Making a transformation from the coordinates $(\bar{z}', s', \tau')$ to the coordinates $(\bar{z}, s, \tau)$, we can have the following relation:

$$\bar{z} = \bar{z}' - 2\Lambda(\sin\beta)\tau'; \quad s = s' + 2\Lambda(\cos\beta)\tau'$$

$$\tau = \tau'$$

The familiar chain rule gives:

$$\frac{\partial}{\partial \tau'} = \frac{\partial}{\partial \tau} - 2\Lambda(\sin\beta)\frac{\partial}{\partial \bar{z}} + 2\Lambda(\cos\beta)\frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial s'} = \frac{\partial}{\partial s}; \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial \bar{z}}$$

If these substitutions are made in equation (36), we obtain:

$$\frac{\partial}{\partial \tau}(\bar{H}\pi_0) + \frac{\partial}{\partial s}\left\{\Lambda(\cos\beta)\bar{H}\pi_0 - \overline{H^3}\pi_0\frac{\partial \pi_0}{\partial s}\right\}$$

$$-\frac{\partial}{\partial \bar{z}}\left\{\Lambda(\sin\beta)\left(2\bar{H} - \left(\frac{\overline{H^{-2}}}{\overline{H^{-3}}}\right)\right)\pi_0 + \left(\frac{1}{\overline{H^{-3}}}\right)\pi_0\frac{\partial \pi_0}{\partial \bar{z}}\right\} = 0$$

which is identical to equation (35) and, consequently, to equation (33).

However, if both opposing surfaces are rough, the term $(\overline{I_1/H^{-3}})$ will not vanish in any coordinate system, but it is possible to transform this term from one coordinate system to another one in a similar manner to that used above.

Practically, $(\overline{I_1/H^{-3}})$ can easily be calculated in terms of $\overline{H^n}$ if both surfaces have the same type of roughness distribution. Suppose that we have the roughness on the moving surface with RMS $= \sigma_m$ and on the stationary surface with RMS $= \sigma_s$. From equation (26), we have:

$$\left(\frac{\overline{I_1}}{\overline{H^{-3}}}\right) = \left\{\frac{1}{\overline{H^{-3}}}\lim\frac{1}{\zeta}\int_0^{\zeta} H^{-3}\delta_m d\zeta'\right\}$$

$$= \left\{\frac{1}{\overline{H^{-3}}}\lim\frac{1}{\zeta}\int_0^{\zeta} H^{-3}(H - H_0 - \delta_s)d\zeta'\right\}$$

$$= \frac{1}{\overline{H^{-3}}}\left\{\overline{H^{-2}} - \overline{HH^{-2}} - \lim\frac{1}{\zeta}\int_0^{\zeta} H^{-3}\delta_s d\zeta'\right\}$$

Since δ_s/σ_s has the same distribution function as δ_m/σ_m,

$$\lim\frac{1}{\zeta}\int_0^{\zeta}\frac{\delta_s}{H^3}d\zeta' = \frac{\sigma_s}{\sigma_m}I_1$$

Therefore,

$$\left(\frac{\overline{I_1}}{\overline{H^{-3}}}\right) = \frac{1}{1 + \frac{\sigma_s}{\sigma_m}}\left\{\left(\frac{\overline{H^{-2}}}{\overline{H^{-3}}}\right) - \bar{H}\right\} \qquad (37)$$

In special cases,

$$\left(\frac{\overline{I_1}}{\overline{H^{-3}}}\right) = 0 \quad \text{for} \quad \sigma_m = 0$$

$$\left(\frac{\overline{I_1}}{\overline{H^{-3}}}\right) = \left\{\left(\frac{\overline{H^{-2}}}{\overline{H^{-3}}}\right) - \bar{H}\right\} \quad \text{for} \quad \sigma_s = 0$$

As far as real roughness is concerned, it is physically realistic to presume that:

$$(\overline{H^n}) = \overline{H^n}$$

Of course, in the case of one-face roughness, the above relation holds for any type of roughness. Therefore, in most cases the temporal average in equation (25) need not be separately calculated. Consequently, we propose use of equation (25) or (24) with

$$\left(\frac{\overline{I_1}}{\overline{H^{-3}}}\right) = \frac{1}{\overline{H^{-3}}}\overline{H^{-3}\delta_m}$$

where, for usual δ_m, the result will, on the right-hand side, automatically be time independent.

Acknowledgment

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References

1 Tzeng, S. T., and Saibel, E., "Surface Roughness Effect on Slider Bearing Lubrication," *A.S.L.E. Trans.*, Vol. 10, 1967, pp. 334–338.
2 Christensen, H., and Tonder, K., "The Hydrodynamic Lubrication of Rough Surfaces," Journal of Lubrication Technology, Trans. ASME, Series F, Vol. 93, No. 3, July 1971, pp. 324–330.
3 Christensen, H., "Stochastic Models for Hydrodynamic Lubrication of Rough Surfaces," *Proc. Inst. Mech. Engrs.*, Vol. 184, Part 1, 1969–1970, pp. 1013–1025.
4 Elrod, H. G., "Lubrication Theory for Newtonian Fluids With Striated Roughness or Grooving," Report #21, Lub. Research Lab., Columbia University, New York, N. Y., 1972.
5 Berthe, D., and Godet, M., "*Mecanique des Fluides.*—Équation de l'écoulement laminaire entre deux parois rapprochées en mouvement relatif," Comptes Rendus Acad. Sc. t. 272, Series A, April 14, 1971, pp. 1010–1013.
6 Vohr, J. H., and Chow, C. Y., "Characteristics of Herringbone-Grooved, Gas-Lubricated Journal Bearings," MTI 64TR15, 1964.

DISCUSSION

D. Berthe[2]

I was very much interested by the analysis presented by the authors. I completely agree with their point of view that some reservations must be made about the location of the roughness either on the stationary or on the moving surface. As we have shown [5], it is necessary to introduce separately the roughness δ_1, δ_2 and the velocities U_1, U_2 of both contacting surfaces. However, we think that equation (25) cannot be considered to be a global-type Reynolds equation for two-sided striated roughness because of the assumptions made in deriving equation (1), particularly when one surface is taken as a reference plane; this assumption obviously is not acceptable when the two surfaces are rough. In the case of two infinitely long cylinders with transverse roughness, the pressure gradient at a time t is given by [5]

$$\frac{dp}{dx} = 6\mu(U_1 + U_2)\frac{h + \lambda(\delta_2 - \delta_1) - h^*}{(h + \delta_2 + \delta_1)^3}$$

where h and $(h + \delta_2 + \delta_1)$ are respectively the mean and the actual film thickness and $\lambda = (U_1 - U_2)/(U_1 + U_2)$ is the slide/roll ratio. For a plain slider using the method described in [5], one obtains

$$\frac{dp}{dx} = 6\mu(U_1 - U_2)\frac{h + \delta_2 - \delta_1 - h^*}{(h + \delta_2 + \delta_1)^3}$$

In these equations, terms δ_1, δ_2 can represent the roughness or the shape defects of the two surfaces; they can be given either by a point-by-point method (from metrology data) by an analytical function or they can be random.

For the particular case of a random height of asperities, taking

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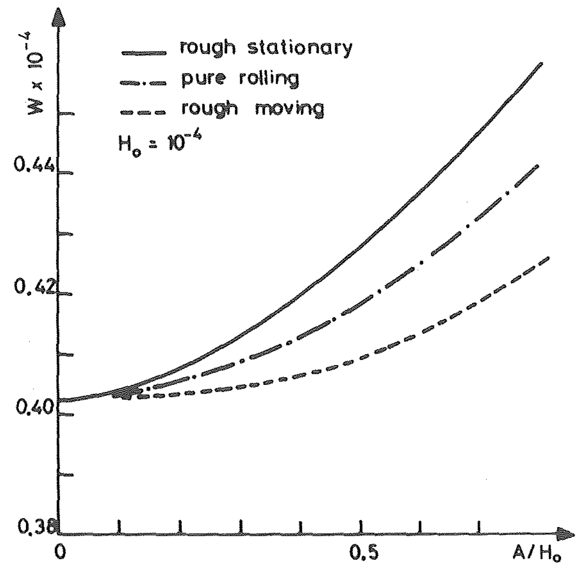
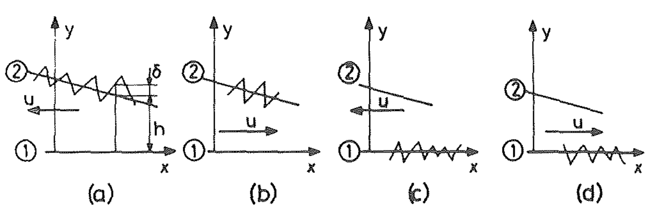


Fig. 3 Load capacity versus roughness ratio (transverse roughness) for two rotating cylinders



$$\frac{dp}{dx} = 6\,\mu u \frac{h + \delta - h^*}{(h+\delta)^3} \qquad \frac{dp}{dx} = 6\,\mu u \frac{h - \delta - h^*}{(h+\delta)^3}$$

Fig. 4 The hydrodynamics of a slider

on average value of pressure gradient by the method previously described by Christensen [3] for transverse roughness, integrating twice one obtains the load-carrying capacity.

For two rigid cylinders (one rough, the other smooth) assuming a roughness distribution given by $f(\delta) = \alpha\,(\delta_{max}^2 - \delta^2)^2$, Fig. 3 shows the variation of nondimensional load-carrying capacity versus roughness ratio δ_{max}/h_0, where h_0 is the mean minimum film thickness. With the same assumptions, for the plane slider studied by the authors, equation (2) shows that we must distinguish four cases (Fig. 2). In the cases (a) and (c) one could expect to find the results shown by the authors in Fig. 4, but for the cases (b) and (d) one could expect to find opposite results, i.e., the greater load will be given when roughness is moving.

For an elastohydrodynamic contact, using the Grubin approach, assuming rigid asperities and isothermal conditions, the mean minimum film thickness for the rough contact h_R can be related to the minimum film thickness calculated for smooth surface h_S by the relation

$$\frac{h_R}{R} = \frac{h_S}{R}\left|1 + (1.17 - 3.55\lambda)\frac{\sigma^2}{h_S^2}\right|$$

where R is the reduced radius of curvature and σ^2 is the standard deviation.

L. S. H. Chow and H. S. Cheng[3]

The authors are to be congratulated for developing a method to determine the effect of surface roughness on the bearing load-carrying capacity. The discussers are currently studying the effect of surface roughness on lubrication in elastohydrodynamic contacts, using the stochastic model proposed by Christensen [3]. In the interest of comparing the authors' approach with the stochastic approach, the discussers have extended Christensen's results to moving rough surfaces.

The Reynolds equation for a purely transverse-roughness surface is

$$\frac{\partial}{\partial X}(H_T{}^3 \frac{\partial P}{\partial X}) = \Lambda(\frac{\partial H_T}{\partial X} + \frac{\partial H_T}{\partial \tau}) \tag{1}$$

where $H_T = h_T/C$; $h_T = h + \delta_1 + \delta_2$; $H = h/C$; h is the average nominal film thickness, which is assumed to be independent of time; δ_1, δ_2 are the roughness parts beyond the nominal level as shown in Fig. 1; $\bar{\delta}_1 = \delta_1/C$; $\bar{\delta}_2 = \delta_2/C$; $X = x/l$; $P = p/p_a$; $\tau = u_1 t/l$; $\Lambda = 6\mu u_1 l^2/p_a C^2$; subscript 1 refers to the moving surface and subscript 2 refers to the inclined surface.

Equation (1) can be simplified to

$$\frac{\partial}{\partial X}[H_T{}^3 \frac{\partial P}{\partial X} + \Lambda(\bar{\delta}_1 - \bar{\delta}_2)] = \Lambda \frac{\partial H}{\partial X} \tag{2}$$

As the right-hand side of equation (2) is a non-random variable, the left-hand side will be a non-random quantity. Using this condition and taking expected values on both sides of equation (2) and simplifying, one obtains

$$\frac{\partial}{\partial X}[\frac{\partial \bar{P}}{\partial X} \frac{1}{E(\frac{1}{H_T{}^3})}] = \Lambda\{\frac{\partial H}{\partial X} - \frac{\partial}{\partial X}[\frac{E(\frac{\bar{\delta}_1 - \bar{\delta}_2}{H_T{}^3})}{E(\frac{1}{H_T{}^3})}]\} \tag{3}$$

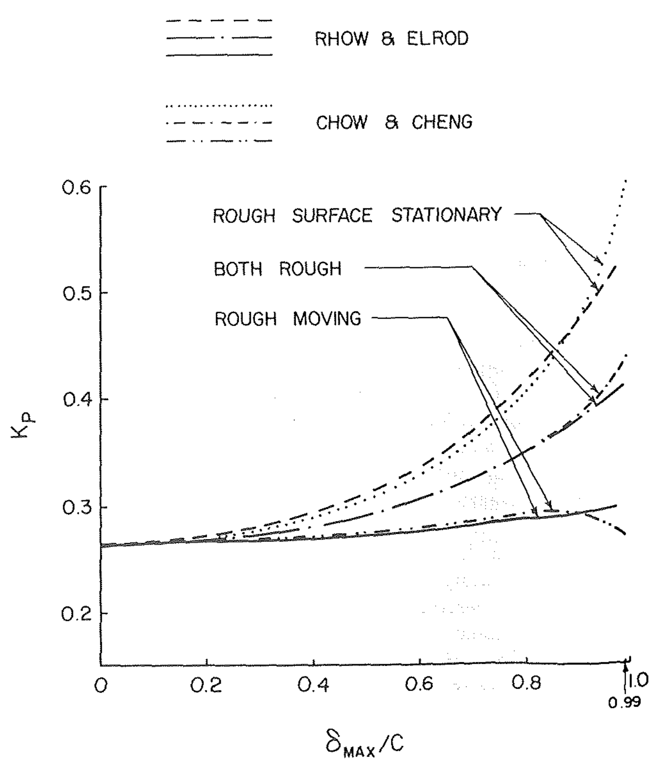


Fig. 5

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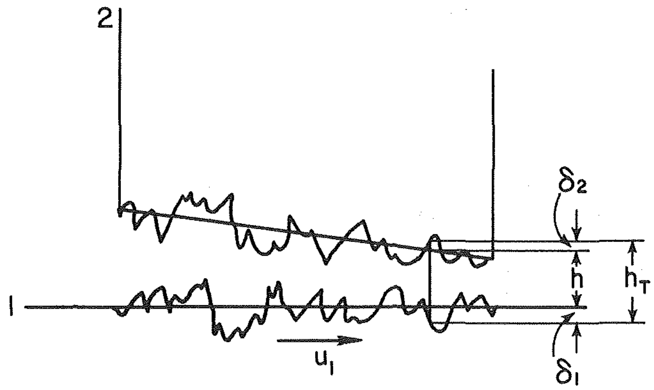

Fig. 6

where $\bar{P}$ is the expected or mean value of P

$$E(\ ) = \int_{-\infty}^{\infty} (\ ) f(\bar{\delta}) d\bar{\delta}$$

$f(\bar{\delta})$ is the probability density function of the random variable $\bar{\delta}$.

With boundary conditions $\bar{P} = 0$ at $X = 0$ and $X = 1$, $\bar{P}$ is solved numerically, and K_p, which is equal to

$$\overline{W}/\Lambda = \int_0^1 \bar{P} dX \tag{4}$$

is obtained for the following three cases: (1) rough surface stationary; (2) both surfaces with equal roughness distribution; (3) rough surface moving (Fig. 5).

A polynomial distribution function [3], which is closely equal to the Gaussian distribution, is chosen as

$$f(\bar{\delta}) = \begin{cases} \frac{35}{96}\left[1 - \frac{1}{9}\left(\frac{\bar{\delta}}{\bar{\sigma}}\right)^2\right]^3 & -3\bar{\sigma} < \bar{\delta} < 3\bar{\sigma} \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

It is shown in Fig. 6 that when the roughness distribution is defined as in equation (5), the discussers' results agree quite well with those obtained by the authors. The only exception occurs when $\delta_{max}/C > 0.9$. In this situation and in the case of rough surface moving, the load-carrying capacity obtained by the authors begins to decrease, while the authors' results are still increasing.

Robert A. Thompson[4]

The authors have presented a theory for the film pressure in bearings where the surface roughness wave length is small compared to the overall linear dimension of the lubricating film. Physically, this type of lubricant film fits the situation existing in journal bearings.

The main argument of the authors' mathematical development is that the load-carrying capacity of a bearing is dependent upon whether a given roughness is distributed on its moving or stationary element. This argument is contrary to intuition, at least for the case of journal bearings. For example, consider a rough shaft running in a perfectly smooth journal. This situation is represented by the solid plot on the authors' Fig. 2. Now suppose the entire bearing system and the machinery to which it is affixed is rotated in a direction opposite to that of the shaft but at a speed equal to the shaft speed. In this case, to an observer in a stationary frame of reference, the shaft would appear to be stationary with the bearing rotating around it. Consequently, the roughness would be on the stationary member, and according to Fig. 2 the load-carrying capacity of the bearing would be increased.

Could the authors clarify this paradox? Is it the result of centrifugal effects or has this writer misinterpreted their results?

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(AUTHORS' CLOSURE APPEARS ON P. 640)

(PAPER CONTINUED FROM P. 560)

Authors' Closure

The authors are grateful to the discussers for their interest and comments, and for this opportunity to clarify certain aspects of the effects of roughness.

We do not agree with M. Berthe's reservations concerning equation (1). Integration across the film of the velocity expressions (equations (2)) in reference [5], for example, yields the following well-known expression for the lineal mass flux in a bearing film:

$$\dot{\mathbf{m}} = \rho h \frac{(\mathbf{U}_1 + \mathbf{U}_2)}{2} - \frac{h^3}{12\mu}\rho\nabla p$$

Mass continuity applied to a little pill-box capped by the two opposing bearing surfaces then yields:

$$\frac{\partial \rho h}{\partial t} + \nabla \cdot \mathbf{m} = 0$$

Or:

$$\frac{\partial(\rho h)}{\partial t} + \frac{(\mathbf{U}_1 - \mathbf{U}_2)}{2} \cdot \nabla(\rho h) = \nabla \cdot \frac{h^3}{12\mu}\rho\nabla p$$

In a coordinate system moving with velocity $\mathbf{U}_2$, this last equation becomes:

$$\frac{\partial(\rho h)}{\partial t} + \frac{(\mathbf{U}_1 - \mathbf{U}_2)}{2} \cdot \nabla(\rho h) = \nabla \cdot \frac{h^3}{12\mu}\rho\nabla p$$

Equation (1) of this paper results when U is identified as the magnitude of the relative velocity $\mathbf{U}_1 - \mathbf{U}_2$ and "x" is measured in that velocity's direction.

An equation showing the individual effects of surface velocity and roughness can be obtained from equations (24) or (25). Following the analysis in the last section of the paper, it is easy to show that:

$$\left(\frac{\overline{H^{-2}}}{\underline{\overline{H^{-3}}}} - \underline{\overline{H}}\right) = \left(\frac{I_1}{\underline{\overline{H^{-3}}}}\right) + \left(\frac{I_2}{\underline{\overline{H^{-3}}}}\right)$$

Substitute this result into equation (25) and let $n = \hat{z}$. The result is:

$$\frac{\partial}{\partial \tau}(\underline{\overline{H}}\pi_0) + \frac{\partial}{\partial s}(\Lambda_s \underline{\overline{H}}\pi_0 - \underline{\overline{H^3}}\pi_0\frac{\partial \pi_0}{\partial s}) + \frac{\partial}{\partial n}(\Lambda_n \underline{\overline{H}}\pi_0 - (\frac{1}{\underline{\overline{H^{-3}}}})\pi_0\frac{\partial \pi_0}{\partial n}) + \frac{\partial}{\partial n}\{\Lambda_n[(\frac{I_2}{\underline{\overline{H^{-3}}}}) - (\frac{I_1}{\underline{\overline{H^{-3}}}})]\pi_0\} = 0$$

Here Λ_s and Λ_n are the components of the dimensionless relative surface velocity along the grooving and perpendicular to the grooving, respectively. Since the divergence of the relative velocity field vanishes, this last equation can be rearranged to become:

$$\frac{\partial}{\partial \tau}(\underline{\overline{H}}\pi_0) + (\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2) \cdot \nabla(\underline{\overline{H}}\pi_0) + \frac{\partial}{\partial n}\Lambda_n\{(\frac{I_2}{\underline{\overline{H^{-3}}}}) - (\frac{I_1}{\underline{\overline{H^{-3}}}})\}\pi_0 = \frac{\partial}{\partial s}\underline{\overline{H^3}}\pi_0\frac{\partial \pi_0}{\partial s} + \frac{\partial}{\partial n}(\frac{1}{\underline{\overline{H^{-3}}}})\pi_0\frac{\partial \pi_0}{\partial n}$$

Finally, transferral to a coordinate system unattached to surface #2 gives:

$$\frac{\partial}{\partial \tau}(\underline{\overline{H}}\pi_0) + (\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \cdot \nabla(\underline{\overline{H}}\pi_0) + \frac{\partial}{\partial n}(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2) \cdot \hat{n}\{(\frac{I_2}{\underline{\overline{H^{-3}}}}) - (\frac{I_1}{\underline{\overline{H^{-3}}}})\}\pi_0 = \frac{\partial}{\partial s}\underline{\overline{H^3}}\pi_0\frac{\partial \pi_0}{\partial s} + \frac{\partial}{\partial n}(\frac{1}{\underline{\overline{H^{-3}}}})\pi_0\frac{\partial \pi_0}{\partial n}$$

The above form of the global Reynolds equation has much to recommend it. It is symmetrical in surface designation, and independent of the senses ascribed to the coordinates "s" and "n" so *long as they are parallel and perpendicular to the grooving, respectively*. It simplifies to the averaged forms of the first two equations given by M. Berthe in the special case of steady, one-dimensional incompressible flow with transverse striations.

Mr. Thompson's discussion, and a portion of M. Berthe's, result from the authors' failure to state clearly the meanings of "stationary" and "moving" in connection with Fig. 2. Here the term "stationary surface" was intended for that surface from which an observer would see no change in gross film thickness. According to this view, all the upper surfaces in cases *(a)* – *(d)* of Fig. 4 would be regarded as stationary, and all lower surfaces moving. On reflection, we believe M. Berthe will agree that the load-carrying capacity of case *(d)* is less than that of case *(b)*.

The comparison offered by Drs. Chow and Cheng is gratifying, since, in view of the slightly different distribution functions employed, the agreement is all that could be expected. Although conceding that various approaches to the roughness problem may each make their own contribution to total understanding, the authors do have some questions concerning the logic underlying the stochastic method of analysis. First, although the expected values appearing in equation (3) are presumably obtained by averaging over many replications of a bearing, the probability density functions will, no doubt, be inferred from the *spatial* topography of one, or a few, bearings. Second, in order to obtain eq. (3) from equation (2) it is necessary to assume that the bearing through-flow is a non-random variable. Such indeed would be the case for short-wavelength, but not long wavelength, roughness. Inasmuch as the spatial character of the roughness, both with respect to amplitude and wavelength, must be invoked in any case, we believe that our multiple-scale analysis, which exploits directly these characteristics, has some advantage over a stochastic analysis in both rigor and generality (though not in brevity!).