$ for the $\mathbf{a}_p$, and $3N$ for the $\mathbf{R}_p$. To find them we have $3N$ scalar equations of motion (above); h equations of geometrical constraints: $f_H(t, \mathbf{r}_p) = 0$ $(H = 1, \ldots, h)$, and m velocity constraints: $\phi_D(t, \mathbf{r}_p, \dot{\mathbf{r}}_p) = 0$ $(D = 1, \ldots, m)$; i.e., so far we have: $6N - (3N + h + m) = (3N - h) - m \equiv n - m \equiv f$ more unknowns. The missing f equations are generated by a *physical* postulate about the $\mathbf{R}_p$'s; e.g., the D'Alembert-Lagrange principle of "ideal constraints," for passive (contact, rolling) reactions; the transformations of the projected inertia terms are important but secondary to Lagrangean mechanics. And this leads inescapably to virtual displacements/work, etc.—the symbol $\delta(\ldots)$ is not the issue. For example, how does the author propose to handle systems under general nonlinear constraints $\phi_D(t, q, \dot{q}) = 0$ $(D = 1, \ldots, m)$ and/or servo/control constraints, without additional physical postulates involving virtual displacements/work?

3 Unfortunately, Cheteav seems unaware of *all* nonholonomic mechanics authors: except Poincaré whose (less than a (total) 2-page paper) deals only with a very special case of nonholonomic "coordinates," and says nothing about such constraints. In addition, Hamel (1949—whose approach is fundamentally different and far superior to Poincaré's) explains clearly that by taking $q_{n+1} = \pi_{n+1} \equiv t$(—ime), his equations extend easily the general *rheonomic*, case; see also Prange (1935).

4 The terminology "generalized speeds" for the $\dot{\pi}^k$ is inconsistent. The $\dot{q}^k$ and $\dot{\pi}^k$ are, respectively, the contravariant holonomic and nonholonomic components of the same (system) velocity vector (see, e.g., Schouten (1954)).

5 Finally, the statement (p. 693) that nonholonomically constrained motions "will ... take place on a lower dimensional point set (of dimension $n_u < n$)" is incorrect. The global motion still takes place in an n-dimensional space; but it is restricted locally.

References

Dobronravov, V. V., 1970, *Fundamentals of Nonholonomic System Dynamics*, Vischaya Shkola, Moscow, pp. 147–184.

Dobronravov, V. V., 1976, *Foundations of Analytical Mechanics*, Vischaya Shkola, Moscow, pp. 201–209.

Ferrarese, G., 1963, "Sulle equazioni di moto di un sistema sogetto a un vincolo anolonmo mobile," *Rendiconti di Matematica, Inst. Nazional di alta Matematica*, Vol. 22, pp. 351–370.

Gugino, E., 1936, "Deduzione unitaria delle equazioni dinamische del Maggi e dell'Appell," *Rend Atti R. Accad. Naz. Roma*, 6th Series, Vol. 23, pp. 406–421.

Hamel, G., 1949, *Theoretische Mechanik*, Springer, Berlin, p. 474.

Kondo, K., editor, *RAAG Memoirs of the Unifying Study of the Basic Problems in Engineering Sciences by Means of Geometry*, 4 Volumes, Gakujutsu Bunken Fukyu-Kai, Tokyo, 1955–1968.

Prange, G., 1935, "Die allgemeinen Integrationsmethoden der analytischen Mechanik," *Encyklopädie der Math. Wissenschaften*, V. 4, Pt. 2, Art's. 12, 13, pp. 505–804, Teubner, Leipzig.

Schouten, J. A., 1954, *Tensor Analysis for Physicists*, 2nd ed., Oxford Univ. Press, Oxford, UK, pp. 194–197 (reprinted by Dover).

Synge, J. L., 1936, "Tensorial Methods in Dynamics, *University of Toronto Press, Toronto*.

Vagner, V., 1941, "Geometrical Interpretation of the Motion of a Nonholonomic Dynamical System," *Trudi Seminara po Vektornomu i Tensornomu analizi*, V. 5.

Vranceanu, G., 1936, *Les espaces nonholonomes*, Gauthier-Villars, Paris.

Author's Closure[12]

1 I do not claim that there is anything mechanically new. What is mathematically new can be discussed. Some novelty is in the notation which makes it clear that the main difference between Lagrange's method for coordinate and for noncoordinate velocity components lies in the noncommutation of certain operators.

2 What Papastavridis says here is essentially that you cannot solve the equations of motion, if you do not have expressions for, or "know," the forces. But in this respect Lagrange's equations do not differ from Newton's. Neither $\dot{\mathbf{P}} = \mathbf{F}$ nor $(d/dt)(\partial T/\partial \dot{q}^a) - (\partial T/\partial q^a) = Q_a$, where $\mathbf{F} \cdot \boldsymbol{\tau}_a \equiv Q_a$, can be solved unless one knows $\mathbf{F}$ or Q_a, respectively.

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True, there is the constitutive relation (or assumption, or idealization) of "ideal" constraints which make the expressions for Q_a easy to get in terms of certain contributions (the "applied" forces) to the vector **F**. But, should the constraints be non-ideal, so that there is sliding (kinetic) friction; $F = \mu N$, where N is the normal force, and μ the coefficient of kinetic friction, then the Lagrange equations, $(d/dt)(\partial T/\partial \dot{q}^a) - (\partial T/\partial q^a) = Q_a$, are still valid and can, for example, help us calculate the sliding friction contribution to Q_a if we know the motion.

In this sense I still think it is wrong to say that there is a "Principle" that must be invoked to get Lagrange's equations. A constitutive relation is not a "principle" but a mathematical model of forces which sometimes holds, sometimes not. A thorough recent study of derivations of Lagrange's equations by Casey (1994) takes the same point of view as the present author as regards these matters.

3 No comment.

4 No comment.

5 My choice of wording may have been unfortunate here. Papastavridis' statement is correct.

Reference

Casey, J., 1994, "Geometrical Derivation of Lagrange's Equations for a System of Particles," *American Journal of Physics*, Vol. 62, pp. 836–847.

Analytical Solution of Hill's Equation[13]

R. T. Shield[14]. The author attempts to solve a nonlinear second-order p.d.e. for a function $u(x, y)$ by introducing a function $\phi(x, y)$. Equations (3) introduce ϕ by defining the derivatives ϕ_x, ϕ_y in terms of the unknown function u. However, in order for ϕ to exist, u must satisfy a compatibility equation obtained from (3) by setting $\partial\phi_x/\partial y$ equal to $\partial\phi_y/\partial x$, giving an additional nonlinear second-order p.d.e. for u which will not hold in general. Thus the method of solution is invalid.

Author's Closure[15]

One of the most difficult things in engineering discipline is to give an analytical solution to nonlinear differential equations encountered in science. However, on the other hand, the simplest thing is to criticize pre-established methods or solutions. I think Professor Shield prefers the second one. After the equation of motion is obtained, the establishment of a solution method becomes a pure mathematical problem. From the mathematical point of view, such transformations as those given by Eqs. (3) in the paper can well be introduced on the condition that they satisfy the equation of equilibrium and the boundary conditions. If there had been a compatibility equation as Professor Shield claimed but not showed, that would surely be Eq. (5), which is a different form of the equation of equilibrium. As can be followed from the references, the comparison of results of the present method with the numerical ones also show that the predictions of the theory are in excellent agreement with those given by numerical methods, without leaving any suspicion on the method. A second paper which is in the pipeline considers nonregular boundaries and also proves that the method gives excellent results and such claims cannot be regarded.

[13]By Y. Pala and published in the Dec. 1994 issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 61, pp. 1000–1001.

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