shear stress resultant) the minimum critical axial compressive stress for a homogeneous, isotropic, circular cylindrical shell is 43 percent of the classical critical stress.

4 It was not the purpose of the paper to study the many interesting results obtained by various research investigators with the aid of nonlinear stability theories. It is known that, based on the Koiter's general theory of initial postbuckling behavior [7, 8], axially compressed cylindrical shells are highly sensitive to imperfections in the sense that imperfections which are small, relative to the shell thickness, will result in a large reduction of the buckling load [9–12]. Accurate investigations of the buckling load require the knowledge of the initial imperfections of the unloaded shell, but, in general, such information is not available beforehand to the engineers, primarily because of the difficulties involved in measuring shell imperfections. However, information as to the relative imperfection sensitivity of various types of shells may be provided from Koiter's theory, thus establishing the validity of the theoretical analysis of the perfect shell configuration. Hence, it is not unexpected that the experimental buckling load of axially compressed cylindrical shells are lower than the theoretical results. The critical stress obtained by the authors for a homogeneous, isotropic, cylindrical shell is much closer to the actual test value compared to the critical stress given previously by the classical theory. If one accepts, as is generally accepted, that the actual buckling load for an axially compressed cylindrical shell is approximately one-third the critical load based on the classical theory, then our result of only 43 percent of the classical critical load is much closer to the actual test value than that of previous investigations.

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Axial Impact of Short Cylindrical Bars[1]

H. F. YUAN.[2] Using Hamilton's principle, the authors treated the problem of two identical bars impacting each other axially with equal but opposite velocities. Second-order corrections due to Love were considered. Assuming zero displacement at the impact end and the initial condition of constant velocity, explicit expressions for displacement, strain, and impact force were obtained.

The authors noted, however, that experimental results indicate a more gradual loading than that predicted by theory because of the presence of air trapped between colliding surfaces that also have microscopic deviation from perfect flatness. Therefore, from a practical standpoint, the zero displacement boundary condition at the impact end is not fulfilled. The actual problem then is not different from that of a bar subject to an impact loading at one end that starts off from zero. For this case, the approach of the paper does not seem to apply. Such a problem can, however, be conveniently handled by first assuming a series solution for the displacement as

$$u(x, t) = \phi_0(t) + \sum_{i=1}^{\infty} \phi_i(t) \cos \frac{i\pi x}{l} \tag{1}$$

By using the virtual displacement technique, Timoshenko[3] arrived at the following classical solution:

$$u(x, t) = \frac{t^2 P(t)}{2Al\gamma} - \frac{2}{\pi\gamma A c_0} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \cos \frac{i\pi x}{l} \times \int_0^t P(z) \sin\left[\frac{i\pi c_0}{l}(t - z)\right] dz \tag{2}$$

where $P(t)$ = impact load.

The writer noted that by using Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_i} - \frac{\partial L}{\partial \phi_i} = Q_i \tag{3}$$

with

$$\begin{aligned} L &= T - U \\ T &= \text{total kinetic energy} \\ U &= \text{total elastic potential energy} \\ Q_i &= P(t)(-1)^i = \text{virtual force} \end{aligned}$$

equation (2) can be derived in a more straightforward way.

When Love's radial inertia term is accounted the equation can be shown as

$$u(x, t) = \frac{1}{M} \sum_{i=1}^{\infty} (-1)^i \cos \frac{i\pi x}{l} \cdot \frac{1}{\eta_i^2 \Omega_i} \int_0^t P(z) \sin \Omega_i(t - z) dz + \frac{t^2 P(t)}{2Al\gamma} \tag{4}$$

with

$$\begin{aligned} M &= \gamma A l/2 \\ p_i &= i\pi c_0/l \\ \Omega_i &= p_i \eta_i \\ \eta_i &= \left[1 + \left(\frac{\sigma k \pi i}{l}\right)^2\right]^{1/2}. \end{aligned}$$

The corresponding strain expression becomes

$$e(x, t) = \frac{\partial u}{\partial x} = \frac{1}{M}\cdot\frac{\pi}{l} \sum_{i=1}^{\infty} (-1)^i \frac{i \sin \frac{i\pi x}{l}}{\eta_i^2 \Omega_i} \int_0^t P(z) \sin \Omega_i(t - z) dz \tag{5}$$

The displacement and strain can thus be calculated by integration once $P(t)$ is prescribed. When a large number of terms of the series is desired for convergence both the second-degree

[1] By H. D. Conway and M. Jakubowski, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 809–813.

[2] Research Structural Engineer, Army Waterways Experiment Station, Vicksburg, Miss.

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quadrature by Filon[4] and the first-degree formula, advanced by Yuan and McElhaney,[5] are accurate.

For experimental purposes the writer thinks it more useful to select the trial impact load $P(t)$ by measuring the strain and then solve the integral equation (5) rather than by choosing simple step or ramp loading as was done in the paper.

Employing the first-degree formula and giving due regard to the propagation speed of axial elastic wave, the numerical solution of the integral equation (5) has indeed been accomplished[5] by an inversion process to which the second-degree Filon's formula does not apply.

Authors' Closure

The authors thank Mr. Yuan for his discussion.

When the strain-time curves for a bar subjected to step loading are obtained using Love's equation, we found that the agreement is generally good except that the theory predicts shorter rise times and more accentuated wakes. In the belief that the discrepancies are mainly due to the fact that the experimental loading is more gradual, we substituted ramp for step loading in the theory and concluded that we were correct in our belief. Mr. Yuan's suggestion that we reverse the procedure and find the theoretical pressure based on the experimentally determined stress-time curves would be a far more complicated way of arriving at the same conclusion.

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Non-Darcy Flows Through Fibrous Porous Media[1]

G. EMANUEL[2] and J. P. JONES.[3] In their paper,[1] the authors contribute greatly to the fundamental understanding of a liquid flowing in porous media. They express the friction factor, f, as $f = R^{-1} + c$, and propose this as a general representation for a class of fibrous porous media. They also inquire whether similar correlations exist for other classes of flow in porous media. In this regard, an important type of flow is that of a gas flowing through a porous medium. Green and Duwez[4] studied the flow of nitrogen gas through sintered porous-metal samples, and indeed found that the foregoing correlation adequately represented their data. However, they explicitly omit compressibility from their analysis, even though some of their experiments involved large mass flow rates where compressibility ought to be important. Emanuel and Jones[5] theoretically studied compressible flow through a porous plate and showed that choking of the flow in the plate can occur and that the Dupuit-Forscheimer relation must be replaced by a compressible analog. The choking aspect has been verified experimentally by Shreeve.[6]

[1] By G. S. Beavers and E. M. Sparrow, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 711–714.

[2] Aerophysics Research Department, The Aerospace Corporation, El Segundo, Calif.

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Authors' Closure

The authors are grateful to Drs. Emanuel and Jones for providing further background on the problem and, in particular, for appending several relevant references.

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Transverse Vibration of a Viscoelastic Column With Initial Curvature Under Periodic Axial Load[1]

P. H. FRANCIS.[2] Professor Stevens has made a very useful contribution to the dynamic theory of thin columns by introducing, simultaneously, the effects of initial curvature and viscous material damping. The initial curvature implies a nonzero lateral response at all dynamic load levels, while the viscoelastic material properties renders this response frequency-dependent. The representation chosen for the viscoelastic modulus, $E(1 + i\eta)$, is particularly good in that the ensuing analysis is much more transparent than would be the case had an operator form been chosen. Also, this representation is likely to be more suitable in many practical applications. The excellent agreement between the theory and the experiments with plexiglas columns gives further veracity to the mathematical model.

The discusser would like to offer two comments which may have bearing on further extensions of this problem. The first is that subharmonic response has been found to be the dominant instability in many physical systems having small or negligible viscous losses and governed by equations of the Mathieu type. It would therefore be of interest (as alluded to by the author) to examine the response characteristics of the column in the neighborhood of $\omega \simeq 2\Omega$. This would enable one to judge whether his system parameters are such that subharmonic resonance is apt to be a problem.

The second comment deals with the form of the initial curvature, which is taken to be the fundamental eigenmode of the system. It would be useful now to examine the effect of a localized "imperfection," to assess its influence on the stability behavior. It would seem that local imperfections would characteristically influence bo h the amplitude response and the location of the stability regions in a weaker manner than is predicted by the present choice of initial curvature. If this is indeed so, the present analysis would represent, in a certain sense, "conservative" design practice.

J. C. WILEY.[3] Professor Stevens is to be complimented on a very lucid and detailed treatment of both analytical and experimental aspects of the problem. The experimental techniques are particularly interesting and can probably be used effectively in

[1] By K. K. Stevens, published in the December, 1969, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 36, TRANS. ASME, Vol. 91, Series E, pp. 814–818.

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