Three-Dimensional Elasticity Solution for the Buckling of Transversely Isotropic Rods: The Euler Load Revisited

K. T. Chau. The author considered the interesting problem of the buckling of transversely isotropic rods using three-dimensional elasticity, and compared his result with the Euler's buckling formula and with two improved formulae including the effect of shearing force proposed by Timoshenko and Gere (1961). The simple formula proposed by the author for the buckling load of isotropic rods, I believe, provides an improvement over the existing formulae.

Actually, the use of three-dimensional elasticity to examine the buckling load for short columns is not a new idea. The author seems not aware of a couple of papers on nearly the same topic published in June 1993 and June 1995 (Chau, 1993, 1995). The omission of the first of these papers leads to the question of whether or not a literature search was done properly by the author. In fact, the author also published one paper (Kardomateas, 1993) in the June 1993 issue of the ASME Journal of Applied Mechanics in which Chau's (1993) paper appeared. (Note that a complimentary issue is sent to all the contributing authors in the same issue by JAM.)

In the 1993 paper, Chau solved the antisymmetric modes of bifurcations in a compressible pressure-sensitive circular solid cylinder under both axisymmetric tension and compression. The material response of the cylinder is modeled by a physical model proposed by Rudnicki (1977). If the solid cylinder is not pressure-sensitive, the six material parameters proposed by Rudnicki (1977) reduces to five, and the conventional constitutive form of transversely isotropic elastic solids is recovered as a special case. The axisymmetric loading considered by Chau (1993) also incorporates the effect of confining stress on the buckling problem (i.e., when compression is considered). Thus, simple compression, which is studied by the author, can be considered as a special case of the more general problem investigated by Chau (1993), obtained by simply setting the confining stress zero and the material being pressure-insensitive. Recently, the results of Chau (1993) were further generalized to transversely isotropic solid cylinders governed by a constitutive law similar to Eq. (8) given by the author (Chau, 1995).

Therefore, the bifurcation analysis by Chau (1993, 1995) provides a more general solution compared to those obtained by the author. But, it also embraces the problem considered by the author as a special case. Although the main emphasis of Chau (1993) is to study the effect of geometric bifurcation modes, which is induced by the free boundary of the cylinders, on the localization of deformation in rock-like materials, Chau (1993) did consider the Euler's buckling as a special case.

In particular, after obtaining the general eigenvalue equations for the elliptic complex and elliptic imaginary sub-regimes (Eqs. (6.11) and (6.16) in Chau, 1993), Chau (1993) continued to consider the so-called long wavelength limit (i.e. $\gamma = m \pi a / L \to 0$, where $m$ is an integer, $a$ the radius of the solid cylinder, and $L$ the length of the cylinder). It was shown analytically that the Euler's buckling formula corresponds to the first term in the series expansion of $\gamma$ when the bifurcation stress is expanded in terms of $\gamma$ (see Eq. (6.21) of Chau, 1993). More importantly, the conclusion applies even if the solid cylinders are pressure-sensitive and transversely isotropic; whereas the author is only able to show this for the isotropic limit. Although the full details in obtaining the Euler's formula were not reported by Chau (1993), the specialization involves a fairly delicate process of algebraic manipulation. Indeed, if one is interested in obtaining an approximate buckling formula for short columns, we can do so by simply retaining the higher order terms in the expansion shown in Eq. (6.18) of Chau (1993).

The main difference between the author's paper and Chau's (1993, 1995) papers is the method of solution. The author followed mainly the method of solution proposed by Elliott (1948) while the general theory proposed by Hu (1954) and used by Chen (1966) was modified and adopted by Chau (1993, 1995). The author also noted that "in general, the roots $s_1$ and $s_2$ (for Eq. (14b)) are either both real or complex conjugates." This classification on the nature of the roots of $s$ is rather incomplete; the roots for $s$, actually, can also be partly real or purely imaginary. And the physical meaning for $s$ being real or complex conjugates is not clear. On the contrary, by adopting Hu's (1956) approach Chau (1993) arrived at a parameter $\nu_p$, which bears the same significance of $s$ in the author's analysis, and the root-type of $\nu_p$ does carry a clear physical meaning (see Eq. (5.4) of Chau, 1993). That is, if a real root of $\nu_p$ exists, the shear band bifurcation or localization of deformation also becomes possible. For the case of no real root, $\nu_p$ can further be either complex conjugate pairs or purely imaginary. Chau (1993) showed analytically that the above conclusion on the validity of Euler's buckling formula as $\gamma \to 0$ applies for both cases. In particular, Euler's was recovered as the long wavelength limit for all the elliptic complex (all roots of $\nu_p$ are complex conjugates), elliptic imaginary (all roots of $\nu_p$ are purely imaginary), and hyperbolic (all roots of $\nu_p$ are real) regimes in the parameter space.

References


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an explanation for this apparent discrepancy would be welcomed.

The author has not provided any physical explanations as to why his Eq. (35) is more accurate than either the Engesser or the Haringx prediction. It is conjectured here that the difference is due to the inclusion of axial flexibility in the author’s analysis, while it was tacitly omitted in Engesser’s work and in the particular version of the Haringx work quoted by Timoshenko and Gere (1961).

References
Engesser, F., 1891, “Die Knickfestigkeit gerader Stabe,” Zentralblatt de

Author’s Closure

The author would like to thank Professor Chau for drawing his attention to his recent papers (Chau 1993, 1995) on the buckling of short columns. These papers were omitted from the reference list unintentionally since they are too recent and very close to the dates of preparation and submission of the present paper to be found in systematic literature searches at the time the manuscript was authored. Professor Chau’s work, which follows a different method of solution, complements the present paper by including compressible pressure-sensitive material and providing an alternative formulation. The discussion on the roots $s_1$ and $s_2$ is also welcome.

In closing, the author would like to emphasize a few of the main objectives and unique contributions of the present paper such as the derivation of a new, simple, improved column buckling formula, which provides an additional term to the Euler buckling load, and the comparison of the three-dimensional elasticity results with the Timoshenko/Engesser/Haringx transverse shear correction column buckling formulas.

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C. W. Bert. Professor Kardomeas’s paper is a welcome addition to the literature of the subject. The two Timoshenko formulas cited in the paper by reference to Timoshenko and Gere (1961) are in fact the well-known Engesser formula (Engesser, 1891) and the Haringx formula (Haringx, 1948), referred to by Timoshenko and Gere as the “modified” formula.

For the case of isotropic materials, there has been extensive criticism of the Haringx formula; see Nanni (1971), Ziegler (1982), and Reissner (1982). In particular, in his extensive three-dimensional elasticity analysis, Nanni concluded that the Haringx analysis is less accurate than the Engesser one. This seems to be in direct contradiction to the conclusion of Kardomeas;

Author’s Closure

The author would like to thank Professor Bert for his gracious comments regarding the paper under discussion. The formulas quoted in the paper as Timoshenko’s (1961) first and second shear correction formulas are indeed the formulas of Engesser (1891) and Haringx (1948) as was pointed out correctly by Professor Bert (Haringx obtained the formula in connection with helical springs and Timoshenko applied Haringx’s approach to bars).

Regarding Nanni’s three-dimensional elasticity approach, a review of his paper (written in German) indicates that Nanni’s (1971) treatment of the problem was based on a stress function approach and an asymptotic scheme with respect to the dimensions of the cross section (e.g., thickness); in this scheme, stresses were expanded in a polynomial series with coefficients that were subsequently determined through compatibility. Therefore, his conclusions (that Engesser’s formula is more accurate than the Haringx analysis) may be influenced by the fact that a certain number of terms were retained (i.e., Nanni’s results can be considered approximations); we may conjecture, therefore, that this conclusion may not hold if more terms in the asymptotic expansion were to be retained.

Instead, our three-dimensional elasticity results for the critical load are derived by numerically solving the determinant of the system of Eqs. (28) (and do not depend on an asymptotic scheme); therefore they can be considered exact. Our conclusion that the Haringx formula is more accurate than Engesser’s was also the feeling of Timoshenko as expressed in Timoshenko and Gere (1961). Nevertheless, it should be noted that in most cases of practical interest, the difference between Engesser or Haringx formulas is small.

Regarding Professor Bert’s conjecture as to why the author’s Eq. (35) is more accurate than either the Engesser or the Haringx formulas, the author would like to thank Professor Bert for offering this reasoning, which seems, indeed, to be a plausible physical explanation.

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