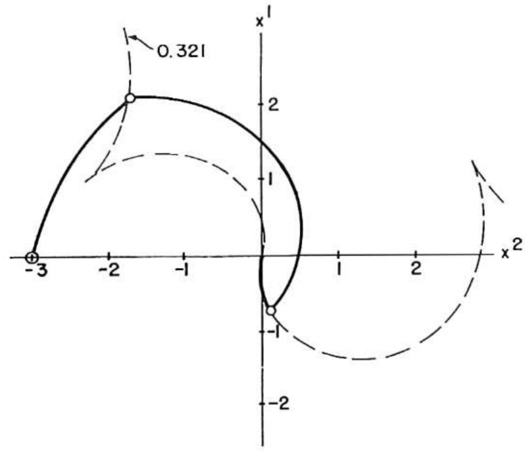

Fig. 5 Projection of switching curve in x^1x^2-plane for Example 5

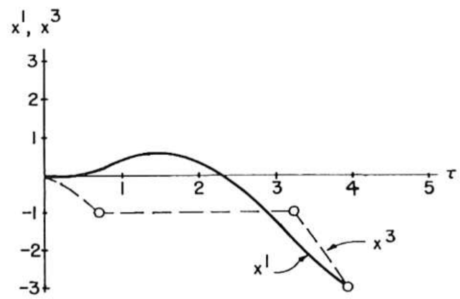

Fig. 6 Variables (x^1, x^3) versus reverse time τ for Example 5

Discussion of Limiting Cases, [see Equation (6)]:

(*a*) $\gamma = 0$, finite ζ. Here the periodic approximation approaches the exact optimum switching times for smaller values of ζ, and for small initial values in the phase space. Indeed the periodic approximation will yield the exact switching times, regardless of the values of the parameters γ and ζ, provided only two switchings are required to reach the origin.

(*b*) $\gamma \to \infty$, finite ζ. This is equivalent to $C' = 0$. Hence for large γ, the periodic approximation becomes more and more exact.

(*c*) For $\gamma = \zeta$, equation (6) becomes

$$u = \text{sgn}\,[C'' - \cos(\nu\tau + \delta')]$$

The periodic approximation is exact.

(*d*) For $\zeta \to 0$, the projection of switching curves into the x^1x^2-plane is composed of spirals which are practically circles. For large γ the function $C'e^{-\gamma\tau} \to 0$ for rather high τ-values, and the switching some distance from the origin will practically lie on the Bushaw curves[5] (see sketch, Fig. 7).

Conclusions and General Discussion

It should be mentioned that, in all of the examples described herein, the periodic approximation procedure yielded the same switching time as obtained using the exact form of equation (6) for the number of significant figures shown. The exact switching times were found using the complete form of equation (6), equation (5*c*), concurrently with the phase plane. One might expect noticeable variance between the periodic approximation and the exact switching times in problems where the initial values of the state variables are at a large distance from the origin. The periodic approximation procedure proved to be a very useful method for obtaining the optimum response of the full third-order system, particularly when equation (7) was used as the basis for the iteration procedure.

[5] D. Bushaw, "Optimal Discontinuous Forcing Terms," Annals of Mathematics Studies Number 41, Princeton University Press, Princeton, N. J., 1958.

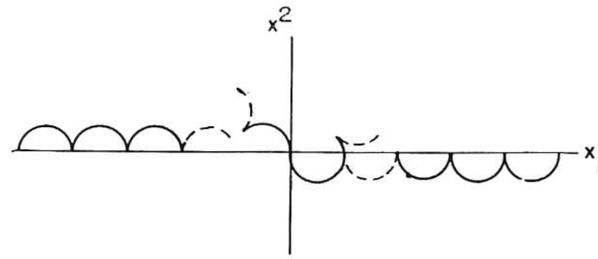

Fig. 7 Sketch of optimum switching curve in x^1x^2-plane for $\zeta \to 0$ and $\gamma \gg 0$

DISCUSSION

Hubert Halkin[6]

This paper is a very useful contribution to the numerical treatment of linear optimal control problems.

In the past few years extensive and successful work has been done on the theoretical aspects of linear optimal control (Bellman, Glicksberg, and Ross; Gamkrelidze; La Salle) but very few nontrivial examples have been completely solved by this method.

The solution of the two-boundaries problem resulting from these theoretical studies leads usually to many difficulties—the more dramatic one being the choice of the adjoint vector at the origin or equivalently of the quantities C' and δ' introduced in this paper. This two-boundaries problem is solved in this paper in a very useful and elegant way.

It is particularly interesting to compare the results obtained in this paper with the synthesizing procedures proposed by Neustadt[7] and the applications on analog computers of these ideas by Paiewonsky.[8]

These different methods complement each other very well: the procedures of solution proposed by Neustadt can be applied to systems of order higher than three, but they will always require an iteration scheme for which the ideas exposed in this paper will be particularly helpful.

Rufus Oldenburger[9]

This paper is a valuable contribution to the study of time optimum bang-bang control, where "optimum" is in the sense that the duration of the transient is to be minimized. The paper is in the area of the minimum time school of Bellman, LaSalle, Pontryagin, Krasovskii, and others. Their choice of the duration of the transient as the parameter to be minimized was dictated by mathematical rather than engineering considerations in that the mathematics of the problem of minimizing this parameter could be solved. This choice of a single measure of performance for nonlinear systems is somewhat analogous to the selection of the rms error by N. Wiener in his optimum theory for linear systems subject to random disturbances. Wiener chose this error in part because the statistical problems involving the minimization of this error could be solved mathematically, at least in a general sense. The choice of the parameter to be optimized is thus often a

[6] Assistant à l'Université de Liège, Belgium.

[7] L. W. Neustadt, "Synthesizing Time Optimal Control Systems," *Journal of Mathematical Analysis and Applications*, vol. 1, 1960, pp. 484–493.

[8] Private communication from B. Paiewonsky, ARAP, Princeton.

[9] Director, Automatic Control Center, Purdue University, West Lafayette, Ind. Mem. ASME.

matter of mathematical convenience rather than engineering necessity.

In 1944 I obtained[10] the optimum response of second-order systems subject to saturation when I was trying to determine whether or not the Woodward governors on Hamilton Standard propellers were designed to give best response. I was not trying to minimize time. An engineer usually must compromise a design to meet several requirements, and looks at many system parameters. The parameter with which I was most concerned was the maximum error in the controlled variable after the application of a sudden disturbance. It is fortunate that in the case of low-order systems the duration of the transient is minimized at the same time as the maximum error. This is an unexpected bonus. One also wishes other properties to be satisfied, such as minimum underswing, minimum area between the response curve and the time axis, etc. For low-order systems these properties are satisfied simultaneously. For higher order systems and certain initial conditions the same is generally, but not always, true. Thus the effort expended on time optimal control is justified, but it must be emphasized that in practice it is not enough to examine or minimize one parameter.

For normal steady-state operation, when the system to be controlled is subject to small disturbances, the controlled variable falls in a certain band about its equilibrium value. The practicing engineer is concerned with how long it will take, after a sudden large disturbance, for the controlled variable to enter and stay within this band. This is related to the time optimal problem.

In my paper[10] referred to above I solved the problem of optimum control of linear systems subject to a saturation limitation on the input, where optimum is in the engineering sense. I carried out the work for the systems that ordinarily occur in engineering practice. I found that, for engineering purposes, little error was introduced when I dropped the first-order term in the authors' Equation (1) for third-order systems. I did not investigate the effect of dropping the γe term in this equation. Whether or not this can be done in engineering applications is an open question.

Authors' Closure

The authors thank the reviewers for their interesting remarks. Only the last section of Dr. R. Oldenburger's discussion needs an answer.

The error introduced by dropping terms in the third-order equation can be evaluated by looking at the zero-trajectories of the different systems. The following figures illustrate the point.

In Fig. 8 the zero-trajectories are given in the e_1e_2 plane for a full third-order system and three possible approximations to it.

The output of the relay is either (+1) or (−1). This determines the scale of the figures. Naturally one can have more and less agreement depending on the values of γ and ζ which here are unity and zero, respectively.

In Fig. 9 projections of these trajectories into the e_2e_3 plane are shown. One can see clearly that only in a rather limited region are the zero-trajectories of the approximations close to the zero-trajectory of the original system.

In Fig. 10 projections of zero-trajectories for several other third-order systems are shown.

If the initial values are not too large and only one switching occurs between start and the reaching of the origin of the phase space, these curves will give an idea of how good the approximation of the optimum control will be, if the complete third-order system is replaced by simpler ones.

[10] R. Oldenburger, "Optimum Nonlinear Control," Trans. ASME, vol. 79, 1957, pp. 527–546.

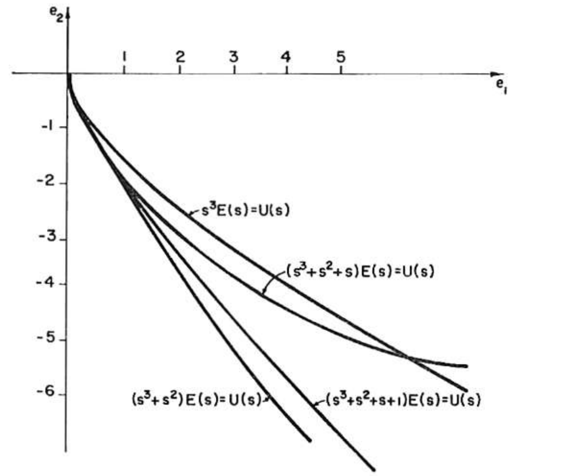

Fig. 8 Projections into the e_1e_2 plane of zero-trajectories of four systems

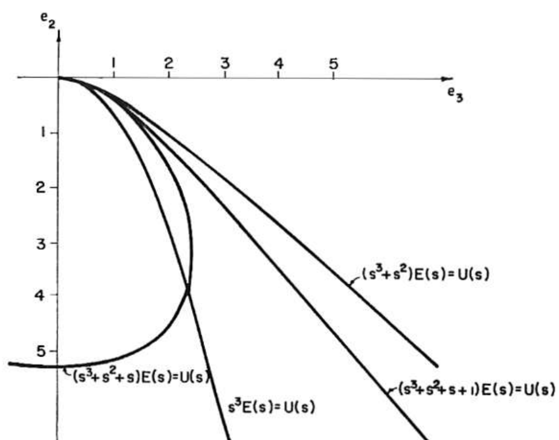

Fig. 9 Projections of the same trajectories into the e_2e_3 plane

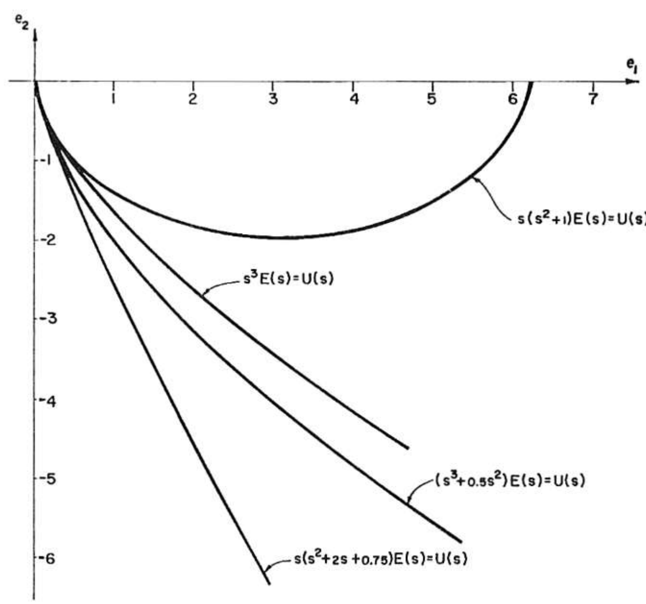

Fig. 10 Projections of trajectories of systems with other parameters