Formation and Interaction of Sonic-Langmuir Solitons

—Inverse Scattering Method—

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We consider behaviour of sonic-Langmuir solitons which are Langmuir oscillations trapped in regions of reduced plasma density caused by the ponderomotive force due to a high-frequency field. In particular, the formation and the interaction of solitons are studied by the inverse scattering technique in the case of the Langmuir waves coupled with ion-acoustic waves propagating in one-direction.

§ 1. Introduction

In Langmuir turbulence the energy of wave field is concentrated in the long-wave part of the spectrum through the nonlinear processes such as nonlinear Landau damping by the electrons and ions, decay of the Langmuir waves with production of ion sound waves and four-plasmon interaction. Since the linear damping mechanisms have little effect on the long waves, what mechanism makes the wave energy dissipate to determine the turbulent spectrum comes into question. In this context, Vedenov and Rudakov showed that Langmuir turbulence with sufficiently long wavelength is unstable to spatial modulation. The nonlinear stage of this instability was examined by Zakharov, who proposed that the three-dimensional focusing of Langmuir waves called “collapse” is the main energy dissipation mechanism of Langmuir turbulence in the long-wave region.

One-dimensional self-modulation for Langmuir waves does not cause their collapse, but leads to soliton formation. Therefore, one-dimensional Langmuir turbulence may be described by an ensemble of solitons. In this connection, Degtyarev, Makhan'kov and Rudakov proposed a new theory on one-dimensional Langmuir turbulence based on the results of numerical computations.

The purpose of this paper is to deal strictly with the interactions of one-dimensional Langmuir waves with sound waves propagating in one-direction, in particular, with the phenomena being concerned with the sonic-Langmuir solitons.

The system of equations for the ion sound wave under the action of the ponderomotive force due to high-frequency field and for the Langmuir wave was formulated by Zakharov:

\[ i \frac{\partial E}{\partial t} + \frac{1}{2} \frac{\partial^2 E}{\partial x^2} - nE = 0, \]  

(1)
\[
\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} - 2 \frac{\partial |E|^2}{\partial x} = 0, \tag{2}
\]

where \( E e^{-i\omega \tau} \) is the normalized electric field of the Langmuir oscillation and \( n \) the normalized density perturbation. The spatial variable \( x \) and the time variable \( t \) are also normalized appropriately. This system of equations has the soliton solution,

\[
E = \sqrt{1 - \nu^2} \frac{1}{2} N \text{sech} \left[ N(x - \nu t) \right] e^{i \nu t + \frac{1}{2} (N^2 - \nu^2) t^2}, \tag{3a}
\]

\[
n = -N^2 \text{sech}^2 \left[ N(x - \nu t) \right]. \tag{3b}
\]

Degtyarev et al.\(^8\) found many interesting phenomena by solving the system of Eqs. (1) and (2) numerically. These are, for example, the soliton formation from a given initial disturbance \((E \neq 0, n = 0)\), soliton scattering with the emission of ion sound waves, the fusion of two solitons and the fission of a soliton by the absorption of ion sound waves. According to this numerical computation Langmuir solitons are created or annihilated through the interactions with ion sound. The selection rule with respect to such interactions of Langmuir soliton with ion sound was investigated by Thornhill and ter Haar\(^9\) with the help of the conservation laws for the system of Eqs. (1) and (2).

The interaction of Langmuir solitons with ion sound can be strictly treated by simplifying Eqs. (1) and (2). We consider the ion sound wave propagating in only one-direction, for example, in the positive \( x \)-direction, then we may assume that

\[
\frac{\partial n}{\partial t} = -\frac{\partial n}{\partial x}, \tag{4}
\]

where the sound speed is normalized to unity. Using Eq. (4) we obtain

\[
\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \left( \frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} \right) \approx -2 \frac{\partial}{\partial x} \left( \frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} \right).
\]

It follows from this that Eq. (2) can be rewritten as

\[
\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} + \frac{\partial |E|^2}{\partial x} = 0. \tag{5}
\]

The soliton solution of Eqs. (1) and (5) can be obtained by replacing \( \sqrt{1 - \nu^2} \) in Eq. (3a) with \( \sqrt{1 - \nu} \) on account of \( \nu \approx 1 \). Applying an asymptotic perturbation method to Eqs. (1) and (5), Karpman\(^6\) studied the dynamics of soliton formation. In the present paper we solve exactly Eqs. (1) and (5) by means of the inverse scattering method which has been recently used in solving nonlinear evolution equations.

In § 2 the eigenvalue equations corresponding to Eqs. (1) and (5) are given and the properties of the scattering problem are described. In § 3 the Gel’fand-Levitan equations are derived and it is shown how the solutions to Eqs. (1) and (5) are represented by the scattering matrix. The zeros of a diagonal element of

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the scattering matrix correspond to the soliton solutions. The N-soliton solutions are discussed in § 4. The phase shifts due to the collisions of solitons are found. The phenomena such as soliton formation and pick-up process of Langmuir wave from a soliton by negative amplitude ion sound are investigated in § 5. Section 6 is assigned to the derivation of the polynomial type of conservation laws.

§ 2. Eigenvalue problem associated with Eqs. (1) and (5)

The inverse scattering method to solve nonlinear evolution equations was first found by Gardner, Greene, Kruskal and Miura7) for the case of the Korteweg-de Vries equation and was generalized by Lax.8) In a previous paper9) the present authors extended the work of Ablowitz, Kaup, Newell and Segur10) to the case of the third order eigenvalue problem. They found the eigenvalue problem associated with Eqs. (1) and (5) in further examinations.

Consider the following eigenvalue equation with \( n(x, t) \) and \( \phi(x, t) \) as the potentials,

\[
\frac{\partial f}{\partial x} + i \frac{2\zeta}{\zeta} \begin{pmatrix} n & -2i\zeta\phi^* & n \\ i\phi & 0 & i\phi \\ -n & 2i\zeta\phi^* & -n \end{pmatrix} f = \begin{pmatrix} 3i\zeta & 0 & 0 \\ 0 & i\zeta & 0 \\ 0 & 0 & -i\zeta \end{pmatrix} f,
\]

(6)

where \( f \) is a column vector with three components, \( \zeta \) the eigenvalue,

\[
\phi(x, t) = E(x, t) e^{i(\zeta t - x)},
\]

(7)

and the asterisk denotes the complex conjugate. Suppose that time evolution of the eigenfunction \( f \) is given by

\[
\frac{\partial f}{\partial t} = \begin{pmatrix} i(2\zeta^2/3 - 2\zeta) & 0 & 0 \\ 0 & -4i\zeta^2/3 & 0 \\ 0 & 0 & i(2\zeta^2/3 + 2\zeta) \end{pmatrix} f + Df,
\]

(8)

\[
D = i \begin{pmatrix} n + |\phi|^2/2 & -2i\zeta \left( -\zeta\phi^* + i\phi_e^* + \phi^* \right) & n + |\phi|^2 \\ -i \left( \zeta\phi + i\phi_e - \phi \right) & 0 & -i \left( -\zeta\phi + i\phi_e - \phi \right) \\ -i \left( n + |\phi|^2/2 \right) & 2i\zeta \left( \zeta\phi^* + i\phi_e^* + \phi^* \right) & -\left( n + |\phi|^2/2 \right) \end{pmatrix}.
\]

(9)

By cross differentiation of Eqs. (6) and (8) we then obtain the system of Eqs. (1) and (5) for \( n \) and \( E \) as the conditions of time invariance of \( \zeta \). The initial value problem for Eqs. (1) and (5) is therefore reduced to the inverse scattering problem for Eq. (6).

The inverse scattering problem of the third order eigenvalue equation was
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solved for a special case by Kaup. We proceed in parallel with Kaup’s work but take care of the singularity $\zeta^{-1}$ in Eq. (6).

It is supposed in this paper that $n$ and $E$ (i.e., $\phi$) tend to zero as $x \to \pm \infty$. We define, for real $\zeta$, the Jost functions $\phi^{(i)}(x, \zeta)$ and $\psi^{(i)}(x, \zeta)$ ($i = 1, 2, 3$) which are solutions of Eq. (6) with the boundary conditions

$$
\phi^{(i)}(x, \zeta) e^{-\zeta i x} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi^{(i)}(x, \zeta) e^{-\zeta i x} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{as } x \to -\infty,
$$

$$
\phi^{(i)}(x, \zeta) e^{-\zeta i x} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi^{(i)}(x, \zeta) e^{-\zeta i x} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{as } x \to +\infty.
$$

We now introduce the Wronskian by

$$
W(f, g, h) = \det \begin{pmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{pmatrix}.
$$

Evidently, $W(f, g, h) \neq 0$ is the necessary and sufficient condition for $f$, $g$ and $h$ to be linearly independent. When $f$, $g$ and $h$ are solutions of Eq. (6), it is shown that

$$
\frac{\partial W}{\partial x} = 3i\zeta W.
$$

From Eqs. (10) $\sim$ (13) we get

$$
W(\phi^{(i)}, \phi^{(e)}, \phi^{(a)}) = W(\psi^{(i)}, \psi^{(e)}, \psi^{(a)}) = e^{3i\zeta x}.
$$

Therefore, $\{\phi^{(j)}(x, \zeta)\}$ and $\{\psi^{(j)}(x, \zeta)\}$ are a set of three linearly independent solutions of Eq. (6). Consequently, $\phi^{(j)}(x, \zeta)$ can be represented by a linear combination of $\{\phi^{(j)}(x, \zeta)\}$, that is,

$$
\phi^{(j)}(x, \zeta) = \sum_{k=1}^{n} a_{jk}(\zeta) \psi^{(k)}(x, \zeta),
$$

where $[a_{jk}(\zeta)]$ is called the scattering matrix. From Eqs. (14) and (15) we obtain

$$
\det(a_{jk}(\zeta)) = 1.
$$

Then, Eq. (15) can be rewritten as

$$
\phi^{(o)}(x, \zeta) = \sum_{j=1}^{n} b_{j}(\zeta) \phi^{(j)}(x, \zeta),
$$

where $[b_{j}(\zeta)]$ is the inverse matrix of $[a_{jk}(\zeta)]$. 
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By using the boundary conditions (10) and (11), it follows from Eqs. (15) and (17) that

\[
\begin{align*}
\alpha_{11}(\zeta) &= \lim_{x \to -\infty} \phi^{(1)}_1 e^{-3i\zeta x}, \\
\alpha_{33}(\zeta) &= \lim_{x \to \infty} \phi^{(3)}_3 e^{i\zeta x}, \\
\beta_{11}(\zeta) &= \lim_{x \to -\infty} \phi^{(1)}_1 e^{-3i\zeta x}, \\
\beta_{33}(\zeta) &= \lim_{x \to \infty} \phi^{(3)}_3 e^{i\zeta x}.
\end{align*}
\]

(19)

If \( n(x) \) and \( \phi(x) \) tend to zero faster than \( |x|^{-1} \) as \( |x| \to \infty \), we can show the following analytical properties by the Neumann series expansions of the Jost functions \( \phi^{(0)}(x, \zeta) \) and \( \phi^{(0)}(x, \zeta); \phi^{(1)}_1 e^{-3i\zeta x}, \phi^{(2)} e^{i\zeta x}, a_{11}(\zeta) \) and \( b_{33}(\zeta) \) are analytic functions of \( \zeta \) in the lower half \( \zeta \)-plane \( (\text{Im}(\zeta) < 0) \) and \( \phi^{(0)} e^{-3i\zeta x}, \phi^{(2)} e^{i\zeta x}, a_{33}(\zeta) \) and \( b_{11}(\zeta) \) are analytic functions of \( \zeta \) in the upper half \( \zeta \)-plane \( (\text{Im}(\zeta) > 0) \).

The asymptotic properties for large \( |\zeta| \) are obtained by the asymptotic expansions of the solutions of Eq. (6):

\[
\begin{align*}
\phi^{(1)}_1 e^{-3i\zeta x} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + O(|\zeta|^{-1}) \quad \text{for } \text{Im}(\zeta) > 0, \\
\phi^{(2)} e^{i\zeta x} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(|\zeta|^{-1}) \quad \text{for } \text{Im}(\zeta) < 0, \\
\phi^{(1)}_1 e^{-3i\zeta x} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + O(|\zeta|^{-1}) \quad \text{for } \text{Im}(\zeta) < 0, \\
\phi^{(2)} e^{i\zeta x} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(|\zeta|^{-1}) \quad \text{for } \text{Im}(\zeta) > 0.
\end{align*}
\]

(20)

From Eqs. (19) and (20), we further obtain

\[
\begin{align*}
\alpha_{33}(\zeta) &= 1 + O(|\zeta|^{-1}), \\
\beta_{11}(\zeta) &= 1 + O(|\zeta|^{-1}) \quad \text{for } \text{Im}(\zeta) > 0, \\
\alpha_{11}(\zeta) &= 1 + O(|\zeta|^{-1}), \\
\beta_{33}(\zeta) &= 1 + O(|\zeta|^{-1}) \quad \text{for } \text{Im}(\zeta) < 0.
\end{align*}
\]

(21)

Though it is difficult to write explicitly the asymptotic forms of the Jost functions for small \( |\zeta| \), it is easily seen that they have merely at most the simple pole at \( \zeta = 0 \) due to the singularity \( \zeta^{-1} \) in the potential matrix of Eq. (6).

The analytical properties of the Jost functions \( \phi^{(0)}(x, \zeta) \) and \( \phi^{(0)}(x, \zeta) \) are not simple. We introduce the functions \( \chi(x, \zeta) \) and \( \chi(x, \zeta) \) with the definite analytical and asymptotic properties, in place of \( \phi^{(0)}(x, \zeta) \) and \( \phi^{(0)}(x, \zeta) \) according to Kaup. Consider the adjoint equation of Eq. (6),

\[
\frac{\partial f^A}{\partial x} - \frac{i}{2\zeta} \begin{pmatrix} n & 3i\phi & 3n \\ -2i\zeta\phi^* & 0 & 0 \\ -n/3 & -i\phi & -n \end{pmatrix} f^A = \begin{pmatrix} -3\zeta & 0 & 0 \\ 0 & -i\zeta & 0 \\ 0 & 0 & i\zeta \end{pmatrix} f^A.
\]

(22)
We define the solutions of Eq. (22), $\phi^{(1)}(x, \zeta)$ and $\psi^{(1)}(x, \zeta)$, satisfying the boundary conditions,

$$\phi^{(1)}(x, \zeta) e^{\imath \zeta x} \to \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad \psi^{(1)}(x, \zeta) e^{\imath \zeta x} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \phi^{(1)}(x, \zeta) e^{-\imath \zeta x} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as $x \to -\infty$.  \hspace{1cm} (23)

$$\phi^{(1)}(x, \zeta) e^{\imath \zeta x} \to \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad \psi^{(1)}(x, \zeta) e^{\imath \zeta x} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi^{(1)}(x, \zeta) e^{-\imath \zeta x} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as $x \to \infty$. \hspace{1cm} (24)

Then it can be shown that $\phi^{(1)}(x, \zeta) e^{\imath \zeta x}$ and $\psi^{(1)}(x, \zeta) e^{-\imath \zeta x}$ are analytic functions of $\zeta$ in the upper half $\zeta$-plane ($\text{Im}(\zeta) > 0$) and $\phi^{(1)}(x, \zeta) e^{-\imath \zeta x}$ and $\psi^{(1)}(x, \zeta) e^{\imath \zeta x}$ are analytic functions of $\zeta$ in the lower half $\zeta$-plane ($\text{Im}(\zeta) < 0$).

The Jost functions $\phi^{(1)}$ and $\psi^{(1)}$ can be represented in terms of $\phi^{(0)}$ and $\psi^{(0)}$:

$$\phi^{(1)} = \frac{1}{2} \sum_{j,k=1}^3 e_{ijk} \sum_{m,p=1}^3 \varepsilon_{mpq} \phi^{(j)}(x, \zeta) \phi^{(k)}(x, \zeta) e^{\imath \zeta x},$$

$$\psi^{(1)} = \frac{1}{2} \sum_{j,k=1}^3 e_{ijk} \sum_{m,p=1}^3 \varepsilon_{mpq} \psi^{(j)}(x, \zeta) \psi^{(k)}(x, \zeta) e^{\imath \zeta x}.$$ \hspace{1cm} (25)

where, $e_{ijk}$ is the alternating tensor and $\alpha_1 = 1/3$, $\alpha_2 = 1$ and $\alpha_3 = -1$. Conversely, it holds that

$$\phi^{(1)} = \frac{1}{2\alpha_n} \sum_{j,k=1}^3 e_{ijk} \sum_{m,p=1}^3 \varepsilon_{mpq} \phi^{(j)}(x, \zeta) \phi^{(k)}(x, \zeta) e^{-\imath \zeta x},$$

$$\psi^{(1)} = \frac{1}{2\alpha_n} \sum_{j,k=1}^3 e_{ijk} \sum_{m,p=1}^3 \varepsilon_{mpq} \psi^{(j)}(x, \zeta) \psi^{(k)}(x, \zeta) e^{-\imath \zeta x}.$$ \hspace{1cm} (26)

Substitution of Eqs. (15) and (17) into Eqs. (26) shows that

$$\phi^{(1)}(x, \zeta) = \sum_{k=1}^3 B_{kj} \psi^{(k)}(x, \zeta),$$

$$\psi^{(1)}(x, \zeta) = \sum_{k=1}^3 C_{kj} \phi^{(k)}(x, \zeta).$$ \hspace{1cm} (27)

We now introduce $\chi$ and $\overline{\chi}$ by the relations

$$\chi_n = \sum_{m,p=1}^3 \varepsilon_{mpq} \phi^{(1)}(x, \zeta) e^{\imath \zeta x},$$

$$\overline{\chi}_n = \sum_{m,p=1}^3 \varepsilon_{mpq} \psi^{(1)}(x, \zeta) e^{\imath \zeta x}.$$ \hspace{1cm} (28)

Then, $\chi_n$ and $\overline{\chi}_n$ are analytic functions of $\zeta$ in the upper and the lower half $\zeta$-plane respectively and $\chi e^{\imath \zeta x}$ and $\overline{\chi} e^{\imath \zeta x}$ are proved to be solutions of Eqs. (6). It follows from Eqs. (25), (27), (28) and (29) that
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\[ \chi(x, \zeta) = e^{-i\zeta x}(b_{21}(\zeta)\phi^{(1)}(x, \zeta) - b_{11}(\zeta)\phi^{(2)}(x, \zeta)), \tag{30a} \]

\[ \bar{\chi}(x, \zeta) = e^{-i\zeta x}(b_{23}(\zeta)\phi^{(3)}(x, \zeta) - b_{33}(\zeta)\phi^{(2)}(x, \zeta)). \tag{30b} \]

By making use of the asymptotic forms of \( \phi^{(1)} \) and \( \phi^{(2)} \) for large \(|\zeta|\), the asymptotic expressions of \( \chi \) and \( \bar{\chi} \) are obtained:

\[ \chi(x, \zeta) = -\left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + O(|\zeta|^{-1}) \text{ for } \text{Im}(\zeta) > 0, \tag{31a} \]

\[ \bar{\chi}(x, \zeta) = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + O(|\zeta|^{-1}) \text{ for } \text{Im}(\zeta) < 0. \tag{31b} \]

These relations will be used to obtain an integral representation of \( \psi^{(0)} e^{-i\zeta x} \) in the next section.

Next, we present two kinds of symmetrical property for the scattering matrix. When \( f(x, \zeta) \) is a solution of Eq. (6), \( f_A(x, \zeta) = B(\zeta)[f(x, \zeta^*)]^\ast \) is a solution of Eq. (22), where

\[ B(\zeta) = \left( \begin{array}{ccc} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{array} \right) = \left( \begin{array}{ccc} 3 & 0 & 0 \\ 0 & 2\zeta & 0 \\ 0 & 0 & 1 \end{array} \right). \tag{32} \]

Taking account of the boundary conditions (10), (11), (23) and (24), we have

\[ \phi_A^{(m)}(x, \zeta) = B(\zeta) [\phi^{(m)}(x, \zeta^*)]^\ast = \alpha_m \beta_m \phi^{(m)A}(x, \zeta), \]

\[ \psi_A^{(n)}(x, \zeta) = B(\zeta) [\psi^{(n)}(x, \zeta^*)]^\ast = \alpha_n \beta_n \psi^{(n)A}(x, \zeta). \]

It follows from these and Eqs. (15) and (27a) that

\[ b_{nm}(\zeta) = \frac{\gamma_m}{\gamma_m} [a_{mn}(\zeta^*)]^\ast, \tag{33} \]

where \( \gamma_m = \alpha_m \beta_m, \gamma_1 = 1, \gamma_2 = 2\zeta, \gamma_3 = -1. \)

It is easily seen that when

\[ f(x, \zeta) = \left( \begin{array}{c} f_1(x, \zeta) \\ f_2(x, \zeta) \\ f_3(x, \zeta) \end{array} \right) \]

is a solution of Eq. (6), then

\[ \tilde{f}(x, \zeta) = \left( \begin{array}{c} f_3(x, -\zeta) \\ -f_2(x, -\zeta) \\ f_1(x, -\zeta) \end{array} \right) e^{2i\xi x} \]

is also a solution of Eq. (6). The considerations on the boundary conditions lead to the relations
We get from Eqs. (19) and (34) the second symmetrical properties:

\begin{align}
\psi^{(0)}(x, \zeta) &= -\tilde{\psi}^{(0)}(x, \zeta), \\
\phi^{(0)}(x, \zeta) &= -\tilde{\phi}^{(0)}(x, \zeta), \\
\psi^{(1)}(x, \zeta) &= \tilde{\psi}^{(1)}(x, \zeta), \\
\phi^{(1)}(x, \zeta) &= -\tilde{\phi}^{(1)}(x, \zeta).
\end{align}

We assume for simplicity the potentials to be on compact support, so that all functions are analytic for all \( \zeta \) in the complex \( \zeta \)-plane except for \( \zeta = 0 \). In the more general case of non-compact support, it is needed only that all the contour integrals in the representation obtained so are reduced to integrals along the real axis plus all contributions due to any poles. We define the contour \( \mathcal{C} \) to be the contour in the complex \( \zeta \)-plane, extending from \(-\infty + i0^+\) to \(0^-\), then from \(0^+\) to \(+\infty + i0^+\) and passing above all zeros of \(a_{23}\) and \(b_{13}\). Similarly, \( \overline{\mathcal{C}} \) is the contour extending from \(-\infty + i0^-\) to \(0^+\), then from \(0^-\) to \(+\infty + i0^-\) and passing under all zeros of \(a_{13}\) and \(b_{23}\) (see Fig. 1). Consider the contour integral.
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for \( \zeta \) above \( \overline{C} \); its value is \( i\pi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) on account of Eqs. (20c) and (21b). Replacing \( \phi^{(1)} \) by \( \sum_{k} a_{k}\psi^{(k)} \) and by using Eqs. (20a) and (34c), we obtain

\[
\phi^{(1)}(x, \zeta) e^{-i\zeta x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2\pi i} \int \frac{d\zeta'}{\zeta' - \zeta} \frac{a_{12}(\zeta')}{a_{11}(\zeta')} \phi^{(2)}(x, \zeta') e^{-i\zeta' x} + \frac{1}{2\pi i} \int \frac{d\zeta'}{\zeta' - \zeta} \frac{a_{13}(\zeta')}{a_{11}(\zeta')} \overline{\phi^{(1)}}(x, \zeta') e^{-i\zeta' x}. \tag{37a}
\]

Similarly, considering

\[
\left[ \begin{pmatrix} d_{\zeta'} \\ -d_{\zeta} \end{pmatrix} \right] \frac{d\zeta'}{\zeta' - \zeta} \psi^{(2)}(x, \zeta') e^{-i\zeta x}
\]

for \( \zeta \) between \( C \) and \( \overline{C} \) and using Eqs. (30), (31) and (34c), we get

\[
\psi^{(2)}(x, \zeta) e^{-i\zeta x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{1}{2\pi i} \int \frac{d\zeta'}{\zeta' - \zeta} \frac{b_{13}(\zeta')}{b_{11}(\zeta')} \psi^{(1)}(x, \zeta') e^{-i\zeta' x} + \frac{1}{2\pi i} \int \frac{d\zeta'}{\zeta' - \zeta} \frac{b_{23}(\zeta')}{b_{21}(\zeta')} \overline{\psi^{(1)}}(x, \zeta') e^{-i\zeta' x}. \tag{37b}
\]

We now assume that \( \psi^{(1)} \) can be represented by

\[
\psi^{(1)}(x, \zeta) e^{-i\zeta x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int s \left[ K(x, s) + \frac{1}{\zeta} L(x, s) \right] e^{i\zeta (x-s)} ds, \tag{38}
\]

where the transformation kernels \( K \) and \( L \) are column vectors independent of \( \zeta \). The structure of the transformation kernels in Eq. (38) was suggested by Kaup \(^{10, 11}\) (see also Ref. 14). The necessary and sufficient condition for Eq. (38) to satisfy Eq. (6) are given by the following equations:

\[
K = \begin{pmatrix} K_1 \\ 0 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_1 \\ L_2 \\ -L_1 \end{pmatrix}, \tag{39}
\]

\[
\frac{\partial K_1}{\partial x} + \frac{\delta K_1}{\delta s} = 4iL_1, \tag{40a}
\]

\[
\frac{\partial L_1}{\partial x} - \frac{\partial L_1}{\partial s} = -\frac{i}{2} nK_1 - \phi^* L_2, \tag{40b}
\]

\[
\frac{\partial L_2}{\partial x} = \frac{1}{2} \phi K_1, \tag{40c}
\]
It can be proved by the method of characteristics that the solution of Eqs. (40) with the boundary conditions (41) and (42) exists and is unique.

We eliminate \( \phi(x, s) \) from Eqs. (37a) and (37b), substitute Eq. (38) and take the Fourier transform with respect to \( \xi \), in which the contour passes under \( \xi=0 \). Thus we obtain the Gel'fand-Levitan equations for \( y>x, \)

\[
K_1(x, y) + 2i \int_y^\infty L_1(x, s) \, ds + G(x, y) + \int_x^\infty L_1(x, s) \{ Q(s, y) + R(s, y) \} \, ds = 0,
\]

\[
F(x) = \frac{1}{\pi} \int d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2itx},
\]

\[
G(x, y) = -\frac{1}{2\pi^2i} \int d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2ity} \int \frac{d\zeta'}{\zeta' - \zeta} b_{11}(\zeta') e^{2ity},
\]

\[
H(x, y) = \frac{1}{\pi} \int d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2itm(x+y)}
\]

\[
+ \frac{1}{2\pi^2i} \int d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2ity} \int \frac{d\zeta'}{\zeta' - \zeta - i\epsilon} b_{11}(\zeta') e^{-2ity},
\]

\[
Q(x, y) = -\frac{1}{2\pi^2i} \int d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2ity} \int \frac{d\zeta'}{\zeta' (\zeta' - \zeta)} b_{11}(\zeta') e^{2ity},
\]

\[
R(x, y) = \frac{1}{\pi} \int d\zeta \frac{a_{12}(\zeta)}{\zeta a_{11}(\zeta)} e^{-2itm(x+y)}
\]

\[
+ \frac{1}{2\pi^2i} \int d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2ity} \int \frac{d\zeta'}{\zeta' (\zeta' - \zeta - i\epsilon)} b_{11}(\zeta') e^{-2ity},
\]

where the limit \( \epsilon \to 0^+ \) is to be taken.

We now see that a zero of \( a_{11}(\zeta) \) corresponds to a soliton solution of Eqs. (1) and (5). Let \( \zeta_0 = x - i\eta \) (\( x, \eta \); real; \( \eta > 0 \)) be a zero of \( a_{11}(\zeta) \). Then, owing to Eqs. (33) and (35), \( a_{12}(-\zeta_0) = 0, b_{11}(\zeta_0^*) = 0 \) and \( b_{12}(-\zeta_0^*) = 0 \). Here we confine ourselves to the case that the zeros of \( a_{11}(\zeta) \) are simple, so that the residue of \( a_{12}(\zeta)/a_{11}(\zeta) \) at the pole \( \zeta_0 \) is \( C = a_{12}(\zeta_0)/a_{11}(\zeta_0) \). The residues of \( b_{12}(\zeta)/b_{11}(\zeta) \)
at $\zeta = \zeta_0^*$ and of $b_{23}(\zeta)/b_{33}(\zeta)$ at $\zeta = -\zeta_0^*$ are both $2\zeta_0^*C^*$ in view of Eqs. (33) and (35). We take into account that $[b_{33}(\zeta)]$ is the inverse matrix of $[a_{33}(\zeta)]$ and Eqs. (33) and (35) to obtain

$$a_{13}(\zeta) = \frac{a_{13}(-\zeta_0^*)}{a_{11}(-\zeta_0^*)} - \frac{a_{33}(\zeta)}{a_{11}(\zeta) b_{33}(\zeta)}.$$ (45)

From this the residue of $a_{13}(\zeta)/a_{11}(\zeta)$ at $\zeta = \zeta_0$ is $-Cb_{23}(\zeta_0)/b_{33}(\zeta_0)$ provided $\zeta_0 \neq -\zeta_0^*$ (i.e., $\xi \neq 0$).

If $a_{11}(\zeta)$ possesses only one zero $\zeta_0$ and if all of the off-diagonal elements of the scattering matrix are equal to zero on the real axis, we obtain

$$F(x) = 2iCe^{-2ix},$$

$$G(x, y) = \frac{4i|C|^2\zeta_0}{\zeta_0^* - \zeta_0} e^{2i(\zeta_0^* - \zeta_0)y},$$

$$H(x, y) = -\frac{4i|C|^2\zeta_0^*}{\zeta_0^* - \zeta_0} e^{2i(\zeta_0^* - \zeta_0)y},$$

$$Q(x, y) = \frac{4i|C|^2}{\zeta_0^* - \zeta_0} e^{2i(\zeta_0^* - \zeta_0)y},$$

$$R(x, y) = \frac{4i|C|^2}{\zeta_0^* + \zeta_0} e^{2i(\zeta_0^* + \zeta_0)y},$$

where

$$C = C_0 e^{-i[\zeta_0^*(\zeta_0^* - 1)t]},$$ (C_0: const).

The Gel'fand-Levitan equations with these kernels can be reduced to easily solvable linear algebraic equations. As the result we obtain

$$n(x, t) = -\frac{16|C_0|^{1/2}e^{2i(\zeta_0^* - \zeta_0)x}}{(\zeta_0^* + \zeta_0)\left[1 - (4|C_0|^2\zeta_0)^{1/2}(\zeta_0^* - \zeta_0)^{1/2}(\zeta_0^* + \zeta_0)\right]} = -4\eta \tanh[2\eta (x - (1 - 2\xi)t) + \delta], \quad (\xi > 0)$$ (46a)

$$\phi(x, t) = \frac{4iC_0 e^{2i(\zeta_0^* - \zeta_0)x}}{1 - (4|C_0|^2\zeta_0)^{1/2}(\zeta_0^* - \zeta_0)^{1/2}(\zeta_0^* + \zeta_0)} = 2\eta \sqrt{2\xi} \tanh[2\eta (x - (1 - 2\xi)t) + \delta] e^{2i(\eta^2 - 1)t - 2i(\xi - 1)\zeta_0^* + \zeta_0}, \quad (\xi > 0)$$ (46b)

where

$$\delta = \log \frac{\sqrt{2\xi} \eta}{|C_0|}, \quad \epsilon = 0 \left[ \frac{C_0 e^{\epsilon \eta}}{|C_0^2 \epsilon|} \right].$$ (48)

These clarify the correspondence of a zero of $a_{11}(\zeta)$ with a soliton solution.

It should be noted that in the case $\xi < 0$, $n(x, t)$ and $\phi(x, t)$ are given by

$$n(x, t) = 4\eta^2 \cosh[2\eta (x - (1 - 2\xi)t) + \delta],$$ (49a)
\[ \phi(x, t) = 2\sqrt{2\xi} \cosh[2\sqrt{(1 - 2\xi)} t] + \delta_4 e^{i(\xi - 1/2) 1 - 2\xi (x - t) + \xi t}, \quad (49b) \]

where

\[ \delta_4 = \log \frac{\sqrt{2\xi \eta}}{|C_{\xi \eta}|}, \quad e^{\theta_4} = i \frac{C_{\xi \eta}}{|C_{\xi \eta}|}. \quad (50) \]

The convergence of the integrals \( \int_{-\infty}^{\infty} |n(x, t)| \, dx \) and \( \int_{-\infty}^{\infty} |\phi(x, t)| \, dx \) is not ensured because of the singularity of \( n(x, t) \) and \( \phi(x, t) \). Therefore, the inverse scattering method must not be applicable to the case of the solution (49). Nevertheless, Eqs. (49) certainly satisfy the system of Eqs. (1) and (5).

When \( \xi \) tends to zero so that the wave velocity tends to the sound velocity, the solution (46) becomes singular unless \( C \rightarrow 0 \). Zeros of \( a_{11}(\zeta) \) and \( b_{11}(\zeta) \) coalesce on the imaginary axis in the lower half \( \zeta \)-plane. In this case \( a_{11}(\zeta) \), \( b_{11}(\zeta) \), \( a_{13}(\zeta) \) and \( b_{13}(\zeta) \) may have double zeros on the imaginary axis. A special consideration is therefore required for this case. However, if \( a_{11}(\zeta) \) is identically zero, \( a_{11}(\zeta) \) has a simple zero \(-i\eta \) \( (\eta > 0) \) and \( a_{13}(\zeta) \) is equal to zero on the real axis, we can easily obtain

\[ n(x, t) = \frac{-8\eta p e^{-4\eta x}}{(1 + (p/2\eta) e^{-4\eta x})^2}, \quad (51a) \]

\[ \phi(x, t) = 0, \quad (51b) \]

where \( p \) is given by

\[ \frac{a_{13}(-i\eta)}{a_{11}(-i\eta)} = -ip, \quad p = p_0 e^{4\eta}, \quad (52) \]

and \( p \) becomes real owing to Eq. (45). If \( p_0 \) is positive, Eq. (51a) is written as

\[ n(x, t) = -4\xi \text{sech}^2[2\eta(x - t) + \delta_\eta], \quad \delta_\eta = \frac{1}{2} \log \frac{2\eta}{p_0}. \quad (53) \]

For negative \( p_0 \) we get a singular soliton like Eq. (49).

It should be noted that the solution (51) is also obtained by taking the limit \( C_\eta \rightarrow 0 \) as well as \( \xi \rightarrow 0 \) in Eqs. (46) so as for \( |C_{\xi \eta}|^2/\xi \) to be constant.

§ 4. **N-soliton solutions**

Here, we study the property of the interaction of \( N \) solitons. Let us assume that \( a_{11}(\zeta) \) has \( N \) zeros in the lower half \( \zeta \)-plane and all of the off-diagonal elements of the scattering matrix equal to zero on the real axis of the \( \zeta \)-plane, that is,

\[ a_{1j}(\zeta) = 0, \quad j = 1, 2, \ldots, N, \]

\[ \zeta_j = \xi_j - i\eta_j, \quad \eta_j > 0. \]

We also assume that all of \( \zeta_j \) are simple zeros of \( a_{11}(\zeta) \) so that the residue of
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\[ a_{12}(\zeta)/a_{11}(\zeta) \text{ at } \zeta_i \text{ is } C_i = a_{12}(\zeta_i)/a_{11}'(\zeta_i). \]
Introduce \( p_n(x), q_n(x) \) and \( r_n(x) \) by
\[
L_1(x,y) = \sum_n p_n(x) \exp(-2i\kappa_n y), \quad
L_1(x,y) = \sum_n q_n(x) \exp(-2i\kappa_n y) \quad \text{and} \quad
K_1(x,y) = \sum_n r_n(x) \exp(-2i\kappa_n y).
\]
The Gel'fand-Levitan equations (43) are reduced to the system of linear algebraic equations,

\[
p_n(x) + \frac{i|C_{n+1}^\rho|^2}{\xi_n q_n} \sum_{\rho \neq 1} e^{2i(\kappa_\rho^* - \kappa_\rho)x} p_m(x) = -2iC_n^\rho, \quad \text{(54a)}
\]

\[
q_n(x) + \xi_n r_n(x) + \frac{i|C_n^\rho|^2}{\eta_n} \sum_{\rho \neq 1} e^{2i(\kappa_\rho^* - \kappa_\rho)x} r_m(x)
+ \frac{i|C_{n+1}^\rho|^2}{\xi_n q_n} \sum_{\rho \neq 1} e^{2i(\kappa_\rho^* - \kappa_\rho)x} q_m(x)
- \frac{2|C_n^\rho|^2}{\eta_n} e^{2i\kappa^* x} = 0, \quad \text{(54b)}
\]

\[
q_n(x) + \frac{i|C_{n+1}^\rho|^2}{\xi_n q_n} \sum_{\rho \neq 1} e^{2i(\kappa_\rho^* - \kappa_\rho)x} q_m(x)
- \frac{|C_{n+1}^\rho|^2}{\xi_n q_n} \sum_{\rho \neq 1} e^{2i(\kappa_\rho^* - \kappa_\rho)x} r_m(x)
- \frac{2|C_n^\rho|^2}{\eta_n} e^{2i\kappa^* x} = 0. \quad \text{(54c)}
\]

Here, we do not write down the explicit form of the \( N \)-soliton solution, but we examine the asymptotic behaviour of the solution as \( t \to \pm \infty \). Putting \( p_n(x) \exp(-2i\kappa_n x) = \tilde{p}_n(x) \) and \( C_n = C_{n0} \exp\{2\eta_n(1-\xi_n) t - 2i(\kappa_n^2 - \eta_n^2 - \xi_n^2) t\} \), we have from Eq. (54a)

\[
\tilde{p}_n(x) + \frac{i|C_{n+1}^\rho|^2}{\xi_n q_n} e^{-4\eta_n x} \sum_{\rho \neq 1} \frac{1}{(\kappa_\rho^* - \kappa_\rho)} \tilde{p}_k(x)
= -2iC_{n0}^\rho \exp\{-2i(\kappa_\rho^* - \kappa_\rho) t + 2(\kappa_\rho^2 - \kappa_\rho^2 - \eta_n^2 t)\} e^{-2i\kappa_n^2 t} e^{-2i\kappa_n x}. \quad \text{(55)}
\]

Assuming \( \xi_1 > \xi_2 > \cdots > \xi_N > 0 \), we consider the asymptotic forms of the \( N \)-soliton solution for \( x - (1-2\xi_m) t = \eta_n/\eta_m = \text{const} \) as \( t \to \pm \infty \). As \( t \to \infty \),

\[ x - (1-2\xi_j) t = x - (1-2\xi_m) t + 2(\xi_j - \xi_m) t \to \begin{cases} \infty & (j < m), \\ -\infty & (j > m). \end{cases} \]

Therefore, Eq. (55) reduces to

\[
\tilde{p}_n(x) = 0 \quad \text{for } n < m, \quad \text{(56a)}
\]

\[
(1 + \frac{|C_{m+1}^\rho|^2}{2\xi_m^2 \eta_m^2} e^{-4\eta_m}) \tilde{p}_m = \Phi_m + \frac{i|C_{n0}^\rho|^2}{\xi_m \eta_m} e^{-4\eta_m} \sum_{k=m+1}^{N} \frac{1}{\eta_k \gamma_k} \tilde{p}_k
\quad \text{(56b)}
\]

\[
\sum_{k=m+1}^{N} \frac{1}{\gamma_k \gamma_k} \tilde{p}_k = -\frac{1}{\gamma_m \gamma_m} \tilde{p}_m \quad \text{for } n \geq m + 1, \quad \text{(56c)}
\]

where

\[ \Phi_m = -2iC_{m0}^\rho \exp\{-2i(\kappa_m^2 - \kappa_m^2 - \eta_m^2 t)\} e^{-2i\kappa_m x}. \]

By using the technique similar to that by Zakharov and Shabat\(^2\) (see the Appendix of their paper) we can verify that
\[ \sum_{k=m}^{N} \left[ a_k(\zeta_k) \right]^* = 1, \quad (l=m, \ldots, N) \]  

\[ \sum_{k=m}^{N} \frac{a_k(\zeta_i)}{(\zeta_i^* - \zeta_k)(\zeta_k^* - \zeta_i^*)} = \delta_{ji}, \quad (l=m, \ldots, N) \]  

\[ \sum_{k=m+1}^{N} \frac{\alpha_k(\zeta_i)}{(\zeta_i^* - \zeta_k)(\zeta_k^* - \zeta_i^*)} = \delta_{ji}, \quad (l=m+1, \ldots, N) \]  

where \( \Pi' \) denotes that the factor equal to zero is omitted from the product. Multiplying Eq. (56c) by \( \left[ a_l(\zeta_i) \right]^* \alpha_n(\zeta_n)/(\zeta_i^* - \zeta_n^*) \) and summing up over \( n \), we get from Eq. (59)

\[ \tilde{p}_l = -\sum_{n=m+1}^{N} \left[ a_l(\zeta_i) \right]^* \alpha_n(\zeta_n)/(\zeta_i^* - \zeta_n^*) \tilde{p}_n. \]  

It follows from Eq. (58) with \( j = m, m+1 \leq l \leq N \) and Eq. (61) that

\[ \sum_{n=m+1}^{N} \frac{1}{(\zeta_i^* - \zeta_n^*)(\zeta_n^* - \zeta_m^*)} \left[ \alpha_n(\zeta_n) \right]^* = \left[ a_m(\zeta_m) \right]^* \frac{(\zeta_i^* - \zeta_m^*)}{(\zeta_m^* - \zeta_i^*)}. \]  

Taking the complex conjugate of this relation and replacing \( \zeta_m \) with \( \zeta_m^* \), we get

\[ \sum_{n=m+1}^{N} \frac{\alpha_n(\zeta_n)}{(\zeta_i^* - \zeta_n^*)(\zeta_n^* - \zeta_m^*)} = \frac{a_m(\zeta_m^*)}{(\zeta_m^* - \zeta_i^*)}. \]  

Substitution of this into Eq. (62) gives

\[ \tilde{p}_l = -\left[ a_l(\zeta_i) \right]^*/(\zeta_m - \zeta_m^*) a_m(\zeta_m^*) \tilde{p}_m. \]  

From Eq. (57) for \( l=m \) we obtain

\[ \sum_{k=m+1}^{N} \frac{a_k(\zeta_k)}{\zeta_k^* - \zeta_m^*} = 1 + \left[ a_m(\zeta_m^*) \right]^*/(\zeta_m^* - \zeta_m^*) \]  

and therefore

\[ \sum_{l=m+1}^{N} \tilde{p}_l = a_m(\zeta_m^*)/(\zeta_m - \zeta_m^*) \tilde{p}_m \left( 1 + \left[ a_m(\zeta_m^*) \right]^*/\zeta_m^* - \zeta_m^* \right) - \left( 1 + \frac{a_m(\zeta_m^*)}{\zeta_m^* - \zeta_m^*} \right) \tilde{p}_m. \]  

It follows from Eqs. (63), (58) for \( l=m \) and (61) that

\[ \sum_{l=m+1}^{N} \tilde{p}_l = -a_m(\zeta_m^*)/(\zeta_m - \zeta_m^*) \tilde{p}_m \left[ \frac{1}{a_m(\zeta_m^*) \sum_{l=m+1}^{N} \left[ a_l(\zeta_i) \right]^* a_m(\zeta_m^*) \right] \]  

\[ = -\frac{\tilde{p}_m}{\zeta_m^* - \zeta_m^*} \left( \frac{\zeta_m - \zeta_m^*}{\zeta_m^* - \zeta_m^*} \right)^2 + 1. \]
Substituting Eq. (65) into Eq. (56b) and using $\tilde{p}_m$ obtained so and Eq. (64), we get

$$
\tilde{p}_m + \sum_{k=m+1}^{N} \tilde{p}_k \frac{a_m(\zeta_m^*)}{\zeta_m - \zeta_k^*} \tilde{p}_m = \frac{a_m(\zeta_m^*)}{[a_m(\zeta_m^*)]^*} \frac{1}{(2i\eta_m/[a_m(\zeta_m^*)]^*)^{1/2}} e^{i\eta_m}. 
$$

In the limit $t \to \infty$ under the condition $\eta_m = \text{const}$ we finally obtain

$$
\phi \to 2\eta_m \sqrt{2 \tilde{p}_m^2} \mathrm{sech} \left[ 2\eta_m + \vartheta' \right] e^{2i(\eta_m^x - \lambda) t + 2i\eta_m^y t + 2i\eta_m^z t}, \quad (66a)
$$

where

$$
\vartheta' = \log \frac{\sqrt{2 \tilde{p}_m^2}}{C_m \eta_m^x} + \log \frac{|a_m(\zeta_m^*)|}{2 \eta_m^x},
$$

$$

\mathrm{e}^{i\theta'} = \left( i \frac{C_m \eta_m^x}{\sqrt{2 \tilde{p}_m^2}} \left[ \frac{|a_m(\zeta_m^*)|}{2 \eta_m^x} \right]^* \right).
$$

Similar calculations yield

$$
n \to -4\eta_m \sqrt{2 \tilde{p}_m^2} \mathrm{sech} \left[ 2\eta_m + \vartheta' \right], \quad (66b)
$$

as $t \to \infty$ ($\eta_m = \text{const}$).

For the case $t \to -\infty$ ($\eta_m = \text{const}$), introducing $\tilde{a}_k(\zeta_k)$ in place of $a_k(\zeta_k)$ by

$$
\tilde{a}_k(\zeta_k) = \prod_{l=1}^{m} (\zeta_k^* - \zeta_l^*),
$$

we can obtain in a similar way the following asymptotic forms:

$$
\phi \to 2\eta_m \sqrt{2 \tilde{p}_m^2} \mathrm{sech} \left[ 2\eta_m + \vartheta' \right] e^{2i(\eta_m^x - \lambda) t - 2i\eta_m^y t + 2i\eta_m^z t}, \quad (67a)
$$

$$
n \to -4\eta_m \sqrt{2 \tilde{p}_m^2} \mathrm{sech} \left[ 2\eta_m + \vartheta' \right], \quad (67b)
$$

where

$$
\vartheta' = \log \frac{\sqrt{2 \tilde{p}_m^2}}{C_m \eta_m^x} + \log \frac{|\tilde{a}_m(\zeta_m^*)|}{2 \eta_m^x},
$$

$$

\mathrm{e}^{i\theta'} = \left( i \frac{C_m \eta_m^x}{\sqrt{2 \tilde{p}_m^2}} \left[ \frac{|\tilde{a}_m(\zeta_m^*)|}{2 \eta_m^x} \right]^* \right).
$$

It is seen from these that the asymptotic forms (66) and (67) as $t \to \pm \infty$ are the same as the forms of single soliton solutions and only the soliton phase $\vartheta$ and the wave phase $\theta$ undergo changes due to the interactions of solitons. These changes throughout the whole process of the interactions are given by

$$
\Delta \vartheta_m = \vartheta_m' - \vartheta_m = \sum_{l=m+1}^{N} \log \frac{\zeta_m - \zeta_l^*}{\zeta_m - \zeta_l^*} - \sum_{l=1}^{m-1} \log \frac{\zeta_m - \zeta_l^*}{\zeta_m - \zeta_l^*}, \quad (68a)
$$

$$
\Delta \theta_m = \theta_m' - \theta_m = \sum_{l=m+1}^{N} \arg \frac{\zeta_m - \zeta_l^*}{\zeta_m - \zeta_l^*} - \sum_{l=1}^{m-1} \arg \frac{\zeta_m - \zeta_l^*}{\zeta_m - \zeta_l^*}. \quad (68b)
$$
The results (68) are as if only paired collisions occur, and such a situation is analogous to those in the Korteweg-de Vries equation and the nonlinear Schrödinger equation.

We note that according to the remark at the end of § 3, the case $\xi_n=0$ can be also included in the above discussion.

§ 5. Some results by means of perturbation method

Let us consider first the possibility that a broad packet of the Langmuir field $|\phi(x)|$ develops to a series of solitons. This depends on whether the zeros of $a_{11}(\zeta)$ exist on the right side of the lower half $\zeta$-plane. Here we study the initial value problem in which $n(x)=0$ at $t=0$ on the assumption that $|\phi(x)|$ is small enough for the perturbation method to be applied.

Putting $\phi^{(0)} e^{-4it_0 x} = F(x, \zeta)$, we obtain the integral equations for $F(x, \zeta)$ from the system of equations (6) and the boundary conditions (10),

\begin{align*}
F_1(x, \zeta) &= 1 - \int_{-\infty}^{x} \phi^*(x') F_2(x', \zeta) \, dx', \\
F_2(x, \zeta) &= \frac{1}{2\zeta} \int_{-\infty}^{x} e^{-i\zeta (x-x')} \phi(x') \{ F_1(x', \zeta) + F_3(x', \zeta) \} \, dx', \\
F_3(x, \zeta) &= \int_{-\infty}^{x} e^{-4i\zeta (x-x')} \phi^*(x') F_2(x', \zeta) \, dx'.
\end{align*}

(69)

If we retain the terms up to second order of $|\phi(x)|$ in the successive approximation for Eqs. (69), we have

$$F_1(x, \zeta) \simeq 1 - \frac{1}{2\zeta} \int_{-\infty}^{x} dx' \phi^*(x') \int_{-\infty}^{x'} e^{-i\zeta (x'-x'')} \phi(x'') \, dx''.$$

Then, we get

$$a_{11}(\zeta) = \lim_{x \to \infty} F_1(x, \zeta) \simeq 1 - \frac{1}{2\zeta} \int_{-\infty}^{\infty} dx' \phi^*(x') \int_{-\infty}^{x'} e^{-i\zeta (x'-x'')} \phi(x'') \, dx''$$

$$= 1 - \frac{1}{2\zeta} \int_{-\infty}^{\infty} dx' \phi(x') \int_{x'}^{\infty} e^{i\zeta (x'-x'')} \phi^*(x'') \, dx''.$$

Taking the average of these two expressions and integrating by parts, we obtain

$$a_{11}(\zeta) \simeq 1 - \frac{1}{8\zeta^2} \sum_{m=0}^{\infty} \frac{1}{(2\zeta)^{2m}} \left[ 2 \int_{-\infty}^{\infty} |\phi^{(m)}(x)|^2 \, dx 
+ \frac{1}{2i\zeta} \int_{-\infty}^{\infty} \{ \phi^{(m)}(x) \phi^{(m+1)}(x) - \phi^{(m)}(x) \phi^{(m+1)}(x) \} \, dx \right],$$

(70)

where

$$\phi^{(m)}(x) = \frac{d^m \phi(x)}{dx^m}.$$
Furthermore, let us assume $\phi(x)$ to take the form
\[ \phi(x) = \varepsilon \Theta(\delta x), \quad (71) \]
where $\varepsilon$ and $\delta$ are small parameters. Then, if $\delta \ll \varepsilon$, in Eq. (70) the terms containing the derivatives with respect to $x$ can be neglected. Therefore, we get
\[ a_{11}(\zeta) \approx 1 - \frac{1}{4i\zeta^2} \int_{-\infty}^{\infty} |\phi(x)|^2 dx, \quad (72) \]
which has a zero in the lower half $\zeta$-plane, that is, the zero $\zeta_0$ is given by
\[ \zeta_0 = \frac{1}{3} e^{-(\pi/4)i} \left( \int_{-\infty}^{\infty} |\phi(x)|^2 dx \right)^{1/2}. \quad (73) \]
Therefore, in such cases one soliton is formed from the initial state consisting of a Langmuir wave only.

Next, let us consider the interaction of a Langmuir soliton with the empty soliton given by Eq. (53) which are initially placed apart from one another by a finite distance $l$. As seen from Eq. (1) the depression of the ion density ($n<0$) works on Langmuir waves as an attractive potential. Therefore, when the empty soliton collides with the Langmuir soliton, there is the possibility that a part of the Langmuir field in the soliton is picked up by the empty soliton, and the latter evolves to a Langmuir soliton. The following eigenvalue problem applies:
\[ \left[ -iA \frac{\partial}{\partial x} + \frac{1}{\zeta} (V_0 + V_1) + W_1 \right] f = \zeta f, \quad (74) \]
where
\[ A = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \frac{n_0}{6} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{n_1}{6} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{i}{2} \phi_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_1 = -\frac{i}{3} \phi_1^* \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
and
\[ n_0(x) = -4\eta_0^2 \text{sech}^2 2\eta_0 x, \quad (75) \]
\[ n_1(x) = -4\eta_1^2 \text{sech}^2 [2\eta_1(x-l)], \quad \phi_1(x) = 2\eta_1 \sqrt{2\xi_1} \text{sech}[2\eta_1(x-l)] e^{-2i\eta_1 x + i\theta}. \quad (76) \]
In the above equations $n_0(x)$ represents the empty soliton with the corresponding eigenvalue $\zeta_0 = -i\eta_0$, and $n_1(x)$ and $\phi_1(x)$ a Langmuir soliton with the corresponding eigenvalue $\zeta_1 = \xi_1 - i\eta_1$, which is placed at $x=l$. If $V_1 = W_1 = 0$ in Eq. (74), then the eigenvalue is $\zeta$, and the corresponding eigenvector $f^{\infty}$ is given by...
This eigenvalue $\zeta_0$ will shift under the effect of $V_1$ and $W_1$. If the shift has the positive real part of the eigenvalue, the empty soliton evolves to a Langmuir soliton through the interaction with the Langmuir soliton (76).

We suppose that $\ell$ is large enough for the overlap of the ion wave (75) and the Langmuir soliton (76) to be small. We expand $f$ and $\zeta$ in Eq. (74) as

$$f = f^{(0)} + F^{(1)} + F^{(2)} + \cdots, \quad \zeta = \zeta_0 + \Delta\zeta^{(1)} + \Delta\zeta^{(2)} + \cdots.$$  (78)

Substitution of these into Eq. (74) gives rise to

$$L_\phi f^{(0)} = -iA \frac{\partial}{\partial x} + \frac{V_0}{\zeta_0} - \zeta_0 I \right) f^{(0)} = 0, \quad (79a)$$

$$L_\phi F^{(1)} = -\left( \frac{1}{\zeta_0} V_1 + W_1 \right) f^{(0)} + \Delta\zeta^{(1)} \left( I + \frac{V_0}{\zeta_0^2} \right) f^{(0)}, \quad (79b)$$

$$L_\phi F^{(2)} = \left\{ -\left( \frac{1}{\zeta_0} V_1 + W_1 \right) + \Delta\zeta^{(1)} \left( I + \frac{V_0}{\zeta_0^2} \right) \right\} F^{(1)} + \Delta\zeta^{(2)} \left( \frac{V_1}{\zeta_0^2} - \frac{V_0}{\zeta_0^3} V_0 \right) f^{(0)} + \Delta\zeta^{(2)} \left( I + \frac{V_0}{\zeta_0^2} \right) F^{(0)}, \quad (79c)$$

where $I$ denotes the unit matrix. The adjoint equation of Eq. (79a),

$$L_\phi A^{(0)} = iA \frac{\partial}{\partial x} + \frac{V_0^T}{\zeta_0^2} - \zeta_0 I \right) A^{(0)} = 0, \quad (80)$$

is easily solved to give the solution,

$$A^{(0)} = \begin{pmatrix} \frac{3}{2} e^{\eta x} \text{sech}^2 2\eta_0 x \\ 0 \\ e^{-\eta x} \text{sech} 2\eta_0 x - e^{\eta x} \frac{1}{2} \text{sech}^2 2\eta_0 x \end{pmatrix}, \quad (81)$$

where $V_0^T$ is the transposed matrix of $V_0$. Taking the inner product of Eq. (79b) with the adjoint solution (81) and using Eq. (80), we get

$$\Delta\zeta^{(1)} = \left\langle f^{(0)}, \left( \left( \frac{1}{\zeta_0} V_1 + W_1 \right) f^{(0)} \right) \right\rangle \left\langle f^{(0)}, \left( I + \frac{V_0}{\zeta_0^2} \right) f^{(0)} \right\rangle^{-1}.$$

(82)
where the inner product of vectors $f$ and $g$, $\langle f, g \rangle$, is defined by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \sum_{i=1}^{3} f_i g_i dx.$$ 

It follows from Eqs. (75), (76), (77), (81) and (82) that

$$\Delta \zeta^{(1)} = -i \eta \int_{-\infty}^{\infty} \sech^2[2 \eta x] \sech^2 2 \eta x dx.$$ 

This is purely imaginary, therefore we must proceed to higher order calculations to obtain the real part of the eigenvalue.

Two other solutions of Eq. (79a) independent of the solution (77) are

$$g = \begin{pmatrix} 0 \\ e^{\nu x} \\ 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} \{2 \eta x (1 - \tanh 2 \eta x) + 1\} \sech 2 \eta x + e^{\nu x} \\ 0 \\ \{2 \eta x (1 + \tanh 2 \eta x) - 1\} \sech 2 \eta x - e^{-\nu x} \end{pmatrix}. \quad (83)$$

Let us expand $F^{(\omega)}$ in Eq. (79b) by means of $f^{(0)}$, $g$ and $h$ as

$$F^{(\omega)} = p(x)f^{(0)} + q(x)g + r(x)h. \quad (84)$$

Substituting this into Eq. (79b), we get

$$\frac{dp}{dx} = \frac{i n_1}{2 \nu_0} e^{-\nu x} (f_1^{(0)} + f_3^{(0)})(h_1 + h_3)$$

$$- i \Delta \zeta^{(1)} e^{-\nu x} \left\{ (3f_1^{(0)}h_3 + f_3^{(0)}h_1) + \frac{n_0}{2 \nu_0^2} (f_1^{(0)} + f_3^{(0)}) (h_1 + h_3) \right\}, \quad (85a)$$

$$\frac{dq}{dx} = \frac{\phi_1}{2 \nu_0} e^{-\nu x} (f_1^{(0)} + f_3^{(0)}), \quad (85b)$$

$$\frac{dr}{dx} = - \frac{i n_1}{2 \nu_0} e^{-\nu x} (f_1^{(0)} + f_3^{(0)})^2$$

$$+ i \Delta \zeta^{(1)} e^{-\nu x} \left\{ 4f_1^{(0)}f_3^{(0)} + \frac{n_0}{2 \nu_0^2} (f_1^{(0)} + f_3^{(0)})^2 \right\}. \quad (85c)$$

It is evident that the right-hand sides of Eqs. (85a) and (85c) and thus $p(x)$ and $r(x)$ are real. Taking the inner product of Eq. (79c) with $f^4$, we can see that the part $p f^{(0)} + r h$ of $F^{(\omega)}$ does not contribute to the real part of $\Delta \zeta^{(\omega)}$. Equation (85b) reduces to

$$q(x) = \frac{i}{4 \eta_0} \int_{-x}^{x} \phi_1(x') \sech 2 \eta x' dx'.$$

Finally we obtain

$$\text{Re}(\Delta \zeta^{(\omega)}) = \frac{\text{Re}(\langle f^4, W \phi(x) g \rangle)}{\langle f^4, (I + (V_0/\nu_0^2))f^{(0)} \rangle} = \frac{1}{8} \left| \int_{-\infty}^{\infty} \phi_1(x) \sech 2 \eta x dx \right|^2. \quad (86)$$
Equation (86) determines the velocity of the Langmuir soliton to which the empty soliton evolves.

§ 6. Conservation laws

Time invariance of $a_{11}(\zeta)$ permits us to make a series of polynomial conservation laws. We define $F=\phi^{(0)}e^{-\kappa x}$, and obtain

\[
 F \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{as} \quad x \to -\infty, \quad F_1 \to a_{11}(\zeta) \quad \text{as} \quad x \to \infty
\]

and

\[
 \frac{dF_1}{dx} = -\frac{i}{2\kappa} n (F_1 + F_3) - \phi^* F_2, \quad (87a)
\]
\[
 \frac{dF_2}{dx} = \frac{1}{2\kappa} \phi (F_1 + F_3) - 2i\zeta F_2, \quad (87b)
\]
\[
 \frac{dF_3}{dx} = \frac{i}{2\kappa} n (F_1 + F_3) + \phi^* F_2 - 4i\zeta F_3. \quad (87c)
\]

For large $\zeta$ we expand $F_1$, $F_2$ and $F_3$ in powers of $\zeta^{-1}$ as follows:

\[
 F_1 = 1 + \sum_{n=1}^{\infty} A_n \zeta^{-n}, \quad F_2 = \sum_{n=1}^{\infty} B_n \zeta^{-n}, \quad F_3 = \sum_{n=1}^{\infty} C_n \zeta^{-n}. \quad (88)
\]

Then, $A_n$, $B_n$, $C_n \to 0$ as $x \to -\infty$. Substituting Eqs. (88) into Eqs. (87) and equating the coefficients of the same powers of $\zeta^{-1}$, we get

\[
 A_1 = -\frac{i}{2} \int_{-\infty}^{\infty} ndx, \quad B_1 = C_1 = 0 \quad (89)
\]

and

\[
 A_2 = -\frac{1}{8} \left( \int_{-\infty}^{\infty} ndx \right)^2 - \frac{1}{4i} \left( \int_{-\infty}^{\infty} |\phi|^2 dx \right), \quad B_2 = \frac{1}{4i} \phi, \quad C_2 = \frac{1}{8} n \quad (90)
\]

and for $n \geq 3$,

\[
 \frac{dA_n}{dx} = -\frac{i}{2} n (A_{n-1} + C_{n-1}) - \phi^* B_n, \quad (91a)
\]
\[
 \frac{dB_{n-1}}{dx} = \frac{1}{2} \phi (A_{n-1} + C_{n-2}) - 2iB_n, \quad (91b)
\]
\[
 \frac{dC_{n-1}}{dx} = \frac{i}{2} n (A_{n-2} + C_{n-2}) + \phi^* B_{n-1} - 4iC_n. \quad (91c)
\]

From Eqs. (91b) and (91c), $B_n$ and $C_n$ are determined in terms of $A_{n-2}$, $C_{n-2}$, $B_{n-1}$, $C_{n-1}$, and then from Eq. (91a) $A_n$ is obtained. Thus, $A_n$, $B_n$ and $C_n$ are successively determined. Since $a_{11}(\zeta) = 1 + \lim_{x \to -\infty} \sum_{n=1}^{\infty} A_n \zeta^{-n}$ and $a_{11} = 0$, $A_n$ is conserved. First five conserved quantities thus obtained are given by
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I_t = \int_{-\infty}^{\infty} n dx, \quad I_2 = \int_{-\infty}^{\infty} |\phi|^2 dx, \quad I_3 = \int_{-\infty}^{\infty} \left\{ n^2 + \frac{1}{i} \left( \phi^* \frac{\partial \phi}{\partial x} - \phi \frac{\partial \phi^*}{\partial x} \right) \right\} dx,

I_4 = \int_{-\infty}^{\infty} \left\{ \left( \frac{\partial \phi^*}{\partial x} \right)^2 + 2n|\phi|^2 \right\} dx, \quad I_5 = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} n^2 - 6|\phi|^2 - \frac{1}{2} \left( \frac{\partial n}{\partial x} \right)^2 \right\} dx

- \frac{6}{i} n \left( \phi^* \frac{\partial \phi}{\partial x} - \phi \frac{\partial \phi^*}{\partial x} \right) + \frac{2}{i} \left( \phi^* \frac{\partial^3 \phi}{\partial x^3} - \phi \frac{\partial^3 \phi^*}{\partial x^3} \right) dx.

The fact that the system of equations (1) and (5) has an infinite number of conservation laws is consistent with the character that the identities of solitons are preserved through the mutual interactions as presented in § 4. This implies that the break-up of solitons and the fusion of solitons cannot occur in this simplified system but can occur in the original system of Eqs. (1) and (2). This seems to relate to the fact that Hirota's direct method\textsuperscript{10} to find exact solutions is applicable to this simplified system but not to the original system.\textsuperscript{10}

Finally, we present remarks on applicability of the system of Eqs. (1) and (5). When the propagation velocity of the Langmuir soliton (3) tends to unity keeping the amplitude of \( E \) non-zero, the ion sound wave \( n \) becomes infinitely large. This divergence is caused by the fact that the nonlinear effect of the ion wave is not taken into account in deriving Eqs. (1) and (2).\textsuperscript{19} In this sense the solutions which we have obtained cannot be compared with phenomena in real plasmas. However, the system of Eqs. (1) and (5) presents an interesting example which can be solved exactly by the inverse scattering method.

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