Hydrostatic Structure of a Hot Plasma in a Cluster of Galaxies

Noriaki Shibazaki, Reiun Hoshi, Fumio Takahara* and Satoru Ikeuchi*

Department of Physics, Rikkyo University, Tokyo 171
*Department of Physics, Kyoto University, Kyoto 606

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Hydrostatic equilibrium solutions for the hot gas in a cluster of galaxies have been studied assuming the polytropic relation. The final equilibrium state is related to the initial state of the gas in the cluster using the mass and energy conservation laws between the above two states. Results are applied to the X-ray source in Coma cluster. We have obtained the central temperature, the central density and the mass of the intracluster gas such as 1.8×10⁴°K, 2.8×10⁻²⁷ gm/cm³ and 5.7×10⁻¹⁵ M⊙, respectively.

§ 1. Introduction

It is well known that there exist extended powerful X-ray sources in clusters of galaxies such as Coma, Perseus and Virgo. Observationally, the extents of these X-ray sources are 0.1–1 Mpc and X-ray luminosities are 10⁴³–10⁴⁵ ergs/sec.

As for the X-ray emission mechanism of these sources, inverse compton and thermal bremsstrahlung theories have been proposed. However, recent spectral analyses of soft X-ray data tend to show a strong preference for the thermal model. Moreover, tailed radio sources and the excess of SO galaxies found in these clusters may be well interpreted, if hot intracluster gases exist.

Concerning the thermal model, a wind model, infall models and static models have been studied. Lea has calculated the hydrostatic equilibrium solution for a polytropic gas in a cluster of galaxies under the boundary condition that the temperature of the gas approaches to zero at infinity. By comparing that solution with observations for the Coma cluster, she shows that the intracluster gas is distributed more widely than the galaxies by a factor of about 2–2.5 and the temperature of the gas given from a best-fit model is 1.5×10³°K.

Takahara et al. have investigated the dynamical collapse of gas in a cluster of galaxies. As a result of numerical calculations, they find that a hot plasma is formed after the bounce and it attains an equilibrium state.

In § 2 the hydrostatic equilibrium solution for a polytropic gas in a cluster of galaxies is derived. In § 3 the hydrostatic equilibrium state which is attained after the dynamical collapse is discussed using the energy and mass conservation laws which relate the final state of the gas to the initial state. In § 4 the central temperature, the central density and the mass of the intracluster gas are obtained.
by comparing our model with observations for the Coma cluster. In the last section, the validities of our assumptions and approximations are discussed.

\section{Hydrostatic equilibrium solution for the intracluster gas}

The equation for the hydrostatic equilibrium of the intracluster gas in the gravitational field of galaxies can be solved analytically in the following way if the polytropic relation is assumed for the gas.

Galaxy counts data of clusters are known to be well represented by the isothermal Emden function. Here we assume King's analytical function for the distribution of galaxies in a cluster which approximates the isothermal Emden function:

\begin{equation}
\rho_m = \rho_{m0} \frac{1}{(1 + r^2/a^2)^{3/2}},
\end{equation}

where \( \rho_m \) is the smoothed-out density of galaxies, \( \rho_{m0} \) the value of \( \rho_m \) at the center of the cluster, \( r \) the radial distance from the center of the cluster and \( a \) the core radius. The mass of galaxies within a sphere of radius \( r \), \( M_m(r) \), is obtained from Eq. (1) as

\begin{equation}
M_m(r) = 3 \left\{ \ln (r/a + \sqrt{1 + (r/a)^2}) - \frac{r/a}{\sqrt{1 + (r/a)^2}} \right\} M_{m,\text{core}},
\end{equation}

where

\begin{equation}
M_{m,\text{core}} = \frac{4}{3} \pi a^3 \rho_{m0}.
\end{equation}

The momentum equation which describes the hydrostatic equilibrium of the gas in a cluster of galaxies is given by

\begin{equation}
\frac{1}{\rho} \frac{dP}{dr} = -\frac{GM_m(r)}{r^2},
\end{equation}

where \( \rho \) is the density of the intracluster gas, \( P \) the pressure and \( G \) the gravitational constant. Here we have neglected the self-gravity of the gas, which will be justified in § 5.

The equation of state is given by

\begin{equation}
P = \frac{k}{\mu H} \rho T,
\end{equation}

where \( T \) is the temperature of the gas, \( \mu \) the mean molecular weight of the gas, \( k \) the Boltzmann constant and \( H \) the mass of 1 amu.

Equations (2) and (4) can be solved if a relation between \( \rho \) and \( P \) is given. In this paper we assume the polytropic relation for the intracluster gas, which is written as
where the polytropic index is denoted by $N$. In terms of Eqs. (2), (4), (5) and (6), the density $\rho$ is given by

$$
\frac{\rho}{\rho_0} = \sigma^x = \left[ 1 + \frac{A}{N+1} \ln \left( \frac{x + \sqrt{1 + x^2}}{x} \right) - \frac{A}{N+1} \right]^N,
$$

where $\rho_0$ is the central density and $x$ the non-dimensional distance normalized by the core radius, $x = r/a$, and $A$ is given by

$$
A = \frac{3G M_{\text{core}}}{a} \frac{n}{kT_0},
$$

where $T_0$ is the central temperature. From Eq. (7) the outer boundary $x_M$ of the intracluster gas sphere is given by $\sigma(x_M) = 0$. However, in some cases the boundaries do not exist. In order for the boundary to exist, the following relation should be satisfied:

$$
A > N+1.
$$

Next, we consider the case of isothermal gas; i.e., $N = \infty$. In this case the analytical expression for the density distribution is given by

$$
\frac{\rho}{\rho_0} = \exp \left\{ A \frac{\ln (x + \sqrt{1 + x^2})}{x} - A \right\}.
$$

In the distribution, Eq. (10), the gas extends to infinity, and the density of the gas $\rho/\rho_0$ gradually approaches $e^{-A}$ for large $x$. It is to be noticed that in the case of no-boundary the total mass of gas becomes infinite, unless the artificial cutoff of gas is introduced. In order to avoid such artificial treatment, in the following, we confine ourselves to the hydrostatic equilibrium solutions which have the boundaries at the finite point from the center of the cluster.

§ 3. Relations between the initial and final states of the gas

Takahara et al. have calculated numerically the hydrodynamical collapse of gas in a cluster of galaxies for the various initial gas densities. They have found that a hydrostatic equilibrium state is attained for most cases which we are interested in.

Here, we investigate the hydrostatic equilibrium state after the dynamical collapse. We assume that the distribution of galaxies in a cluster does not change through the collapse of the gas, that is, the distribution is given by Eq. (1) irrespective of time.

If the dissipative process of energy is neglected in the course of the dynamical collapse, the total energy of the gas is conserved between the initial state and the final equilibrium state. The conservation of energy is written as
where $U_i$ and $U_f$ are the thermal energies of the gas in the initial and final states and $\psi_i$ and $\psi_f$ are the gravitational potential energies of the gas due to galaxies in the cluster, respectively. Here we have neglected the kinetic energy of the gas in the initial state and the gravitational potential energy of the gas itself. The gravitational potential energy of the gas per unit mass at a distance $x$ in the gravitational field of galaxies is given by

$$\Phi(x) = \int_{r}^{\infty} \frac{G M_m(r)}{r^2} dr = \frac{3 G M_m}{a} \frac{\ln (x + \sqrt{1 + x^2})}{x}.$$  

(12)

As for the initial state, for simplicity, we assume a uniform gas sphere of radius $r_{bi}(x_{bi})$ at rest with density $\rho_i$. Using Eq. (12), the gravitational potential energy $\psi_i$ in the initial state is obtained as

$$\psi_i = \int_{0}^{r_{bi}} \Phi(r/a) \cdot 4\pi r^2 \rho_i dr$$

$$= \frac{G M_m \rho_i}{a} \frac{M_{core}^i J(x_{bi})}{a},$$

(13)

where

$$M_{core}^i = \frac{4}{3} \pi a^3 \rho_i$$

(14)

and

$$J(x_{bi}) = 9 \left\{ \frac{x_{bi}^2}{2} + \frac{1}{4} \right\} \ln (x_{bi} + \sqrt{1 + x_{bi}^2}) - \frac{1}{4} x_{bi} \sqrt{1 + x_{bi}^2}.$$  

(15)

Here we assume that the final equilibrium state after the bounce is represented by the polytropic solutions obtained in § 2. In terms of Eqs. (7) and (12), the gravitational potential energy of the gas in the final state is given by

$$\psi_f = \int_{0}^{r_{bf}} \Phi(r/a) \cdot 4\pi r^2 \rho_f dr$$

$$= \frac{G M_m \rho_f}{a} \frac{M_{core}^f I_s(A, N)}{a},$$

(16)

where

$$M_{core}^f = \frac{4}{3} \pi a^3 \rho_0$$

(17)

and

$$I_s(A, N) = 9 \int_{0}^{A} \left[ 1 + \frac{A}{N+1} \frac{\ln (x + \sqrt{1 + x^2})}{x} - \frac{A}{N+1} \right]^x$$

$$\times \ln (x + \sqrt{1 + x^2}) \cdot x \, dx.$$  

(18)
In terms of Eqs. (5)–(7), the thermal energy of the gas in the final state is given by

$$U_f = \int_{r_0}^{r_f} \frac{3}{2} \frac{kT_0}{\mu H} 4\pi r^2 dr = \frac{3}{2} \frac{kT_0}{\mu H} M_{\text{core}}' \cdot I_3(A, N),$$

(19)

where

$$I_3(A, N) = 3 \int_{r_0}^{r_{bs}} \left[ 1 \cdot \frac{A}{N+1} \ln \left( \frac{x + \sqrt{1 + x^2}}{x} \right) \frac{A}{N+1} \right]^{x+1} x^2 dx. \quad (20)$$

If there is no ejection of the gas from the cluster at the time of the bounce, the total mass of the gas $M$ is also conserved between the initial and final states:

$$M_{\text{core}} x_b^i = M_{\text{core}} I_1(A, N) = M,$$

(21)

where

$$I_1(A, N) = 3 \int_{r_0}^{r_{bs}} \left[ 1 \cdot \frac{A}{N+1} \ln \left( \frac{x + \sqrt{1 + x^2}}{x} \right) \frac{A}{N+1} \right]^{x} x^2 dx.$$

(22)

Substituting Eqs. (13), (16) and (19) into Eq. (11) and using Eq. (21), we have

$$\frac{9}{2} \frac{1}{A} \frac{I_1 - I_3}{I_1} = - \frac{J(x_b)}{x_b^3}.$$  

(23)

where the thermal energy of the gas in the initial state is neglected, since the initial density and temperature are very low. If a polytropic solution is assumed for the final state, the value of the parameter $A$ of the final equilibrium state can be calculated from Eq. (23) as a function of the radius $x_b$ of the initial state. In Fig. 1 are shown the relations between the parameters $A$ and $x_b$ for the various polytropic indices. As is seen from Fig. 1, the value of $A$ strongly depends on $x_b$ in the range $x_b \leq 15$, but in the range $x_b \geq 15$, $A$ is determined almost independently of $x_b$. This is due to the fact that the initial gravitational potential energy on the right-hand side of Eq. (23) decreases very rapidly with increasing $x_b$ in the range $x_b \leq 15$ but varies slowly in the range $x_b \geq 15$. It is noted that the value of $A$ gradually approaches $N+1$ as $x_b$ becomes larger. Filled circles in Fig. 1 indicate critical values of $x_b$, where $A$ satisfies the relation $A=N+1$. On these circles, if $N>3$, the total masses of polytropic gas spheres remain finite though outer boundaries of gas spheres become infinite. For the $x_b$'s larger than the critical values no equilibrium solutions for the final state exist, since initial total energies become larger than possible maximum energies of final equilibrium states with finite outer boundaries (or infinite boundaries just discussed).
§ 4. Application to the Coma cluster

Here, we apply the results of the above considerations to the X-ray source in Coma cluster. As for the parameters for the distribution of galaxies in Coma cluster, we use $a=6.4$ and $\rho_{m0}=1.9 \times 10^{-25} \text{ gm/cm}^3$, which are obtained from the data by Rood et al.\textsuperscript{12} with the Hubble constant of $H_0=50 \text{ km/sec/Mpc}$. The composition of the gas is taken as 90% Hydrogen and 10% Helium by number.

As shown in the previous section, the value of the parameter $A$ in the final equilibrium state is almost independent of the initial radius of the gas sphere if $x_{bi}>15$, but depends on the polytropic index. Taking this fact into account, we study particularly the cases with $x_{bf} \approx 30$ ($r_{bi} \approx 200'$). In Table I are shown the

<table>
<thead>
<tr>
<th>$N$</th>
<th>$A$</th>
<th>$x_{bf}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.3</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>4.2</td>
<td>137</td>
</tr>
<tr>
<td>4</td>
<td>5.0</td>
<td>951</td>
</tr>
</tbody>
</table>

As shown in Fig. 1, the number on each line is the polytropic index assumed for the final state. Filled circles indicate the critical values of $x_{bf}$ beyond which Eq. (23) cannot have the equilibrium solution.

Fig. 2. Surface brightness distributions of X-ray radiations emitted from the polytropic gas spheres. The abscissa $\theta$ is the angular distance from the center and the ordinate is the X-ray surface brightness $\gamma(\theta)$ defined by Eq. (24). Open circles indicate the X-ray surface brightness distribution of Coma cluster obtained from the best-fit model by Lea et al.\textsuperscript{13}
values of $A$ and $x_M$ derived from Eqs. (23) and (7) for the polytropic indices $N=1$, 3 and 4, respectively. In Fig. 2 we show the surface brightness distribution of X-ray radiations from the hot plasma, which is distinguished by a set of parameters $N$ and $A$ as shown in Table I. The abscissa is the angular distance from the center of the Coma cluster and the ordinate is the X-ray surface brightness $\eta(\theta)$ defined by

$$\eta(\theta) = \frac{\int_0^\theta \varepsilon(\theta)d\theta}{\int_0^\infty \varepsilon(\theta)d\theta} \times 100 \%, \quad (\%) \quad (24)$$

where $\varepsilon(\theta)$ is the free-free emission rate at angle $\theta$ in ergs/radian/sec.

It is to be noticed that X-ray surface brightness distributions thus determined must be compared with the observational data corrected to the finite resolution of X-ray detectors. Lea et al.\textsuperscript{13} have fitted their models, which assume the density distribution of the hot plasma, to the observed X-ray source profile, taking account of this correction. Their best-fit model is represented in Fig. 2 by open circles. Among the polytropic indices considered here, the case $N=3$ ($A=4.2$) is seen to fit their result. Substituting the values of the cluster parameters $a$ and $\rho_{10}$ and $A=4.2$ (for $N=3$) into Eq. (8), the central temperature of the intracluster gas is obtained as $kT_c=16$ KeV. This temperature is comparable to the bremsstrahlung temperature, $12 \pm 12, -4$ KeV, which is given from the spectral analysis\textsuperscript{14} of the X-ray data with the energy-dependent Gaunt factor.

The total X-ray luminosity $L_x$ from a polytropic gas sphere is given by

$$L_x = \int_{r_c}^{\infty} \varepsilon_{ff} \cdot 4\pi r^2 dr$$

$$= \varepsilon_{ff} \cdot \frac{4}{3} \pi a^3 \cdot 3 \int_0^{2\pi} 1 + \frac{A}{N+1} \ln(1 + x^2) - \frac{A}{N+1} x^2 \cdot e^{-x^2}$$

$$\times x^2 dx, \quad (25)$$

where the rate of free-free emission is

$$\varepsilon_{ff} = 4.8 \times 10^{10} \rho T^{1.5}, \quad \text{ergs/sec/cm}^3 \quad (26)$$

and $\varepsilon_{ff}$ is the value of $\varepsilon_{ff}$ at the center of the cluster. In order for Eq. (25) with $N=3$ and $A=4.2$ to give the observed luminosity $5.7 \times 10^{44}$ ergs/sec\textsuperscript{15} from the Coma cluster, the central density of the hot intracluster gas is $\rho_c=2.8 \times 10^{-27}$ gm/cm$^3$. Substituting this density into Eq. (17) and using Eqs. (21) and (22), we have $M=5.7 \times 10^{44}$ $M_\odot$ for the total mass of the intracluster gas in the Coma cluster.

§ 5. Discussion

In this section the validity of basic assumptions and approximations used in the previous sections is examined, especially on the application to the X-ray source
of the Coma cluster. The physical parameters determined from the fitting of our model to X-ray observations are listed in Table II, where assumed parameters are \( x_{BI} \sim 30 \) and \( N = 3 \).

Table II. Physical parameters for the X-ray source in Coma cluster, where \( x_{BI} \sim 30 \) and \( N = 3 \) are assumed.

<table>
<thead>
<tr>
<th>( T_0 ) (°K)</th>
<th>( \rho_0 ) (gm/cm(^3))</th>
<th>( M ) (( M_\odot ))</th>
<th>( M_\odot(x_{BI}) ) (( M_\odot ))</th>
<th>( \tau_{fr} ) (year)</th>
<th>( \tau_{cond} ) (year)</th>
<th>( \tau_{dy}n ) (year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8x10^8</td>
<td>2.8x10^{-17}</td>
<td>5.7x10^{14}</td>
<td>2.8x10^{15}</td>
<td>6.3x10^{10}a)</td>
<td>~10^4</td>
<td>5.1x10^6</td>
</tr>
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</table>

\( a) \) Value of \( \tau_{fr} \) estimated at the center of the cluster.

As is shown in Table II, the mass of galaxies \( M_\odot(x_{BI}) \) within a sphere of radius \( x_{BI} \) is larger by a factor of 5 than the total mass of the intracluster gas. Therefore, the gravity of the system is mainly determined from the distribution of galaxies, and the contribution of gas to the gravity can safely be neglected.

The thermal bremsstrahlung emission, the heat conduction and the ejection of gas at the bounce are considered to be the main energy loss mechanisms from the cluster. Cooling time scales, \( \tau_{fr} \) and \( \tau_{cond} \) due to thermal bremsstrahlung and heat conduction estimated using the Coma cluster parameters are also listed in Table II. These time scales may be compared with the dynamical time scale at the outer boundary of the initial distribution of gas, which is given by

\[
\tau_{dy}n \approx \frac{x_{BI}^{-2}}{\sqrt{\ln(x_{BI} + \sqrt{1 + x_{BI}^2})}} \frac{1}{\sqrt{8\pi G \rho_{mg}}}.
\]  

One can see from Table II that the bremsstrahlung cooling time is much longer than the Hubble time \((\sim 10^{10} \text{ years})\) so that the energy loss due to this process can thoroughly be neglected. On the other hand, the energy loss due to heat conduction cannot be neglected even in the period of the dynamical time scale. However, if closed or small scale magnetic fields of \( B > 10^{-18} \) gauss are embedded in the intracluster gas, \( \tau_{cond} \) becomes longer than the Hubble time.

According to Takahara et al.,\(^9\), the outermost shell of the gas still moves outward at the termination of their calculation. However, in most cases we are interested in, that shell is found to be bound in the gravitational field of galaxies. The ejection of gas from the cluster after the bounce can thoroughly be neglected, so that the conservation laws of mass and energy can safely be used.

Throughout our calculations we have applied the energy conservation law only to the gas under the assumption that the distribution of galaxies does not change with time. However, in real clusters of galaxies it is quite likely that the distribution of galaxies changes largely with time, especially in the initial collapsing phase.\(^{10}\) If this is the case, the energy conservation law must be applied to the whole system which includes both gas and galaxies. However, the basic features
of the final equilibrium state discussed in this paper may be preserved without serious modifications. In order to derive a definite conclusion it is necessary to investigate the dynamical behavior of a composite system of a gas and galaxies, where the distribution of galaxies is, of course, described by hydrodynamic equations.

References