Function $f(x)$ being odd and analytic can be represented by a power series expansion involving only odd-power terms:

$$f(x) = ax + bx^3 + cx^5 + \dots$$

Applying (3) and (4) and setting $x_1 = k \cdot A$, $k < 1$, we find

$$k = \tfrac{1}{2} + \psi(A^2, \dots)$$

For small amplitude oscillations the first-order approximation $k = \frac{1}{2}$ is adequate. Substitution of $x_1 = \dfrac{A}{2}$ into formula (4) yields (1).

It should be observed that due to a typographical error in Denman's paper [formula 10] the symbol A was omitted in the denominator of (1).

Author's Closure

Mr. Jonckheere's remarks reflect the fact, well known to applied mathematicians, that Chebyshev polynomial approximations have nearly minimum-maximum error. While the two methods give the same result as $A \to 0$, the graphical procedure is better when the nonlinear restoring function is known only graphically, and it is so simple it can be applied by a draftsman. The approximate equivalent linearization technique is preferable when the function is known analytically, and can be extended to higher-order approximations.

In reference to Mr. Jonckheere's last remark, since $T_1(x') = T_1(x/A) = x/A$, equation (10) in my paper is correct.

The Hydroelastic Stability of a Flat Plate[1]

E. H. Dowell.[2] Weaver and Unny have recently presented an interesting study of the title problem. It is the purpose of the present Discussion to point out that the problem has been studied in considerable greater generality by the present author in reference [1].[3] In particular, the effects of fluid compressibility, plate two-dimensionality, and nonlinear structural behavior have been taken into account. There is also an experimental literature [2]. To answer a major question asked by Weaver and Unny, the more comprehensive theory and the most recent as well as earlier experiments all indicate a simple static divergence with no flutter. For a recent survey of the aeroelastic stability of plates and shells (of which hydroelastic stability is a special case) the reader may refer to reference [3].

References

1 Dowell, E. H., "Nonlinear Oscillations of a Fluttering Plate II," *AIAA Journal*, Vol. 5, No. 10, 1967, pp. 1856–1862.

2 Gislason, T., Jr., "An Experimental Investigation of Panel Divergence at Subsonic Speeds," *Princeton University AMS Report* No. 921, 1970.

3 Dowell, E. H., "Panel Flutter: A Review of the Aeroelastic Stability of Plates and Shells," *AIAA Journal*, Vol. 8, No. 3, 1970, pp. 385–399.

Authors' Closure

The authors would like to thank Dr. Dowell for his interest in their work and for bringing to their attention the recent experimental report (Dowell's reference [2]).

It seems to us that the problem of the hydroelastic behavior of a plate is not entirely solved. While Dowell's reference [1] gives a very comprehensive analysis, its emphasis is on supersonic flow. Furthermore, if the fluid is a liquid, compressibility may well be neglected and it is to be hoped that reasonable results for the hydroelastic problem could be obtained with a computational effort short of that required in this reference.

There seems to be little doubt about the occurrence of static divergence. However, it is difficult to understand Dr. Dowell's remark about no flutter. Flutter has been predicted for higher velocities by our theory as well as, apparently, by Dowell (his reference [1, p. 1859], Dowell [4],[4] and Dugundji, et al. [5]. The latter, dealing with long panels on an elastic foundation, was confirmed experimentally. In addition, the early paper by Jordan [6] discussed the occurrence of flutter in his experiments and a survey of U. S. companies [7] reported a number of incidents of subsonic panel flutter experienced on flight vehicles. Perhaps experiments in a water tunnel with the consequent higher dynamic pressures, and hence lower critical flow velocities would shed more light on the question.

References

4 Dowell, E. H., "Flutter of Infinitely Long Plates and Shells—Part 1: Plate," *AIAA Journal*, Vol. 4, No. 8, 1966, pp. 1370–1377.

5 Dugundji, J., Dowell, E. H., and Perkin, B., "Subsonic Flutter of Panels on Continuous Elastic Foundation," *AIAA Journal*, Vol. 1, No. 5, 1963, pp. 1146–1154.

6 Jordan, P. F., "The Physical Nature of Panel Flutter," *Aero Digest*, Feb. 1956, pp. 34–38.

7 Mirowitz, L. I., et al., "Panel Flutter Survey and Design Criteria," Aerospace Industries Association of America, Report ARTC-32, Aug. 1962.

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[1] By D. S. Weaver and T. E. Unny, published in the September, 1970, issue of the JOURNAL OF APPLIED MECHANICS, TRANS. ASME, Vol. 92, Series E, pp. 823–827.

[2] Associate Professor, Department of Aerospace and Mechanical Sciences, Princeton University, Princeton, N. J.

[3] Numbers in brackets designate References at end of Discussion.

[4] Numbers in brackets designate References at end of Closure.

The Free Plastic Compression of Pure Metals[1]

M. J. HILLIER.[2] The authors have set out to provide an extension of empirical laws of plastic behavior to a wider range of conditions. For practical purposes they appear to have succeeded and are to to congratulated. The following opinions, therefore, should be taken primarily as a criticism of our present state of fundamental knowledge.

In order to describe the plastic behavior of materials it appears necessary, and it may be sufficient, to recognize the validity of the following points of view:

1 The so-called quasi-static "strain-hardening" curve is unlikely to represent basic data for there is little evidence that it has ever been obtained under either isothermal or adiabatic conditions. At low rates of deformation the apparent rate of strain hardening is often controlled by the rate of heat conduction. At high rates of deformation a significant proportion of the apparent strain hardening may often be accounted for by material inertia.

2 The so-called "stress relaxation" has not been shown to be

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[1] By V. S. Shankla and R. F. Scrutton, published in the December, 1970, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 92, Series E, pp. 1121–1133.

[2] Associate Professor, Faculty of Engineering and Applied Science, Memorial University of Newfoundland, St. John's, Newfoundland, Canada.

the result of (volume constant) plastic deformation. It is most probably the consequence of volume changes consequent upon cooling following plastic deformation, and volume changes associated with the diffusion out of vacancies in a stress field.

3 The so-called "strain-rate effect" does not explicitly represent any physical property. "Strain rate" is itself a dependent variable formed from the independent variables, strain, and time. The strain-rate effect is an empirical and implicit result of an explicit dependence on strain and time separately. A physical description of behavior in terms of the strain-rate variable often violates the principle of time reversal invariance.

4 The sensitivity of yield (flow) stress to increases in temperature occurring as a result of plastic deformation is an order of magnitude greater than the corresponding sensitivity to changes in initial or ambient temperature.

5 True strain hardening is time-independent and is associated only with isothermal plastic strain.

6 Irreversible thermal softening associated with an increase in temperature requires time in order to take effect. The strain-rate effect is a consequence of this time-dependence. Strain hardening is not increased by an increase in the rate of deformation. It is the thermal softening exhibited during the time scale of deformation that is reduced by an increase in the strain rate.

7 The current yield stress $\sigma(t)$ at time t is strain (e) and temperature (T) history-dependent through a functional relationship of the form

$$\sigma(t) - \sigma(t_0) = \int_{t_0}^{t} \left[\frac{\partial S}{\partial e} + \frac{\partial S}{\partial T} \phi(t - t') \right] dt'$$

where

$$S = S(e, T)$$

and ϕ is a function of time only.

It is the present writer's opinion that the apparently baffling jigsaw of contradictory plastic phenomenon can only be fitted together with the aid of the overall picture provided by the foregoing qualitative description.

J. J. JONAS.[3] The authors have undertaken the analysis of a very difficult problem in metal plasticity, one in which both the strain rate and the temperature are varying during the deformation. The effect on the flow stress of the adiabatic temperature rise is clearly of considerable importance when the strain rate is $\simeq 10^2$ sec^{-1}, and this point is brought out in the paper. However, the authors conclude that the flow stress is relatively insensitive to a threefold change in strain rate during adiabatic plastic flow. I would like to ask whether this conclusion follows from the manner in which the rate sensitivity is taken into account in their equations (e.g., equation (10)), or whether the lack of rate sensitivity is actually a property of the material investigated.

This question comes to mind because numerous determinations of the rate sensitivity have now been made for the case of constant strain-rate deformation at homologous temperatures greater than 0.5.[4] These lead to values of about 0.2 for the exponent m in the relation $\sigma = K\dot{e}^m$. If the steady-state rate sensitivity can be considered to apply, at least approximately, to conditions of changing strain rate, a threefold change in strain rate can be expected to lead to something like a 25 percent change in flow stress. Even if the rate sensitivity is only 0.1 under the conditions used by Slater, et al., a flow stress change of about 12 percent would still be expected.

J. G. WAGNER.[5] As interpreted by the discusser, the authors propose the free compression test as a valuable means of evaluating empirical uniaxial constitutive relations. The free compression test is defined as the homogeneous adiabatic compression of a test specimen by the impact of a falling weight. The objective of the authors is well founded; however, the discusser would like to make several points in relation to the approach adopted.

The fact that no universal stress-strain relation capable of describing quasi-static isothermal behavior, as well as dynamic "adiabatic" behavior, has not empirically emerged is not surprising. Conceptually, the reasons are clear. Empirical relations, of the type considered, have for the most part no fundamental basis. They are mathematical models of uniaxial behavior under rather limited and sometimes unknown conditions. Many times several functional forms can adequately model the same data. The authors have chosen to test the universality of two specific forms (equations (4) and (10)) of these empirical relations. The description of adiabatic behavior through the use of presumably isothermal relations embodies two key assumptions:

1 The total energy of deformation is completely converted to heat.

2 A mechanical equation of state governs the deformation process.

The first assumption may be very nearly correct for lead under certain conditions [1];[6] however, its applicability should be examined for the test conditions of interest. The assumption of greater interest is that of the existence of a mechanical equation of state. The utility of such an equation is at best limited to monotonically increasing variations in the test parameters. Basically, flow stress is structure sensitive and governed by dislocation arrays which are not in thermodynamic equilibrium. The dislocation structure regulating the flow stress is, for any given strain and temperature, a path function of the strain-temperature history. It may well be that, even for monotonically increasing strains, a mechanical equation of state does not fundamentally exist.

The conclusion that a power law, of the form given by equation (4), is inadequate due to a lack of temperature dependence may not be warranted. The fact that the coefficient σ_0 must exhibit a strong temperature dependence is evident from the data from which it emerged ($\sigma = \sigma_0$ at $e = 1$). This functional dependence must be accounted for in the manipulations of equation (7). Moreover, σ_0 and n are, in general, also dependent on the strain rate ($\dot{e}$). In ruling out these considerations, a meaningful comparison of equations (4) and (10) may not be possible.

It should also be noted that equations (4) and (10) are not necessarily unique to that class of metals exhibiting face-centered cubic crystal structures. Both the power-law form of equation (4) and a "velocity modified temperature" form [2], as exhibited by equation (10), have proven useful in describing the uniaxial behavior of certain BCC materials.

The elementary analysis of the free compression test assumes homogeneous deformation of the test specimen. Wave propagation within the specimen and its interaction with the test apparatus is ignored. In view of the authors objective, consideration of wave-propagation effects would seem essential. Unfortunately, the inclusion of potentially inhomogeneous deformation introduces considerable mathematical complexity into the analysis. Such complexity would considerably distract from the convenience of using the free compression test for the purpose proposed.

References

1 Loizou, N., and Sims, R. B., "The Yield Stress of Pure Lead in Compression," *Journal of the Mechanics and Physics of Solids*, Vol. 1, 1953, pp. 234–243.

2 MacGregor, C. W., and Fischer, J. C., "A Velocity Modified Temperature for the Plastic Flow of Metals," JOURNAL OF APPLIED MECHANICS, Vol. 13, TRANS. ASME, Vol. 68, 1946, pp. A11–A16.

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[3] McGill University, Montreal, Canada.

[4] Jonas, J. J., Sellars, C. M., and McTegart, W. J., "Strength and Structure Under Hot Working Conditions," *Metallurgical Reviews*, Vol. 14, No. 130, 1969, pp. 1–24.

[5] Assistant Professor, Department of Mechanical Engineering, University of Pittsburgh, Pittsburgh, Pa.

[6] Numbers in brackets designate References at end of Discussion.

Authors' Closure

The authors wish to thank the contributors for their valuable comments and criticisms. Perhaps the queries raised by Professor Jonas may be considered as being answered in part by the third and sixth of Professor Hillier's comments. In this case the lack of rate sensitivity would be a consequence of the mathematical form of the equations and of the properties of the material. In any case the effect is somethat obscured by the experimental errors associated with the drop-weight technique. Although exact numerical results are important it is considered that the main bearing of the paper has to do with the philosophical interpretation of plastic data.

In reply to Professor Wagner the authors agree that the effects of wave propagation were neglected. However, by a suitable choice of impact velocity, specimen size, and anvil dimension these effects may be minimized. Inhomogeneous deformations associated with frictional effects usually cause some errors when compressing plastic specimens. However, more recent and exact analyses of the barrelling tendency suggest that it may be possible in the future, to introduce a correction term to take care of this source of error. In hot-working operations little energy is lost in internal lattice distortion or specimen hardening. The more common energy loss is usually attributable to thermal conduction. This is a serious matter when straining at slow speeds or when using hot specimens and cold platens.

The existence of an equation of state is frequently refused credence on the ground that the instantaneous flow stress is dependent on the past history of the specimen (and thus on its structure). It is suggested that this past history of straining or heat-treatment itself may have been subject to the rules of an equation of state. However the position as outlined in the paper regarding the existence of an all-embracing equation of state is clear. No such law was postulated and no support was given to any particular mathematical form. It is agreed that mathematical models are merely based on empirical results which have been plotted on graph paper. However, engineers are usually optimists and would prefer to think that practical problems may be based on the basis of well-established physical laws. Where these do not exist it is natural to clutch at straws and to use or seek to extend the range of usefulness of whatever data may be available.

The authors wish to thank Professor Hillier for his perceptive comments regarding the plastic flow of metals which comments have contributed significantly to the value of the paper. Presumably a number of the conclusions stated are based on the results of experimental work performed by Professor Hillier and it is to be hoped that these results may be published in the near future. It would seem from comment number four that the extension of an empirical isothermal equation to take account of adiabatic heating may be fundamentally unjustified even though it would appear to be practically convenient. Presumably the much larger temperature sensitivity during plastic flow is to be attributed to the very localized nature of dislocation movement and to the accumulation of slip in submicroscopic lamellas. It is to be hoped that this discussion may lead to a critical reappraisal of existing plasticity concepts.

Low-Speed Slip Flow Over a Wedge[1]

Y. C. L. WU.[2] The authors presented a very useful method in treating the incompressible slip flow over a wedge. The agreement with numerical calculations certainly would inspire workers to seek solutions of boundary-layer flows by means other than purely numerical.

As pointed out by the authors the study of rarefied gas flow is motivated from the high-speed flight in the upper atmosphere, therefore, it would be of great interest to extend the present work to treat the compressible boundary layer. That the slip velocity agrees so well with experiments at high Mach number ($M = 10$) is somewhat surprising. Does this mean that the compressibility effect and variation in transport properties in hypersonic flow are not important in determining the slip velocity? Is this also true for skin friction?

Authors' Closure

The authors wish to thank Professor Wu for her interest and generous comment in regard to our work in low-speed slip flow over a wedge [1].[3]

It should be pointed out that the experimental data on slip velocity are available only for cases of flat plate flow within the hypersonic regime. Comparison with the low-speed theoretical analysis is possible only after these results are reinterpreted through compressibility transformations and this has been carried out in an early study [2]. A linear viscosity-temperature relationship has been introdued for the expedience of decoupling the momentum and energy equations. However, in another study of hypersonic flow past a flat plate [3], where a different viscosity-temperature relationship ($\mu \sim \sqrt{T}$) and a slightly different velocity-slip coefficient have been employed, the analytical results of velocity slip are not much different and they all compare favorably with the same set of experimental data. Unfortunately, experimental data of velocity slip under a variety of flow conditions have not been produced and a thorough evaluation of the theoretical results is not possible at the present. The authors fully appreciate Professor Wu's concern and cannot answer her question in a satisfactory manner although the limited amount of available experimental data does indicate that a simple account of transport properties for hypersonic flows, as suggested in these early studies, might be adequate under these flow conditions. For the same reason, the authors cannot also answer adequately the question concerning the skin friction, since the experimental information on skin friction within this flow regime is practically nil.

Perhaps it is pertinent to point out that analyses of slip-flow problems by the present scheme would yield a slip-velocity larger than 50 percent of the free stream close to the leading edge of the plate. This is possible only when specular reflection occurs there. Recent experimental data on heat transfer [4, 5] indicated that as the leading edge is approached, the rate of heat transfer does not tend to that of the diffused free molecular flow. The theoretical results produced from reference [3][4] now gives a better agreement with these new data. Again, additional abundant experimental data are needed to clairfy the flow conditions close to the tip of the plate.

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2 Chow, W. L., and Chow, B. T., "Slip Flow Past a Semi-In-

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[1] By K. E. Kasza and W. L. Chow, published in the June, 1970, issue of the Journal of Applied Mechanics, Vol. 92, Series E, pp. 454–460.

[2] Associate Professor, The University of Tennessee Space Institute, Tullahoma, Tenn.

[3] Numbers in brackets designate References at end of Closure.

[4] These results are misquoted reference [4]. Also the merged layer analysis has been further extended so that the results from reference [3] now tend asymptotically to strong interactions limit.