Synergetic Approach to the Phenomena of Mode-Locking in Nonlinear Systems

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The computer experiments are carried out on the phenomena of self-synchronization in a many-mode system described by the van der Pol type equation. The results are successfully explained in terms of the perturbation theory based on a mean field approximation proposed in a previous article. The extension of the theory to further complicated phenomena of mode-locking is briefly discussed.

§ 1. Introduction

Generally speaking, the oscillation of macrovariable originates from the breakdown of Onsager's reciprocity in the non-equilibrium open system. Living system is a typical example of the open systems. Thus a variety of oscillating phenomena can be seen in the biological processes, and many of them seem to be synchronous on the functional order of organism. The synchronized rhythm in living bodies is characterized by two factors. One is the externally driven oscillation (or forced synchronization) such as circadian rhythm. This type of rhythmical processes is studied in various fields of biology. The other is the internal synchronization caused by the oscillating chemical reaction, which is not studied theoretically enough though such self-synchronizations are observed frequently in biochemical processes.

The dynamical models describing biological processes are classified into two groups; the conservative and dissipative (or non-conservative) dynamics. The population variation, morphogenetic processes and neural activity are analysed by the use of the conservative dynamical models. In particular, the statistical theory of many-body system described by the conservative dynamics was proposed by Kerner in analogy with Gibbs ensemble in statistical mechanics. The phenomenon of synchronization has not yet been explicitly lightened for the conservative dynamical system, but the phenomena concerning the instability in such conservative systems are recently studied in connection with the problems of ergodic hypothesis, and the result assures the adequacy of the ensemble theory by Kerner. On the other hand, the latter case of the dissipative dynamics with many degrees of freedom was not studied till quite recently in disregard of the enormous number of dissipative examples observed in biological systems. A typical equation describing the
dissipative dynamics is so-called van der Pol equation. The same type of equation is used as the biological model of the heart beat and of the repetitive excitation of nerve membrane.\textsuperscript{3,0} The basic equation was

$$\frac{d^2X}{dt^2} = -\omega^2 X + 2\lambda (1 - 4X^2) \frac{dX}{dt},$$

where $\lambda$ is a constant, $\omega$ a frequency and $X$ a variable sub-ordinated to the membrane potential. The remarkable characteristic of the above equation is the competition of two effects; i.e., the negative resistivity and the suppression caused by nonlinear term, which leads to the occurrence of the oscillating orbit $X(t)$ with a limit cycle. The many-body system represented by the following coupled equation:

$$\frac{d^2X_n}{dt^2} + \omega_n^2 X_n - 2\lambda (1 - 4X_n^2) \frac{dX_n}{dt} = 2\lambda \varepsilon F_n(\{X_k, \dot{X}_k\}),$$

$$n=1, 2, \cdots, N$$

reveals the phenomena of self-synchronization under an appropriate interaction represented by $\varepsilon F_n(\{X_k, \dot{X}_k\})$. Here $\dot{X}_k$ is the time derivative of $X_k$. In the practical problems $X_k$ denotes such macrovariables as membrane potential and the concentration of chemical component in the system under consideration. The heart beat, the oscillating protoplasmic flow of a mycetozoa\textsuperscript{9}, and the spectrum of brain waves\textsuperscript{8} are understood in terms of the self-entrainment of the coupled many modes. In these phenomena, an essentially significant point lies on the fact that these systems are constituted by a great number of the unit system, such as a heart muscle, an actomyosin filament and a neural circuit.

Oscillation in nonlinear systems with many degrees of freedom has been extensively studied in various fields of physical science. In spite of great success of perturbational approach to the nonlinear phenomena, many unsolved problems remain in the region of far from linear branch. The self-synchronization or self-entrainment is one of the most interesting unsolved phenomena observed in the many-body dissipative system with nonlinear feedbacks. When $\lambda$ is small, by means of the rotational wave approximation\textsuperscript{9} Eq. (2) is transformed to give

$$\frac{dW_n}{d\tau} = i\lambda W_n + W_n (1 - |W_n|^2) + \varepsilon F_n(W_1, W_2, \cdots, W_N),$$

$$n=1, 2, \cdots, N$$

where $W_n e^{i\omega_n \tau} = \sqrt{2} \{ (\dot{X}_k / \omega_n) + iX_k \}$ is a state variable with complex value. In the derivation of Eq. (3) we have assumed $\omega_n = \omega + \lambda \omega_n$ and $\tau = \lambda t$. The above equation has transparent structure compared with Eq. (2), thereto the same qualitative characters as Eq. (2) are retained in Eq. (3). In the present paper we are concerned with the system described by Eq. (3). Since it is not easy to get exact solution of orbits in nonlinear system in general, the so-called “synergetic” approach is employed. Synergetic method means a cooperation of computer simulation and
mathematical analysis of the results. This method is shown to be useful both in practical and in basic problems in physics.10

The basic idea and results of the perturbation theory proposed in the previous paper11 are stated briefly in § 2. Details of calculation are described in Appendices I and II. The results of the computer simulations are discussed in § 3, and are compared with the results of the perturbation theory. Section 4 is devoted to the illustration of the numerical experiments, and Section 5 is for discussion.

§ 2. Perturbation theory of self-synchronization

We study the self-entrainment of the coupled modes or coupled oscillator systems described by the following generalized equation of Eq. (3):

\[ \frac{dW_n}{dt} = i\omega_n W_n + P_n(W_n) + \varepsilon F_n(W_1, W_2, \ldots, W_N), \quad (n=1, 2, \ldots, N) \]  

(4)

where \( F_n \) is the interaction between the \( n \)-th mode and others, and \( P_n \) is the nonlinear term. Here we assume that \( P_n \) satisfies the relation \( P_n(1) = 0 \). This limitation on \( P_n \) is not essential but conventional in our theory and is discarded if necessary. \( W_n \) and \( \omega_n \) are the state variable and the native frequency of the \( n \)-th mode respectively, and \( \varepsilon \) is a positive constant. For the sake of brevity, \( F_n(W_1, W_2, \ldots, W_N) \) is abbreviated as \( F_n(\{W_k\}) \), and \( (n=1, 2, \ldots, N) \) is abridged throughout the subsequent arguments. \( W_n \) is a complex value, but \( \omega_n \) is real. In this paper except for the general discussion in Appendix I, we limit our discussion to the following case:

\[ P_n(W_n) = W_n(1 - |W_n|^2) \]  

(5)

and \( F_n(\{W_k\}) \) is assumed to be a linear function of \( W_k \) and satisfies the following constraint: \( F_n(\{W_0\}) = 0 \), where \( W_0 \) is an arbitrary complex value. These limitations are not essential for the theory in this paper, and these points are discussed in § 5.

When the distribution function of the native frequency, \( f(\omega) \), is approximated by the \( \delta \)-function with the peak at \( \omega_0 \), i.e., \( f(\omega) = \delta(\omega - \omega_0) \), the exact synchronized solution is obtained as follows:

\[ W_n = e^{i\omega_0 t + \phi_0}. \]  

(6)

Here \( \phi_0 \) is a constant phase. Though the state represented by Eq. (6) is not the synchronized solution any more in the case where the distribution deviates from the \( \delta \)-function, the another synchronized solution \( W_0^* \) can be seen near the solution (6) provided that the attractor (or stable solution) given by Eq. (6) is strong enough. On the assumption that the distance between these two solutions (\( W_0 \) and \( W_0^* \)) is relatively small, we have derived the approximated form of Eq. (4) in the case when the variance of the native frequency, \( \langle (\omega - \omega_0)^2 \rangle \), has a non-zero
value, where \( \omega_0 \) is the mean frequency;

\[
\omega_0 = \int_{-\infty}^{\infty} \omega f(\omega) \, d\omega \quad \text{(see Appendix I)}.
\]

In the case where \( F_n(\{W_k\}) = (1/N) \sum_{k=1}^{N} (W_k - W_n) \) and \( f(\omega) = (\gamma/\pi) \left( \frac{i^2}{\gamma} + (\omega - \omega_0)^2 \right)^{-1} \) Eq. (4) is transformed to give

\[
\frac{dW_n}{dt} = i\omega_n W_n + P_n(W_n) - \varepsilon W_n + \Delta_0 |W_n| e^{i\omega_0 t}
\]

(7)

with \( \Delta_0 = \varepsilon e^{-\gamma t} \). The derivation of Eq. (7) is given in Appendix I. From a statistical viewpoint, Eq. (7) is regarded as a mean field approximation for the \( N \)-body system described by Eq. (4). In other words, the self-locking of Eq. (4) was transformed to the forced locking of each mode under the effective field with the frequency \( \omega_0 \). Though this perturbation theory is not self-consistent, we use the terms of “mean field” in this paper by reason that the effect from the many surrounding modes is replaced by a synchronized behavior. In what follows, we study the analytical solution of Eq. (7):

As derived in Appendix II, the order parameter \( \sigma \) defined by the relative fraction of the entrained modes, and the motion of the locked modes \((W_n = \rho_n e^{i\phi_n})\) are represented as follows:

\[
\rho_n = \sqrt{1 - \varepsilon \left(1 - e^{-\gamma t} \cos\left(\sin^{-1} \frac{\rho_0}{\Delta_0}\right)\right)},
\]

\[
\phi_n = \omega_0 t + \sin^{-1} \frac{\rho_0}{\Delta_0} + \bar{\phi}_0,
\]

\[
\sigma = \frac{2}{\pi} \tan^{-1} \left( \frac{\varepsilon}{\gamma} e^{-\gamma t} \right),
\]

(8)

where \( \bar{\phi}_0 \) is a constant phase. On the other hand, the phase \( \phi_n \) of the \( n \)-th mode which is not locked on the frequency \( \omega_0 \) drifts as follows:

\[
\phi_n = \phi_n - \omega_0 t = 2 \tan^{-1} \left( \frac{\Delta_0}{\rho_0} \frac{\sqrt{\rho_0^2 - \Delta_0^2}}{\tan \frac{\sqrt{\rho_0^2 - \Delta_0^2}}{2} (t - t_0)} \right)
\]

(9)

and the effective period \( \tau_n \) is given by

\[
\tau_n = \frac{2\pi}{\sqrt{\rho_0^2 - \Delta_0^2}},
\]

(10)

where \( t_0 \) is the initial time. In consequence, the effective frequency distribution \( \tilde{f}(\omega) \) is different from the Lorentzian, and is obtained by Eqs. (8) and (9) as follows:

\[
\tilde{f}(\omega) = \sigma \delta(\omega - \omega_0) + f'(\omega - \omega_0),
\]

(11)

where
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\[ f'(d) = \frac{\gamma}{\pi} \frac{|d|}{\sqrt{d^2 + A_n^2 (y^2 + d^2 + A_n^2)}}. \] (12)

In the case when the native frequency distribution is not simple Lorentzian but has two dominant peaks:

\[ f(d) = \frac{1}{2} (\delta(d - 2\pi D) + \delta(d + 2\pi D)) \] (13)

the mean field approximation described in Appendix I leads to

\[ \dot{\psi}_k = A_k - A_k^* \sin(\psi_k), \] (14)

where \( A_k = \pm 2\pi D \) and \( A_k^* = \varepsilon \cos(2\pi D / \varepsilon) \). In the case \( |D| < A_k^*/2\pi \), all the modes are locked on the frequency \( \omega_0 \), and the phases \( \psi_k \) are fixed as follows:

\[ \psi_k = \psi_k^* = \pm \arcsin \left( \frac{2\pi D}{A_k^*} \right) \quad \text{for} \quad A_k = \pm 2\pi D. \]

While in the case \( |D| > A_k^*/2\pi \), the mean velocity of phase drift calculated by Eq. (10) is given by

\[ \langle \dot{\psi}_k \rangle = \langle \dot{\psi}^* \rangle = \pm \sqrt{(2\pi D)^2 - (A_k^*)^2} \quad \text{for} \quad A_k = \pm 2\pi D. \]

The relative phase difference \( A\psi = \psi^* - \psi^- \) and the drift velocity \( A\dot{\psi} = \langle \dot{\psi}^* \rangle - \langle \dot{\psi}^- \rangle \) given by

\[ A\psi = 2 \sin \left( \frac{2\pi D}{A_k^*} \right), \]

\[ A\dot{\psi} = 2 \sqrt{(2\pi D)^2 - (A_k^*)^2} \] (15)

are compared to the results of the simulation in the next section.

§ 3. Comparison with the computer simulation

Computer simulation was carried out with the system given by Eqs. (4) and (5) in the case \( N=100, \varepsilon=0.2 \) and \( T=2\pi/\omega_0 = 1 \). \( T=1 \) is not the essential point in the present analysis, since \( \omega_0 \) is able to be abridged always by the condition \( F_n(\{W_n\}) = 0 \).

(A) Lorentzian distribution of the native frequency

The frequency of the \( n \)-th mode is given by

\[ \omega_n = \gamma \tan \left( \frac{n}{100} \pi - \frac{50.5}{100} \pi \right) + 2\pi. \] (16)

Figures 1 show the time course of \( k \)-th mode in the case \( \gamma = 0.04 \). At the initial state \( (t=0) \), each phase \( \phi_n (= \phi_n - \omega_0 t) \) and amplitude \( \rho_n \) are distributed at random. At \( t=40T \), both phases and amplitudes intend to aggregate on a branch (this is called the entrained branch in this article), and when \( t > 80T \) the branch is fixed almost invariantly for \( 8 \leq n \leq 93 \). Therefore, we can understand that the system
Fig. 1. Phase space diagram of the $k$-th mode. 
(a) is the chaotic state for $t=0$, (b) and (c) are for $t=40T$ and for $t=80T$, respectively. The black spots denote the state of each mode obtained by the numerical calculation.
Fig. 2. Phase space diagram of the self-entrained state. (a), (b), and (c) are the phase space relation for the cases $\tau=0.01$, 0.06, and 0.1, respectively. The black spots are the results of the simulation and the solid lines are the theoretical curves given by Eq. (8).
reached a self-entrained state at $t=80T$. In all the calculations performed on the present article, $200T$ was sufficient for the system to reach the self-entrained state. In what follows, we illustrate the self-synchronized states for various cases with $\gamma=0.04$, 0.06 and 0.1 in Figs. 2(a), (b) and (c), respectively. For the sake of convenience, each phase $\varphi_n$ is measured in relative to the phase of the 50-th mode. The solid lines in Fig. 2 show the theoretical curves obtained by Eq. (8). The theoretical results are well in line with the results of computer simulation when the parameter $\gamma$ is relatively small, i.e., $\beta=\gamma/\varepsilon<0.3$, but the mean field approximation does not hold when the value of $\beta$ is large. The order parameter $\sigma(\beta)$ is shown in Fig. 3. The solid line (a) in the figure is the theoretical results obtained by Eq. (8) and the line (b) is the following analytical curve:

$$\sigma(\beta) = \frac{2}{\pi} \tan^{-1} \sqrt{1 - \frac{2\beta}{\beta}}$$

(17)

which was obtained by Kuramoto under another simplification of Eq. (4), and the dotted line (c) denotes the numerical results.

(B) Frequency distribution with two dominant peaks

The system with the frequency distribution given by Eq. (13) is numerically studied in the case $\gamma=0.04$. Other parameters of the system are unchanged, i.e., $N=100$, $\varepsilon=0.2$ and $T=1$. Figures 4(a) and (b) show the phase difference $\Delta \phi$ and the mean drift velocity $\dot{\phi}$, respectively. The solid lines denoted in the figures are the theoretical curves given by Eq. (15). Near critical point satisfying $2\pi D = D^*$, numerical values of $\Delta \phi$ and $\dot{\phi}$ are not exactly determined, since the drift velocity of phases ($\dot{\phi}^+$ and $\dot{\phi}^-$) becomes very slow. Far from the critical point the mean field approximation explains the results of simulation very well.

![Fig. 3. Order parameter of the self-synchronization. The black spots are the results of the simulation. The line (a) is the theoretical result given by Eq. (8), and (b) is the result obtained by Kuramoto mentioned in the text.](https://academic.oup.com/ptp/article-abstract/56/3/703/1935391)
§ 4. Mode-locking phenomena beyond the mean field theory

As mentioned in the previous section, the mean field theory breaks asymptotically when the distribution of the native frequency becomes flat. In this section the other types of breakdown of the theory are qualitatively stated.

Let us consider the case where the frequency distribution is given by the following superposition of the Lorentzian distribution, i.e.,

\[
f(\Delta) = \frac{1}{2\pi} \left( \frac{1}{\gamma^2 + (\Delta + 2\pi D)^2} + \frac{1}{\gamma^2 + (\Delta - 2\pi D)^2} \right),
\]

\[\tag{18}
\]

Fig. 5. Phase space diagram in the case of partial self-synchronization. Each dot denotes the results of numerical calculation at \( t = 220T \), under the same initial state as in Fig. 1(a).
Figure 5 is the result for the case N=100, \( \varepsilon = 0.2 \), \( \tau = 0.02 \) and \( 2\pi D = 0.2 \). Two locking branches appear: One includes the native mode for \( 4 \leq n \leq 48 \) and another for \( 53 \leq n \leq 97 \). The angular drift velocity of these branches \( \omega^\pm \) are approximately given by Eq. (10) as follows: \( \omega^\pm \sim \omega_0 \pm 2\pi D \). The partial self-locking as illustrated in Fig. 5 is commonly observed in the laser system of multi-modes,\(^{10}\) and the partial locking with multi-branches are easily derived in the case when the frequency distribution \( f(\omega) \) has several dominant peaks,

\[
f(\omega) = \frac{1}{M\pi \sum_{i=1}^{M} \frac{\gamma_i}{\gamma_i^2 + (\omega - \omega_i)^2}},
\]

where \( \gamma_i \) is a positive constant, \( \omega_i \) a dominant frequency and \( M \) an arbitrary integer. Our perturbation method developed in this paper, however, is not available to the phenomenon of partial self-locking, since the mean field approximation is based on the assumption with a single entrained branch.

At the last of this section, we mention the transient phenomena of mode-locking. The organizing process of the entrained state is represented by the total amplitude \( E \) defined by

\[
E = \left| \sum_{k=1}^{N} W_k \right|.
\]

Figures 6(a) and (b) show the transient states of the systems given by Eqs. (4) and (7), respectively. The parameters are taken as \( N=100, \varepsilon = 0.2 \) and \( \tau = 0.02 \). At the entrained state, both behaviors shown in Fig. (6) are almost the same except for the small fluctuation around the mean level, but the relaxation time of the transient process is quite different. This shows that the adequacy of the
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mean field approximation is limited in the case the system is in the entrained state.

§ 5. Discussion

In this paper we have explored a mean field theory of the mode-locking phenomena in many-body system with nonlinear feedbacks. By the theory, the phenomenon of self-locking is analyzed in terms of the forced locking under the external periodic stimulus with a given frequency \( \omega_0 \). As mentioned in § 4, there are some breakdowns in the perturbation theory. The time dependent behavior represented by Eq. (7) should be considered in taking account of the irregular force into Eq. (7), consequently which should be regarded as the Langevin type equation. The partial mode-locking will be treated in the framework of the mean field theory if we use the local mean field approximation for each entrained branch. These analysis is left in the future study. Nevertheless, results of the synergetic approach are well reproduced by the perturbation theory on the whole so far as the value of \( \beta \) is small.

The theory of the present paper can be easily extended to the other nonlinear oscillators system represented by

\[
P_n(W_n) = -W_n(|W_n|^4 - |W_n|^2 + A),
\]

where \( A \) is a constant. The series coupling of oscillators are given by putting as, \( F_n(\{W_k\}) \sim W_{n+1} + W_{n-1} - 2W_n \), which is an approximation to the one-dimen­sional diffusion process of \( F_n(\{W_k\}) \sim \partial^2 W_n/\partial n^2 \). In this series coupling the number of entrained mode is much smaller than in the case of parallel coupling given by \( F_n(\{\}) = \sum_{k=1}^N (W_k - W_n) \). Under the condition \( f(\omega) = \delta(\omega - \omega_0) \), the diffusion model has the synchronized wave solution other than Eq. (6). We can take the wave solution as the unperturbed reference state instead of Eq. (6). These problems in further complicated models will be studied in another paper.

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Appendix I

---The Mean Field Theory of the Self-locking System---

When we put \( W_n = \rho_n e^{i\phi_n} \), Eqs. (4) and (5) are rewritten as follows:

\[
\frac{d\rho_n}{dt} = P_n(\rho_n) + \varepsilon \rho_n \text{Re}[F_n(\{W_k\})/W_n],
\]

\[
\frac{d\phi_n}{dt} = \omega_n + \varepsilon \text{Im}[F_n(\{W_k\})/W_n].
\]
If we limit the discussion to the neighborhood of the synchronized solution (or an attractor \( W_0^* \)), Eq. (A·1) is approximated as follows:

\[
\rho_n = 1,
\]

\[
\frac{d\phi_n}{dt} = \omega_n + \varepsilon \text{Im}[e^{-i\phi_n}F_n'(\{e^{i\phi_n}\})], \tag{A·2}
\]

provided that the frequency distribution function \( f(\omega) \) has a sharp peak at the mean frequency \( \omega_0 = (1/N)\sum_{k=1}^{N} \omega_k \). Under this condition Eq. (A·2) is rewritten as

\[
\frac{d\psi_n}{dt} = J_n - \varepsilon \sum_{m=1}^{N} A_{nm}\psi_m + O(\{\psi_n\}), \tag{A·3}
\]

where \( \psi_n = \phi_n - \omega_n t - \phi_0 \), \( J_n = \omega_n - \omega_0 \), and \( A_{mn} = \text{Re}[\partial F_m(\{W_k\})/\partial W_n]_{W_k=W_n} \).

Here the higher order terms of \( O(\{\psi_n\}) \) have been neglected. As mentioned above it is necessary for the occurrence of self-entrainment that the attractor given by Eq. (6) is stable, i.e., the eigenvalues of matrix \( A_{mn} \) are non-positive, and that the eigenstate against the zero eigenvalue is not degenerate. Then the synchronized motion is derived uniquely from the steady solution of \( \psi_n \) that will be written by \( \psi_n^s \). By the following approximation for the interaction term: \( \rho_n = 1 \) and \( \psi_n = \psi_n^s \), Eq. (A·1) is transformed to give

\[
\frac{d\rho_n}{dt} = P_n(\rho_n) + \varepsilon \rho_n \text{Re}[e^{-i\phi_n}F_n'(\{e^{i\phi_n}\})],
\]

\[
\frac{d\psi_n}{dt} = J_n - \varepsilon \text{Im}[e^{-i\phi_n}F_n'(\{e^{i\phi_n}\})], \tag{A·4}
\]

where \( F_n'(\{\}) \) means that all \( \psi_k \) except \( \psi_n \) are replaced by \( \psi_n^s \) in the curly bracket of \( F_n(\{\}) \). If we put \( F_n'(\{e^{i\phi_n^s}\}) = F_n'(e^{i\phi_n^s}) \), Eq. (A·4) is simply written as

\[
\frac{dW_n}{dt} = i\omega_n + P_n(W_n) + \varepsilon |W_n|F_n'e^{i\phi_n^s} + i\omega_n t \tag{A·5}
\]

and the synchronized behavior of the entrained modes which satisfies the condition\(|J_n| < \varepsilon F_n'\) is given by \( d\rho_n/dt = d\phi_n/dt = 0 \), i.e.,

\[
\rho_n = \sqrt{1 + \varepsilon F_n'\cos(\psi_n - \phi_n^0)},
\]

\[
\phi_n = \sin^{-1}\frac{J_n}{\varepsilon F_n'} + \phi_n^0, \tag{A·6}
\]

where both \( F_n' \) and \( \phi_n^0 \) are definite functions of \( n \).

**Appendix II**

---Derivation of Eqs. (8) and (15)---

In the case \( F_n(\{W_k\}) = (1/N)\sum_{k=1}^{N}(W_k - W_n) \), the asymptotic solution \( \psi_n^s \)
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(= \psi_k(t \rightarrow \infty)) of Eq. (A·3) is given by $A_k/\varepsilon + \bar{\psi}_n$. Here $\bar{\psi}_n$ is a constant. In the limit of the continuous spectrum ($N^{-1} \rightarrow 0$), the mean field approximation of the interaction term in Eq. (A·4) becomes as follows:

$$e^{-i\phi_x} F_n' \{ \{ e^{i\phi_x} \} \} = \int_{-\infty}^{\infty} f(\lambda) \left( e^{i\phi_x(\lambda)} - e^{i\phi_n} - 1 \right) d\lambda$$

$$= \begin{cases} 
\frac{1}{\pi} \frac{1}{A^2 + \gamma^2} & \text{for the Lorentzian case} \\
\frac{1}{\sqrt{\pi}} e^{-\lambda^2} & \text{for the Gaussian case} \\
\cos(2\pi D/\varepsilon) e^{-i\phi_x} - 1 & \text{for the case of } f(\lambda) = \frac{1}{2} (\delta(\lambda - 2\pi D) + \delta(\lambda + 2\pi D)) 
\end{cases}$$

The above relation leads to Eqs. (8) and (15) in the text. The order parameter $\sigma$ defined by the number of entrained modes is represented as follows:

$$\sigma = \frac{1}{N} \sum_{n=1}^{N} \theta(\varepsilon F_n' - |A_k|)$$

$$= \frac{2}{\pi} \tan^{-1} \left( \frac{1}{\beta} \right) \text{ (for the Lorentzian case)}$$

where $\theta$ denotes the unit step function and $\beta$ stands for $\gamma/\varepsilon$.

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