$$\alpha_j - \gamma_i \geq \beta_k - \alpha_l \tag{17}$$

$$\alpha_l - \gamma_i \geq \beta_j - \alpha_k \tag{18}$$

$$\alpha_l - \gamma_i \geq \beta_k - \alpha_j \tag{19}$$

$$\gamma_i + \beta_j \leq \alpha_k + \alpha_l \tag{20}$$

$$\gamma_i + \alpha_j \leq \beta_k + \alpha_l \tag{21}$$

Equations (14), (15), and (21) are automatically satisfied since

$$\gamma \leq \alpha$$

and (22)

$$\alpha \leq \beta$$

irrespective of the subscript involved. Equation (20) is a direct statement of the classification law and thus makes Grashof's law a necessary condition. Equation (16) is automatically satisfied since β is the largest and γ is the smallest twist angle. Thus their difference must be larger than the difference of the other two sides. The remaining three equations can be converted to direct statements of the classification law by adding $\gamma_i + \alpha_l$ to equation (17), $\gamma_i + \alpha_k$ to equation (18), and $\gamma_i + \alpha_j$ to equation (19). Therefore the conditions of Grashof's law are both necessary and sufficient for the existence of class I linkages.

If the linkage satisfies the reverse inequality, one or more of equations (17) through (20) will not be satisfied. If the resulting linkage can be assembled, it will therefore be a rocker-rocker type for which no link can make a full rotation with respect to the other links. To assemble a linkage its largest link must be smaller than the sum of the other three. This condition is included in Grashof's law for class I linkages since

$$\gamma + \beta \leq \alpha + \alpha \tag{23}$$

is a stricter limitation on the magnitude of β than

$$\beta < \alpha + \alpha + \gamma. \tag{24}$$

Conclusions

The spherical four-bar linkage is one of the simplest space mechanisms in construction and in that it produces a plane motion output from a plane motion input. Because of this and the fact that all RCCC linkages with the same twist angles have the same angular relations, the spherical four-bar linkage is worthy of much study. An RCCC linkage is a spatial four-bar which contains one revolute joint and three cylindrical joints. It is one of the simplest mechanisms to obey the Kutzbach criterion for the possession of a single degree of freedom (10). Of particular interest here is the fact that the spherical four-bar linkage is an RCCC linkage and can be constructed as one. Thus the spherical four-bar stands as a stepping stone for both the analysis and the synthesis of the general RCCC mechanism which produces true spatial motion of the output.

The main value of a complete set of unique models for the spherical four-bar lies in the clearer visualization of the spherical four-bar and its related mechanisms which is made possible by the lack of redundant models. Thus the application of Grashof's law to the classification of the spherical four-bar and therefore to the general RCCC linkage is made possible. By the same token, design procedures for both mechanisms should be more easily obtainable with the use of this model set.

More generally, this work indicates the lack of and need for complete sets of unique models in three-dimensional kinematics. Such a set exists in plane kinematics once the link lengths are converted to link ratios, but it is believed that this is the first application of such a set in the study of a spatial mechanism. This complex field of study can only benefit from the use of such sets.

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DISCUSSION

A. H. Soni[3]

Type Determination of a Spherical Four-Bar Mechanism. The authors are to be complimented for proposing the solution of the problem on type-determination of spherical four-link mechanisms. While the authors' presentation does provide a significant contribution to the problem, it is believed, however, that the proposed solution of type-determination of spherical four-link mechanisms appears to be inadequate.

1 For the completeness of transformation routine, it is necessary to include the following classes of linkages:

Class *G*:	σ_j,	σ_k,	σ_l,	σ_k
Class *H*:	α_j,	σ_k,	σ_l,	α_i
Class *I*:	σ_j	α_k,	σ_l,	α_i
Class *J*:	α_j,	σ_k,	σ_l,	α_i
Class *K*:	α_j,	α_k,	σ_l,	α_i

where 270 deg $\geq \sigma \geq$ 180 deg

Mechanisms of classes G, H, and I are found to exist as "complementary" solutions of the authors' Class *A*. [11, 12][4]

2 The proposed Grashof Law appears to be inadequate in type determination of the linkages shown in Table 1. Note that although the linkage dimensions satisfy the authors' criteria of class 2 mechanisms, these linkages do exist as crank-rocker and drag-link mechanisms.

The writer will greatly appreciate receiving comments from the authors on the following techniques for the type-determination of spherical four-link mechanisms:

1 A spherical four-link mechanism with input-link $= a$, coupler-link $= b$, output-link $= c$, and the fixed-link $= d$ is shown in its four characteristic positions in Figs. 8–11. Figs. 8 and 9 describe the positions in which the mechanism has the maximum and the minimum transmission angles. Figs. 10 and 11 describe the positions in which the mechanism has limit positions.

If μ_{max} and μ_{min} are the maximum and the minimum transmission angles, then the application of the Cosine-Law to the spherical triangles A_1B_1Q and A_2B_2Q yield

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[4] Numbers in brackets designate Additional References at end of discussion.

Table 1 Type-determination of "nongrashof" spherical four-link mechanisms

Case	a	b	c	d	Type of Spherical Mechanism
	Link-Lengths of spherical four-link mechanisms				
1 *	70	90	90	160	Drag-link
2	60	85	80	140	Drag-link
3	60	80	80	170	Drag-link
4	170	75	85	60	Crank-Rocker

* See reference [14]

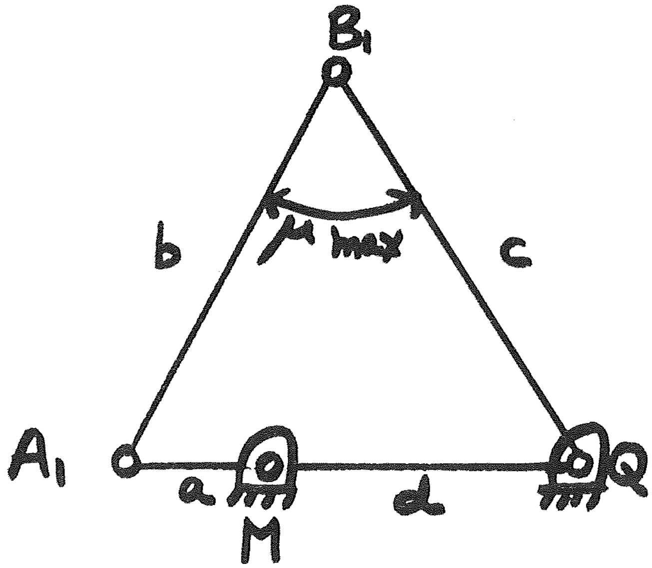

Fig. 8

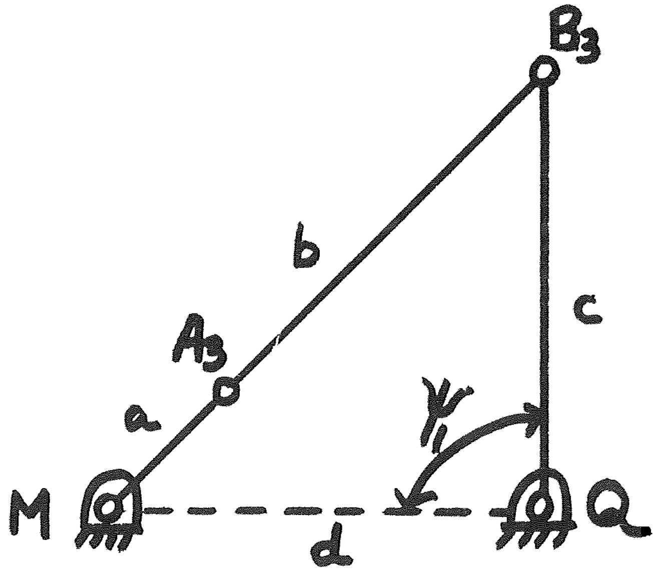

Fig. 10

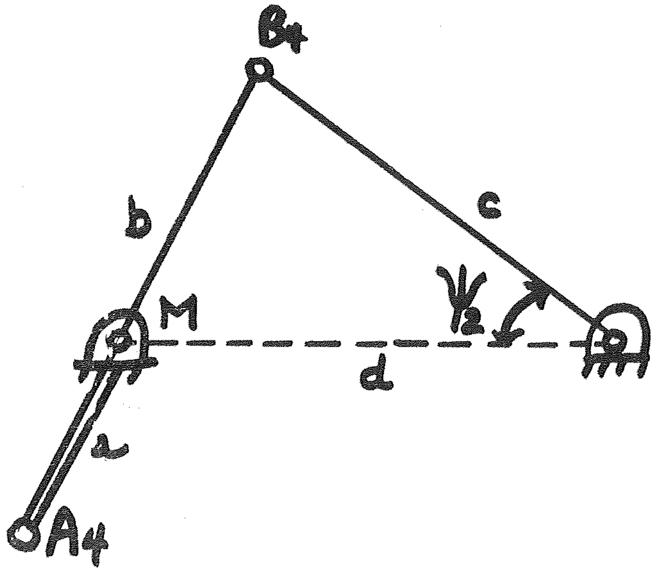

Fig. 9

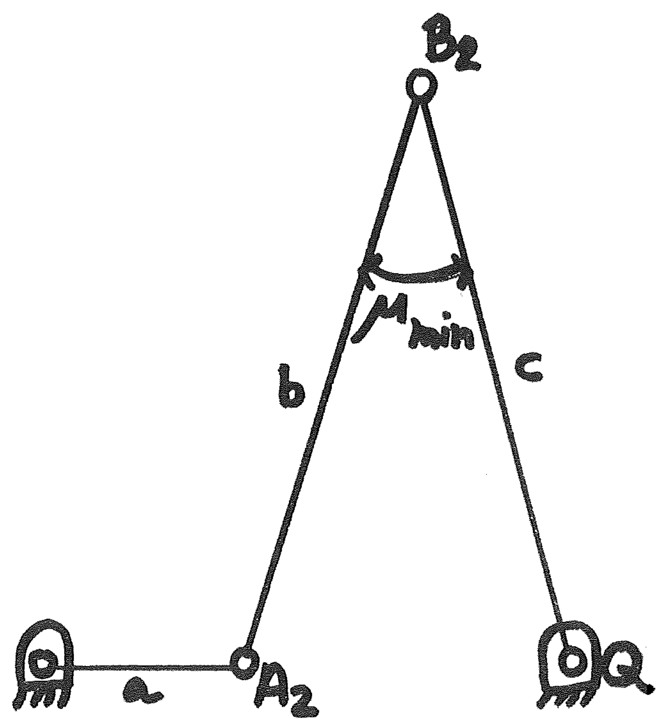

Fig. 11

$$\cos b \cos c + \sin b \sin c \cos \mu_{\max} = \cos (a + d) \quad (25)$$

$$\cos b \cos c + \sin b \sin c \cos \mu_{\min} = \cos (a - d) \quad (26)$$

Note that μ_{max}[5] and μ_{min}[5] vary within the ranges defined by

$$90° \geq \mu \text{ min} \geq 0° \quad (27)$$

$$180° \geq \mu \text{ max} \geq 90° \quad (28)$$

Because of these specific variations, equations (25) and (26) will yield the following inequalities:

$$-1 \leq \left\{ \text{Cos } \mu_{max} = \frac{\text{Cos } (a + d) - \text{Cos } b \text{ Cos } c}{\text{Sin } b \text{ Sin } c} \right\} \leq 0 \quad (29)$$

$$+1 \geq \left\{ \text{Cos } \mu_{min} = \frac{\text{Cos } (a - d) - \text{Cos } b \text{ Cos } c}{\text{Sin } b \text{ Sin } c} \right\} \leq 0 \quad (30)$$

Considering the left side of the inequalities, we get

$$\text{Cos } (a + d) \geq \text{Cos } (b + c) \quad (31)$$

and

$$\text{Cos } (a - d) \leq \text{Cos } (b - c) \quad (32)$$

Application of the Cosine-Law to triangles MB_3Q and MB_4Q describing the limit positions of a spherical four-link mechanism in Figs. 10, 11, we get

$$\text{Cos } c \text{ Cos } d + \text{Sin } c \text{ Sin } d \text{ Cos } \psi_1 = \text{Cos } (a + b) \quad (33)$$

and

$$\text{Cos } c \text{ Cos } d + \text{Sin } c \text{ Sin } d \text{ Cos } \psi_2 = \text{Cos } (b - a) \quad (34)$$

or

$$\text{Cos } \psi_1 = \frac{\text{Cos } (a + b) - \text{Cos } c \text{ Cos } d}{\text{Sin } c \text{ Sin } d} \quad (35)$$

and

$$\text{Cos } \psi_2 = \frac{\text{Cos } (b - a) - \text{Cos } c \text{ Cos } d}{\text{Sin } c \text{ Sin } d} \quad (36)$$

For a crank-rocker mechanism $[\psi_1]_{max} \leq 180°$

and

$$[\psi_2]_{min} \geq 0°$$

That is,

$$\frac{\text{Cos } (a + b) - \text{Cos } c \text{ Cos } d}{\text{Sin } c \text{ Sin } d} \geq -1 \quad (37)$$

$$\frac{\text{Cos } (b - a) - \text{Cos } c \text{ Cos } d}{\text{Sin } c \text{ Sin } d} \leq +1 \quad (38)$$

equations (37) and (38) yield

$$\text{Cos } (a + b) \geq \text{Cos } (c + d) \quad (39)$$

$$\text{Cos } (b - a) \leq \text{Cos } (c - d) \quad (40)$$

Note that the inequalities given by equations (31), (32), (39) and (40) provide the necessary and sufficient conditions to extend Grashof-Law for spherical mechanisms. The transformations described in classes *B, C, D, G, H,* and *I* are satisfied by these inequalities.

Each of these four Cosine inequalities will provide five different inequalities relating directly the link-lengths of spherical mechanisms. For example, equation (36) will provide the following angular inequalities:

$$(a + d) \leq (b + c) \quad (41)$$

$$\Pi - (a + d) \geq \Pi - (b + c) \quad (42)$$

$$\Pi + (a + d) \geq \Pi - (b + c) \quad (43)$$

$$\Pi + (a + d) \geq \Pi + (b + c) \quad (44)$$

$$\Pi - (a + d) \geq \Pi + (b + c) \quad (45)$$

Similarly, equation (37) will provide the following angular inequalities.

$$|a - d| \geq |b - c| \quad (46)$$

$$\Pi - |a - d| \leq \Pi - |b - c| \quad (47)$$

$$\Pi + |a - d| \leq \Pi - |b - c| \quad (48)$$

$$\Pi + |a - d| \leq \Pi + |b - c| \quad (49)$$

$$\Pi - |a - d| \leq \Pi + |b - c| \quad (50)$$

A systematic study of equations (41) and (46) will lead to the Grashof criterion proposed by the author. Equations (42) and (47) will include the transformations given in classes *B, C, D*. Equations (44) and (49) will include the transformations given in classes *G, H,* and *I*. Equations (43), (45), (48) and (50) will include the transformations required for the classes *E, F, J* and *K*. For the completeness of the Grashof Law, we feel that equations (46–50) should be considered. Accordingly, the following Grashof Law is proposed.

1 If equations (31) and (32) are satisfied, then the linkage is either a crank-rocker or a drag-link mechanism.

2 If equations (31), (32), and either equation (39) or equation (40) is satisfied, then the linkage is a crank-rocker mechanism.

3 A rocker-rocker mechanism is obtained when one of the following conditions is satisfied:

(*a*) Either equation (31) or equation (32) is satisfied.

(*b*) Either equation (39) or equation (40) is satisfied.

2 The previous procedure for the type-determination may appear to be "unconventional." If one were to propose Grashof-Law similar to that proposed by the author, then the inequalities (41) through (50) must be examined simultaneously. The transformations of author's classes *B, C, D* are severely restricting when a type-determination of a mechanism is under consideration. For instance, the mechanisms with links *a, b, c, d* and $\Pi \pm a, b, \Pi \pm c, d$ will have the same input-output relationship. For the purpose of type-determination, one is only required to examine the type, (for example, crank-rocker, drag-link, etc.), of the mechanism rather than examining "piece wise" the total performance of the linkage. It is because of this reason it appears that the "appropriate" transformations become unobtainable for the linkages of classes *E* and *F* to be transformed into linkages of class *A*.

This writer has made a pilot study to examine the inequalities given by equations (41)–(50). Since classes *B, C, D, G, H,* and *I* can be transformed into class *A*, and classes *E, J,* and *K* can be transformed into class *F*, the inequalities giving classes *A* and *F* linkages must be thoroughly examined. This examination leads to the following propositions:

If l_{max}[6], l_{min}, m, and n are four link-lengths of a spherical mechanism, (l_{max}[7] $< l_{min} + m + n$, and $l_{min} \leq \frac{\Pi}{2}$, $m \leq \frac{\Pi}{2}$, $n \leq \frac{\Pi}{2}$ and $0 \leq l_{max} \leq 2\Pi$), then all the spherical linkages can be grouped into two groups of linkages, and the Grashof-Law for their type-determination can be written in a following manner:

$$(L). \quad l_{max} + l_{min} \leq \Pi$$

1 A spherical four-link mechanism belongs to class 1 linkage provided:

$$l_{max} + l_{min} \leq m + n$$

[5] $\mu_{max} = 90 + \lambda$ and $\mu_{min} = 90 - \lambda$

where $\quad \text{Sin } \lambda = \dfrac{\text{Sin } d \text{ Sin } a}{\text{Sin } c \text{ Sin } b}$

[6] l_{max} = largest link
l_{min} = smallest link

[7] This condition is necessary in order to build a spherical four-link mechanism.

The type-determination of class 1 linkage is worked out in the following manner:

(*a*) The class 1 linkage is a drag-link when the fixed link is the smallest link.

(*b*) The class 1 linkage is a crank-rocker when the input-link is the smallest link

(*c*) The class 1 linkage is a rocker-rocker for the configuration other than (*a*) or (*b*).

2 A spherical four-link mechanism belongs to class 2 linkage provided:

$$l_{\max} + l_{\min} > m + n$$

A class 2 linkage is a rocker-rocker under these conditions.

$$(M). \quad l_{\max} + l_{\min} > \Pi$$

1 A spherical four-link mechanism belongs to class 3 linkage provided

$$[(l_{\max} + l_{\min}) - \Pi] \geq [\Pi - (m + n)]$$

The type-determination of class 3 linkage is worked out in the following manner:

(*a*) The class 3 linkage is a drag-link when the fixed link is the largest link.

(*b*) The class 3 linkage is a crank-rocker when the input-link is the largest link.

(*c*) The class 3 linkage is a rocker-rocker for the configuration other than (*a*) or (*b*).

2 A spherical four-link mechanism belongs to class 4 linkage provided:

$$[(l_{\max} + l_{\min}) - \Pi] < [\Pi - (m + n)]$$

A class 4 linkage is a rocker-rocker.

Note that class 2 and class 4 are satisfying the condition stated by the author for his class 2 linkages. That is,

$$l_{\max} + l_{\min} > m + n$$

Thus, in reality, there are three different classes of linkages, classes 1, 2, and 3. The classes 1 and 3 are capable of providing linkages which are drag-link, crank-rocker or rocker-rocker.

Finally, this writer would like to call attention to reference [13]. It is important to note that Grashof himself, (see pp. 152–153) has proposed a law for the type-determination of spherical mechanism. This law is the same as that proposed by the authors. It appears that Grashof did not propose the transformations described by the authors. However, these transformations are partially described in authors' references [15, 17].

The authors have uniquely contributed to the present state of the art in proposing the transformations of all spherical linkages into class *A* and *F*. For their contributions, the authors are to be congratulated.

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Authors' Closure

The authors would like to thank Professor Soni for his interest in spherical four-bar linkages and their gross-motion classification. It is apparent from his discussion that the main point of this paper is not adequately emphasized. Unique model sets of linkages simplify the analysis of linkage properties as well as the descriptions of the properties themselves. The search for and description of a unique model set are more difficult than those of a corresponding arbitrary and redundant model set produced by general notation. This search is rewarded, however, by detailed knowledge of the linkage and its properties. In addition, the statements of the properties are apt to be more concise, offsetting the more involved notation used to describe the linkage.

Specifically, in the case of spherical four-bar linkages, all possible linkages can be described by groups of four twist angles which comply with the restrictions given in the paragraph containing equations (4–6). The use of this set of unique descriptions enables the direct application of Grashof's law to the gross motion classification of all spherical four-bar linkages. The uniqueness of the models and the completeness of the set as well as the validity of the gross-motion classification are proven in the paper.

Concerning the examples of Professor Soni, the following can be stated:

(1) By using the first transformation (equations (8)) his classes *G* through *K* can be converted to classes *B*, *C*, *D*, *C* and *F*, respectively. Note that this transformation is one of description only. A "linkage" of class *G* and one of class *B* are merely two "redundant" descriptions of a linkage whose "unique" description is in class *A*. Obviously the selection of class *A* as the describing class is arbitrary, the choice being caused by the relative simplicity and ease of visualization of its models.

(2) In way of repetition, the complimentary "solutions" mentioned are merely duplicate representations or descriptions of identical linkages and are not separate linkages. This means that any spherical four-bar can be constructed with a cross link similar to the coupler of the universal joint.

(3) The "proposed Grashof Law" appears to be inadequate only becuase it is applied to descriptions which are not in the proposed set of unique descriptions. The proof and application of this law are based on the use of the set of unique descriptions presented in the first part of the paper.

(4) The linkage of Figs. 8–11 possesses maximum and minimum values of μ and ψ if and only if it can be represented as a linkage in which no two adjacent sides possess twist angles which add up to an angle greater than 180 deg. Thus the additional classification is based on properties which are not claimed.

This fourth point illustrates the need for the simplest possible valid model set in linkage analysis. The more detailed a description is, the easier it is to visualize. And visualization is a very useful tool in engineering analysis.

The authors would like to thank Professor Soni again for his interest in spherical four-bar linkages and this paper. It is hoped that this additional discussion clarified the vagueness in the paper which precipitated the discussion.