The geometry of space-time in dual theory is studied through the coupling of the graviton, scalar and antisymmetric tensor states to the Ramond-Dirac system (fermionic string). The requirement of super-gauge algebra severely restricts the contents of the spin-connection coefficient and forbids the presence of torsion. The study of a model action in terms of local fields indicates that the dual theory in low energy limit is closely related to the Brans-Dicke theory with $\omega=1$.

§ 1. Introduction

From the viewpoint of particle physics, the quantum theory of gravity is no more than the theory of graviton and its interaction with itself and other particles. If the structure of the interaction is interpreted in terms of the geometrical concepts, such as the metric tensor, affine connection, etc., then by using this link we can go over to the geometric theory of gravity.

In this context, the dual theory is a very interesting model, since it is a unified theory of infinitely many particles, including the graviton which is indeed described, in a certain low energy limit, by the Einstein Lagrangian.\(^3\)\(^4\) It is quite important to notice that the contents of the dual theory, i.e., particle spectrum, amplitudes, the dimension of space-time and so on, are essentially determined by its own algebraic structure which originates from the internal geometric property of strings. Since the coupling scheme of the graviton in this model is thus uniquely determined, we can further expect that the dual theory has its own space-time geometry. It is then an amusing question to ask what it is. In the present note, we are going to explore this problem by examining the interaction of massless closed string states (the graviton, the scalar and the antisymmetric states) with the Ramond-Dirac system from the geometric point of view. The coupling of the graviton was already studied in a previous paper\(^3\) which is hereafter referred to as I.

Scherk and Schwarz\(^5\) suggested that the geometry of space-time in the dual theory is non-Riemannian by showing that the action corresponding to the zero-slope limit of the closed string sector is expressible in terms of an affine (non-metric) connection with nonvanishing torsion. However, their procedure is quite formal and seems to be without any compelling reason. Firstly, although their affine connection is of non-metric type, there is no homothetic curvature and by a suitable Weyl transformation we can define new metric tensor with respect to which the
connection is of metric type. Secondly the identification of the metric tensor and of the affine connection should be performed through the coupling of these geometrical quantities to matter. They did not study this problem.

§ 2. D-bein formalism for the Ramond-Dirac system

We here briefly summarize the D-bein formalism\(^b\) for the Ramond-Dirac system for the purpose of self-containedness of this note.

At each point \(x^a\) in space-time, tangent space \(T(x)\) exists. As the coordinate basis in the tangent space \(T(x)\), we introduce D-bein field \(V^a(x)\) (\(a=1, \ldots, D\)) and its inverse such that \(g_{\mu \nu} = V^a \eta_a \eta^b \), \(V^a V^b = \eta^{ab}\), \(\eta^{00} = -1\), \(i=1, \ldots, D-1\) where \(g_{\mu \nu}\) is the metric tensor. We define a string variable \(x^a(\sigma)\) and a conformal spinor \(S^a(\sigma)\), parametrized by \(\sigma(0<\sigma<\pi)\) such that \(\partial x^a(\sigma)/\partial \sigma = 0\), \(S^a_S^a = S^a_S^a\) at \(\sigma=0, \pi\). The index \(a\) of the conformal spinor \(S^a(\sigma)\) is referred to in each tangent space \(T(x(\sigma))\), and the index \(i\) is referred to in the special orthogonal gauge in which the “zweibein” of Iwasaki and Kikkawa\(^b\) is the unit matrix. Quantization is done by imposing \([x^a(\sigma), p_i(\sigma')] = i\pi \delta^a_i \delta(\sigma - \sigma')\), \(\{S^a(\sigma), S^b(\sigma')\} = -4\pi \delta^a_b \delta(\sigma - \sigma')\). The Ramond equations in curved space-time are then given by

\[
\tilde{F}_a|\phi\rangle = 0, \quad (n \geq 0)
\]

where

\[
\tilde{F}_a = \frac{1}{2\sqrt{2\pi}} \int_0^\pi \left( G_+(\sigma) e^{in\sigma} + G_-(\sigma) e^{-in\sigma} \right) d\sigma,
\]

\[
G_+(\sigma) = S^a_S^a \left[ V^a(x(\sigma)) \partial_x^a(x(\sigma)) + V_a^a(x(\sigma)) \frac{\partial x^a(\sigma)}{\partial \sigma} \right],
\]

\[
G_-(\sigma) = S^a_S^a \left[ V^a(x(\sigma)) \partial_x^a(x(\sigma)) - V_a^a(x(\sigma)) \frac{\partial x^a(\sigma)}{\partial \sigma} \right],
\]

\[
\partial_x^a(x) = p^a(x) + \frac{1}{2} i A^a(x),
\]

\[
M_{ab}(\sigma) = \frac{1}{8} \left[ S^a_S^a, S^b_S^b \right].
\]

The \(\tilde{F}_a\) so defined is formally invariant under the infinitesimal transformation\(^*)\)

\[
\delta x^a(\sigma) = f^a(x(\sigma)),
\]

\[
\delta S^a_S^a = \omega_{ab}(x(\sigma)) S^b_S^a, \quad (\omega_{ab} = -\omega_{ba})
\]

because the D-bein and the spin connection coefficient \(A^a_{ab}(x)\) are supposed to transform as

\(^*)\ We take this opportunity to point out that the Ref. 3) contains some misprints, especially, in § 2; e.g., \(\exp[\omega \beta]\) should read \(e^{\omega \beta}\) which is the local Lorentz transformation matrix.
The covariant functional derivative defined by (4) corresponds to the following prescription for the infinitesimal parallel transport of the state functional

$$X''(x) = X'(x) + \omega'(x) dX'(x).$$

This is a natural generalization of that for local fields.

\section{3. The contents of the D-bein and the spin-connection coefficient}

In order to make the system defined by (1) \~(5) self-consistent and compatible with the dual-theory world, the $\tilde{F}_n$ have to satisfy the super-gauge algebra (SGA). Our idea in exploring the space-time geometry in the dual theory is that this requirement will restrict the contents of the D-bein and the spin-connection coefficient and will consequently determine the space-time geometry embodied in the dual theory.

We summarize our results. The verification of these results are given in the Appendix.

(A) The correct on-shell vertices compatible with the SGA and with the factorization property of the amplitudes for the graviton and scalar states correspond to the D-bein and the spin-connection coefficient given by

$$V_{a \mu} = \gamma(\phi) (\gamma_{a \mu} + \sqrt{8 \pi G} h_{a \mu}),$$

$$A_{ab \mu} = \frac{1}{2} V_{a \mu} (C_{dab} - C_{abd} - C_{bad}),$$

where

$$C_{dab} = (V_a^* V_b - V_b^* V_a) \delta_{d \mu},$$

$$\gamma(\phi) = 1 + \sqrt{8 \pi G \frac{D-2}{D-2} \phi + O(\phi^2)},$$

and $\phi, h_{ax}$ are the asymptotic fields (on the mass shell) corresponding to the massless scalar and the graviton states, respectively. The antisymmetric part of the D-bein is unphysical and does not couple to the spinning string. (See the Appendix.)

The relation (11) between the D-bein and the spin-connection coefficient amounts to the torsionless (i.e., symmetric) affine connection $\Gamma_{\mu \nu}^{\rho}$ of metric type, Christoffel symbol. If torsion $\Gamma_{\nu \rho}^{\mu}$ does not vanish, (11) should be replaced by

$$A_{ab \mu} = \frac{1}{2} V_{a \mu} (C_{dab} - C_{abd} + T_{dab} - T_{bad} - T_{bda}),$$

where
provided that the spin-connection coefficient is compatible with the $D$-bein.\(^7\)

(B) The torsion part in (14) is not compatible with the SGA. In particular, the identification of the torsion $\Gamma^{t}_{[\nu \lambda]}$ with the $g^{\mu \nu} F_{\lambda \mu}$, where $F_{\lambda \mu} = \partial_\lambda A_\mu + \partial_\mu A_\lambda - \partial_\nu A_{\lambda \mu}$ with $A_\mu = -A_\mu$ being the asymptotic field corresponding to the massless antisymmetric state, does not reproduce the correct vertex for that state.

The result (A) implies that if we define the metric tensor to be $g_{\mu \nu} = g(\phi)^2 \times (\eta_{\mu \nu} + \sqrt{32\pi} G h_{\mu \nu})$, where $\phi$ and $h_{\mu \nu}$ are the scalar and the graviton field respectively, the affine connection should be of metric type. The result (B) implies that the affine connection should be symmetric, in contrast with the suggestion by Scherk and Schwarz.\(^9\) Then, how to interpret the antisymmetric state of the closed string sector? We find that

(C) the correct vertex for the antisymmetric state is obtained as the minimal interaction associated with the following gauge transformation of the state functional

$$\delta |\psi\rangle = \left[ i \int_0^\pi \zeta_a e^{i k z(\sigma)} \frac{\partial x^a(\sigma)}{\partial \sigma} d\sigma + \frac{i}{16} \int_0^\pi (k_{a \sigma} \gamma_a - k_{b \sigma} \gamma_b) (S_4^a(\sigma) S_b^b(\sigma) - S_b^a(\sigma) S_4^b(\sigma)) e^{i k z(\sigma)} d\sigma \right] |\psi\rangle , \quad (16)$$

where $\zeta_a$ is an arbitrary infinitesimal vector. Equation (16) is a generalization, to the spinning string, of the gauge transformation considered by Ramond and Kalb\(^8\) and by Nambu\(^9\) for the ordinary string for which the second term in (16) is not necessary. It seems impossible to relate this transformation to the geometry of external space-time. As in the Ref. 8) one needs, besides the antisymmetric tensor field, another gauge vector field which couples to the ends of the string. These gauge fields are to be regarded as being independent of the space-time geometry, as the electromagnetic and Yang-Mills fields are.

Thus we can conclude that the geometry of space-time in the dual theory is Riemannian. This followed from the proper algebraic structure of the dual theory.

We have used the external field approach. If we start, instead, from the second-quantized field theory for strings,\(^10\) one may expect the following situation: We will not need to introduce the $D$-bein and the spin connection externally in order to make the theory invariant under the transformations (6) and (7), and the compensation of the term coming from the functional derivative will be achieved by the term coming from the non-linear interaction terms. A similar phenomenon, indeed, occurs in a non-linear spinor theory.\(^11\) Much work will still be necessary to establish this conjecture. We hope our work will be a preliminary step which will lead to more complete approach in this problem.
§ 4. Local Lagrangian model

To get a further insight on our results we consider a local Lagrangian model. Aside from the term associated with the antisymmetric tensor field, a natural model action for the system of the graviton, the massless scalar and the massless Dirac field reflecting the results of the previous section will be

\[ S = - \int dx \sqrt{\bar{g}} \left( \frac{\bar{R}}{16\pi G} + \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) + i \int dx \sqrt{\bar{g}} \bar{\psi} \gamma^a V^a_\nu D_\nu \phi, \]  

(17)

where \( \bar{g}_{\mu\nu} = g_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu} \) with \( h_{\mu\nu} \) being the graviton field, \( \bar{R} \) the corresponding scalar curvature. The first term in (17) is the zero-slope limit of the symmetric closed string sector.\(^p\) For the second term which describes the coupling of the graviton and the scalar to the Dirac field, we adopt the D-bein given by (10) and the corresponding covariant derivative without the torsion term in view of the results (A) and (B). The relation between the D-bein given by (10) and the metric tensor \( g_{\mu\nu} \) is

\[ V^a_\nu \frac{\partial}{\partial x^a} = g_\nu^a V^a_\nu. \]

The massless Dirac field is of course the substitute for the Ramond string. In I, it was shown that the massless antisymmetric tensor does not couple directly to the ground state of the Ramond string which reduces to the local Dirac field in the zero-slope limit.

To obtain the exact form of \( g(\phi) \), we would have recourse to the zero-slope limit and perform lengthy calculation. It is beyond the scope of the present note. For the purpose of further discussion, however, let us tentatively set \( g(\phi) = \exp(\sqrt{8\pi G/D-2})\phi \) which is compatible with (13). Then, by using the relation

\[ \{ \ell, \mu \} = \{ \ell, \mu \} \sqrt{\frac{8\pi G}{D-2}} \left( \bar{g}^{\mu\nu} \partial_\nu \phi + \bar{g}^{\mu\nu} \partial_\nu \phi - g^{\mu\nu} g_{\mu\nu} \partial_\nu \phi \right), \]

(18)

where the l.h.s. is the Christoffel symbol constructed from \( g_{\mu\nu} \), we find

\[ \int dx \sqrt{\bar{g}} \left( \frac{\bar{R}}{16\pi G} + \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) = \int dx \sqrt{\bar{g}} \left( R\sigma - \frac{1}{\sigma} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right), \]

(19)

where \( R \) is the scalar curvature constructed from the \( g_{\mu\nu} = V_\nu^a V^a_\mu \) and

\[ \sigma = \frac{1}{16\pi G} \exp[-\sqrt{8\pi G(D-2)}\phi]. \]

We observe that (19) corresponds to the Brans-Dicke theory\(^m\) of \( \omega = 1 \). This value of \( \omega \) is relevant to the dual theory\(^n\) at least to the first order with respect to the scalar field, because \( g(\phi) \) is correct to that order. Thus the gravity contained, as the low energy limit, in the dual theory deviates considerably from the Einstein theory (\( \omega \to \infty \)), while the geometry of space-time is still Riemannian.

\(^p\) The value \( \omega = 1 \) is different from that given by Scherk and Schwarz\(^n\) who, however, only study the coupling of the scalar to super heavy particles in the zero-slope limit.
Appendix

If $\hat{F}_n - F_n$, ignoring the seagull terms, is factorized into the combination of the form

$$\int_{\sigma_0}^{\sigma} \left[ U(\sigma) V(-\sigma) e^{i n \sigma} + U(-\sigma) V(\sigma) e^{-i n \sigma} \right] d\sigma = W_n$$

such that

$$\left[ F_n, V(\sigma) \right] = -\frac{i}{\sqrt{2}} \frac{d}{d\sigma} \left[ e^{i n \sigma} U(\sigma) \right]$$
$$\left\{ F_n, U(\sigma) \right\} = -\sqrt{2} e^{i n \sigma} V(\sigma)$$

the super gauge algebra,

$$\left\{ F_n, \tilde{F}_m \right\} \simeq -2 \tilde{L}_{n+m} - \frac{1}{2} D_m \tilde{\delta}_{m,-n}$$

is satisfied. Here \( \simeq \) means equality ignoring the seagull terms. Neglecting the seagull terms is not an approximation but is justified in the dual theory since the super gauge symmetry, when interaction is present, is only the symmetry of the on-shell amplitudes for which the seagull terms are always analytically defined to be zero in the dual theory due to the exact pole duality of the on-shell amplitudes.\(^a\) Mathematically, this corresponds to define the amplitudes as the finite part of the integral. In the zero-slope limit, the seagull and the self-interaction terms automatically appear\(^b\) owing to the effect of the propagation of super heavy states, as required by the gauge invariance. In this connection we emphasize that the zero-slope limit is not simply to erase the super heavies. For this reason it is, in general, not possible to directly take the zero-slope limit on the second quantized Lagrangian for strings.\(^c\)

Using (10)\(\sim\)(13) in (2) we find that the vertices which describe the emission of the graviton and the scalar out of the fermionic string are given by, respectively,

$$W_n^g = \sqrt{\frac{G}{4\pi}} \int_{\sigma_0}^{\sigma} \epsilon^{ab} (U_a(\sigma) V_b(-\sigma) e^{i n \sigma} + U_a(-\sigma) V_b(\sigma) e^{-i n \sigma}) d\sigma$$

$$W_n^s = \sqrt{\frac{G}{4\pi(D-2)}} \int_{\sigma_0}^{\sigma} \eta^{ab} (U_a(\sigma) V_b(-\sigma) e^{i n \sigma} + U_a(-\sigma) V_b(\sigma) e^{-i n \sigma}) d\sigma$$

$$+ \sqrt{2} \pi G (D-2) \left[ F_n, \int_{\sigma_0}^{\sigma} e^{i n \sigma} d\sigma \right] \delta(0)$$

with $\epsilon^{ab} k_a = \epsilon^a = -k^2 = 0, \epsilon^{ab} = \epsilon^{ba}$,

where

\(^a\) Usually, this is called the cancelled propagator argument.
\[ V_a(\sigma) = \left[ P_a(\sigma) + \frac{1}{4} \Gamma_a(\sigma) (k \cdot \Gamma(\sigma)) \right] \exp \frac{i}{2} \frac{k}{Q(\sigma)}, \quad (A\cdot7) \]

\[ U_a(\sigma) = \Gamma_a(\sigma) \exp \frac{i}{2} \frac{k}{Q(\sigma)} Q(\sigma). \quad (A\cdot8) \]

The second term in \((A\cdot7)\) comes from the spin-connection coefficient. \((A\cdot7)\) and \((A\cdot8)\) satisfy \((A\cdot2)\) and \((A\cdot3)\) when they are contracted with an arbitrary polarization tensor \(\epsilon^{ab}\) satisfying \(k_a \epsilon^{ab} = k_b \epsilon^{ab} = 0\). The operators used to define \((A\cdot7)\) and \((A\cdot8)\) are the familiar ones satisfying \(x(\sigma) = (Q(\sigma) + Q(-\sigma))/2\), \(P(\sigma) = dQ(\sigma) / d\sigma\), \(S_v(\sigma) = S_v(-\sigma) = \Gamma(\sigma)\). The vertex \((A\cdot5)\) is the correct vertex for the graviton state which was studied in detail in I. The second term in \((A\cdot6)\) is singular but only a pure gauge since it is a commutator with \(F_a\) and does not couple to the amplitudes because the amplitudes are constructed by using the vertex \(W_0\) and the propagator \(F_0^{-1}\). It is easy to check that the pure gauge term does not violate the SGA. By introducing a non-covariant polarization tensor \(\tilde{\epsilon}^{ab} = \eta^{ab} - (k^a \tilde{k}^b + k^b \tilde{k}^a) / k \cdot \tilde{k}\), where \(k\cdot\tilde{k} \neq 0\), \(k^a = \tilde{k}^a = 0\). The \(W_a^\sigma\) is then written as \(W_a^\sigma = \sqrt{G / 4\pi (D-2)} \int_0^\sigma d\sigma' \epsilon^{ab} (U_a(\sigma') V_b(-\sigma') e^{i\sigma} + U_a(-\sigma) V_b(\sigma) e^{-i\sigma}) + \text{pure gauge term.} \quad (A\cdot9) \)

Since now \(k_a \epsilon^{ab} = k_b \epsilon^{ab} = 0\), \((A\cdot9)\) is compatible with the SGA. This shows that \((A\cdot6)\) is the correct scalar vertex. The factor \(\sqrt{D-2}^{-1}\) in \((A\cdot6)\) is required from the consistency with the factorization. Thus we see that the SGA and the factorization require \((11)\) and \((13)\) at least to the first order of the external asymptotic fields. However, once it turns out that the spin-connection is written in terms of the \(D\)-bein, the relation \((11)\) is unique by the transformation property \((8)\) and \((9)\) apart from the term which cannot be written in terms only of the \(D\)-bein. This proves \((A)\).

By \((14)\) we find that the vertex for the torsion is proportional to

\[ \int_0^\sigma d\sigma f_{abc} \left( \Gamma^a(\sigma) (\Gamma^b(\sigma) \Gamma^c(\sigma) + \Gamma^a(\sigma) \Gamma^b(-\sigma)) e^{i\sigma} + \Gamma^a(-\sigma) (\Gamma^b(\sigma) \Gamma^c(\sigma) + \Gamma^a(-\sigma) \Gamma^b(-\sigma)) e^{-i\sigma} \right) \exp i k x(\sigma), \]

where \(f_{abc} = -f_{acb}\). This cannot be brought to the form \((16)\) and \((18)\) irrespective of the content of the \(f_{abc}\). By direct calculation we can easily check that in this case the algebra of \(\tilde{F}_a\) does not close. The identification of the torsion with \(\tilde{\epsilon}^{\sigma F_{\epsilon\sigma}} \) is not compatible with the SGA. Thus the result \((B)\) follows.

\textsuperscript{(*)} For the proof, see I.

\textsuperscript{(***)} Note that the difference between \((11)\) and another spin-connection satisfying \((9)\) must be a tensor.
The correct vertex for the antisymmetric state is obtained from \((A \cdot 5)\) by replacing the polarization tensor \(\epsilon^{ab} = \epsilon^{ba}\) by an antisymmetric polarization tensor \(A^{ab} = -A^{ba}\), \(k^a A^{ab} = 0\). Note however that this does not mean that state corresponds to the antisymmetric part of the \(D\)-bein which is proved in I not to couple to the amplitudes. In obtaining the form \((A \cdot 5)\) the symmetric property of the asymptotic graviton field is used. It is easy to see that the transformation \(A^{ab} \rightarrow A^{ab} + (k^a \zeta^b - k^b \zeta^a)/2\pi\) compensates the change of \(F_0\) under the transformation \(16\) except at the end points where the minimally coupled vector field\(^9\) achieves the compensation.

References

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Note added in proof: The result \((B)\) in \(\S 3\) only means that torsion, if any in dual string model, cannot be associated to the antisymmetric tensor state. It does not necessarily forbid the possibility that we obtain the Weyl-Cartan type torsion,\(^9\) for the zero slope-limit of four fermion amplitude, which would manifest itself as a point four-fermion interaction.