The problem of $\pi^-$ condensation within the framework of the highly simplified Hamiltonian (in which only the negative pions are contained) is discussed by using the method of the normal mode. The contradiction between the conclusions presented by Sawyer et al. and Migdal with respect to the $\pi^-$ condensation in neutron star matter is resolved.

§ 1. Introduction

The possibility of pion condensation in high density nuclear matter has attracted special interest recently. It is very important to investigate the characters of nuclear matter from the viewpoint of density. There are some different approaches to this problem.

Sawyer and Scalapino\(^3\sim^5\) have worked on the problem of $\pi^-$ condensation in neutron star matter on the basis of the highly simplified Hamiltonian in which the condensed $\pi^-$ field has been replaced by the mean field. They have suggested that the ground state energy of neutron star matter may be lowered by the appearance of $\pi^-$ condensed phase at the critical density. On the other hand, by using the pion Green function in nuclear matter Migdal\(^4\) has shown that an electrically neutral ($\pi^+\pi^-$) pair and $\pi^0$ condensate appear. Baym et al.\(^5\) have discussed the problem of pion condensation using the chirally symmetric Lagrangian. There is, however, a serious contradiction between their conclusions with respect to the possibility of $\pi^-$ condensation in neutron star matter. Migdal\(^6\) has claimed that Sawyer et al. have used a wrong procedure in the variational calculations of the energy, and that the correct calculation does not lead to the formation of $\pi^-$ condensate. On the contrary, Sawyer\(^7\) have maintained that Migdal’s remarks on the critical condition are not correct. Au, Baym and Flowers\(^8\) have discussed the criteria for the appearance of pion condensation in terms of the Green’s function in matter. In a recent paper, Kanai\(^9\) supported Migdal’s situation. At the present stage, however, there are several uncertain points. The problems are focused on the chemical potential and the configuration of new protons (quasi-particle).

In our previous paper\(^10\) we derived the instability condition of neutron star matter from a viewpoint of the normal mode method. The purpose of this paper is to investigate the physical meaning of $\pi^-$ condensation within the framework of the model Hamiltonian in which $\pi^-$ mesons are treated as the Schrödinger field.
In § 2 the critical density for the instability of normal neutron star matter is derived.

In § 3 the model of Sawyer and Scalapino is reformulated by the normal mode method.

In § 4 the critical condition is discussed in detail.

§ 2. Instability of neutron star matter

First of all, we briefly mention a necessary condition for the stability of the Hartree-Fock solution in the many-body problem. In former times Sawada and Fukuda have found that there exists an important relation between the instability of the Hartree-Fock state and the frequency of normal mode. The instability of the Hartree-Fock state is characterized by the complex frequencies of the corresponding collective eigenmode $S'$ which satisfies the following equation:

$$[H, S'] = \omega S',$$  \hspace{1cm} (2.1)

where $H$ is the total Hamiltonian of the system. In this case we can construct another ground state solution with lower energy which is to be obtained by the variational principle by making use of the trial function

$$|\Psi(\lambda)\rangle = e^{i(S(\lambda)+S(\lambda))}|\Phi_0\rangle,$$  \hspace{1cm} (2.2)

where the function $S(\lambda)$ has the same structure as $S$, but the coefficients contain some variational parameters.

For definiteness, we discuss the problems within the framework of the following simple model Hamiltonian:

$$H = \sum_q \varepsilon_q \left( n_q^+ n_q^- + p_q^+ p_q^- \right) + \omega_k a_k^+ a_k^- - i M_k \sum_q (p_{q-k}^+ \sigma \cdot n_q a_k^- - h. c.),$$  \hspace{1cm} (2.3)

where $n_q^+ = (n_q^+, n_q^-)$ and $p_q^+ = (p_{q^+}, p_{q^-})$ are respectively the creation operators for neutrons and protons of momentum $q$, $a_k^+$ is the creation operator for $\pi^-$ of momentum $k = -k^\perp$, $\varepsilon_q = q^2/2m$ is the kinetic energy of nucleon and $\omega_k = (k^2 + m^2)^{1/2}$ is the energy of pions. The last term in the Hamiltonian is the $p$-wave part of pion-nucleon interaction, in which $M_k$ has the form

$$M_k = \frac{f k}{m_s (\omega_k Q)^{1/2}},$$  \hspace{1cm} (2.4)

where $f$ is the coupling constant and $Q$ the volume. This Hamiltonian contains only the negative pion and corresponds to the description of pions by the Schrödinger equation. The Hartree-Fock state in our case is the normal state of pure neutron matter in which all the single-particle states are filled with neutrons up to the Fermi momentum $q_F$ set by the total nucleon density $(3\pi^2 \rho)^{1/3}$;

$$|\Phi_0\rangle = \prod_{q < q_F} n_{q^+, \sigma}^+ |0\rangle,$$  \hspace{1cm} (2.5)
where $|0\rangle$ is the true vacuum which does not contain pions. In order to find an eigenmode associated with $\pi^-$, we define an operator as

$$S_k^{(-)} = A_k a_k^+ + \sum_q \xi_k(q) n_q^+ \sigma_q \rho_{q-k},$$

(2.6)

where the second term of the right-hand side is the particle-hole operators coupled with pion field. The coefficients $A_k$ and $\xi_k(q)$ are to be determined from Eq. (2.1). One has in this way

$$(\omega - \omega_k) A_k - i M_k \sum_q N_q(n) \xi_k(q) = 0,$$

(2.7)

where $N_q(n) = \langle \Phi_0 | n_q^+ n_q | \Phi_0 \rangle$. From Eq. (2.7) we obtain the eigenvalue equation for $\omega$:

$$1 = -\frac{m q_F f^2 k^2}{2\pi^2 m^*} \frac{\phi(k, \omega)}{\omega_k (\omega - \omega_k)},$$

(2.8)

where

$$\phi(k, \omega) = \frac{2\pi^2}{m q_F \omega} \sum_n \frac{N_q(n)}{\omega + \varepsilon_{q-k} - \varepsilon_q},$$

$$= \frac{\omega + \varepsilon_k - q_F}{2\varepsilon_k} \left\{ \frac{(\omega + \varepsilon_k)^2}{k^2 v_F^2} - 1 \right\} \ln \left| \frac{\omega + \varepsilon_k + k v_F}{\omega + \varepsilon_k - k v_F} \right|. \quad (2.9)$$

The eigenvalue equation (2.8) can be evaluated explicitly. The right-hand side is shown as a function of $\omega$ in Fig. 1, in the case that the stability is preserved. The intersection points between A and B correspond to the continuum states of particle-hole in the limit $Q \to \infty$. The points C and D correspond to the collective motion of neutron particle-proton hole pairs and the $\pi^-$ state, respectively. As the density increases, the loop between C and D is lifted up and therefore the points C and D come closer to each other. At the critical density $\rho_c$ they coincide with each other (double pole). The complex eigenfrequencies appear at the densities above $\rho_c$. For $\omega + \varepsilon_k \gg k v_F$, we have

$$\phi(k, \omega) \approx -\frac{2\pi^2 \rho}{m q_F (\omega + \varepsilon_k)}.$$

In this case the points C and D coincide with each other at $\omega = \omega_k/2$ and the
corresponding critical density is
\[ \rho_c = \frac{m^* \omega_k (\omega_k + \xi_k)^2}{4 \pi^2 k^2} \approx \frac{m^* \omega_k^2}{4 \pi^2 k^2}, \]
where we neglect finally the difference of the particle-hole energy.

On the other hand, the eigenmode for the proton particle-neutron hole pairs \( (p\bar{n}) \) is equal to \( S_k^{(-)} \). The eigenvalue is, therefore, given by \( -\omega \). The instability condition is the same as that described for \( \pi^- \). According to the criterion on instability, the state \( |\Phi_0\rangle \) is no longer stable with respect to these kinds of collective oscillations. This instability leads to formation of an electrically neutral \( \pi^- \), quasi-boson (particle-hole pair) condensate.²

§ 3. Rederivation of the results by Sawyer and Scalapino

In the previous section we found that the normal neutron sea \( |\Phi_0\rangle \) becomes unstable with respect to the collective oscillation described by \( S_k^{(-)} \) at the critical density \( \rho_c \). Thus we have to look for another ground state solution with lower energy which is to be obtained on the basis of the variational principle by making use of the following trial function:

\[ |\Psi\rangle = \exp\left[i(S_k^{(-)} + S_k^{(-)})\right]|\Phi_0\rangle = U|\Phi_0\rangle, \]
\[ S_k^{(-)} = S_k^t + S_k^n, \]  \( (3.1) \)

where \( S_k^{(-)} (\alpha, \beta) \) has the same structure as \( S_k^{(-)} \) in Eq. (2.6) with respect to operators \( a^t, a, p^t \sigma_n, \) and \( n^t \sigma_d p \). We decompose the unitary operator \( U \) as follows:
\[ U = U_s U_n, \]
\[ U_s = \exp\left[i(S_k^t + S_k^n)\right] = \exp\left[\alpha a_k^t - \alpha^* a_k\right], \]
\[ U_n = \exp\left[i(S_n^t + S_n^n)\right] = \exp\left[\sum_q \beta_q (p_{q-k}^t \sigma_q n_q) - \beta_q^* (n_q^t \sigma_q p_{q-k})\right]. \]  \( (3.2) \)

In this place we shall consider the physical meaning of the trial function. The state constructed by operating \( U_\pi \) to the meson vacuum is the so-called “coherent state”.²

\[ |\alpha\rangle = U_\pi |0\rangle = \exp \left(-\frac{1}{2} |\alpha|^2 \right) \sum_n \frac{1}{n!} (\alpha a_k^t)^n |0\rangle = \sum_n \exp (in\varphi) \sqrt{\omega_n} |n\rangle, \]  \( (3.3) \)

where

² If the pion fields which are described by the Klein-Gordon equation are used, the eigenmode for \( \pi^- \) is defined by
\[ S_k^{(-)} (\pi^-) = A_k a_k^t + B_k b_{-k} + \sum_q \xi_k (q) \left( n_q^t \sigma_q p_{q-k} \right). \]
The eigenmode \( S_k^{(-)} (\pi^-) \) for \( \pi^- \) is, therefore, equal to \( S_k^{(-)} \), and has the eigenvalue \( -\omega \). Hence, the condensate of \( \pi^- \) and particle-hole pairs appear at the critical density.
\[ |n\rangle = \frac{1}{\sqrt{n!}} (a_k^\dagger)^n |0\rangle, \]
\[ \sqrt{\omega_n} = \frac{1}{\sqrt{n!}} |\alpha|^n \exp\left(-\frac{1}{2} |\alpha|^2\right), \]
\[ \alpha = |\alpha| \exp(i\varphi_\alpha). \]

This form shows that the average occupation number of the \( n \)-th state is given by a Poisson distribution with a mean value \( |\alpha|^2 \). The expectation value of \( a_k \) is

\[ \langle \alpha | a_k | \alpha \rangle = \alpha. \]

Next, the state \( |\Psi \rangle = U_N |\Phi_0\rangle \) is represented as the coherent state of proton particle-neutron hole pair (\( p\pi \)-pair) or the Hartree-Fock state of the new neutron particles (\( u_q \)):

\[ |\Psi \rangle = \prod_{q, q'} \left( \cos |\beta_q| + (-1)^{\langle q | q \rangle} e^{i\varphi_q} \sin |\beta_q| \right) |\phi_0\rangle \]
\[ = \prod_{q, q'} u_{q q'}^\dagger |0\rangle, \]

where \( u_{q q'}^\dagger = U_N \beta_{q-k} \beta_{q-k}^\dagger U_N^{-1} \). This wave function is reduced to the normal neutron star matter at \( \cos |\beta_q| = 1, \sin |\beta_q| = 0 \). In order to maintain overall charge neutrality, the number of protons must be equal to the number of pions. The condition of charge neutrality is given by

\[ \langle \Psi | \sum_q p_q^\dagger p_q | \Psi \rangle = \langle \Psi | a_k^\dagger a_k | \Psi \rangle. \]

The constraints on the state are conveniently dealt with by introducing Lagrange multiplier \( \mu \). The effective Hamiltonian is, therefore,

\[ \mathcal{H} = H + \mu \left( \sum_q p_q^\dagger p_q - a_k^\dagger a_k \right). \]

For the calculation of the expectation value of \( \mathcal{H} \) it is convenient to transform \( \mathcal{H} \) by the unitary operator \( U \). The results of transformations of \( a_k^\dagger, p_{q-k}^\dagger \) and \( n_q^\dagger \) are as follows:

\[ a_k^\dagger \rightarrow U_n^{-1} a_k^\dagger U_n = a_k^\dagger + \alpha^*, \]
\[ p_{q-k}^\dagger \rightarrow U_n^{-1} p_{q-k}^\dagger U_n = \cos |\beta_q| \beta_{q-k}^\dagger + e^{-i\varphi_q} \sin |\beta_q| n_q^\dagger \sigma_z, \]
\[ n_q^\dagger \rightarrow U_n^{-1} n_q^\dagger U_n = \cos |\beta_q| n_q^\dagger - e^{i\varphi_q} \sin |\beta_q| p_{q-k}^\dagger \sigma_z. \]

With this transformation, the new Hamiltonian has the form

\[ \mathcal{H} \rightarrow \mathcal{H}_{\text{trans}} = U^{-1} \mathcal{H} U = H_1 + H_2 + H_3 + E_0, \]
\[ H_1 = (\omega_k - \mu) a_k a_k^\dagger - i \mu a_k^\dagger \sum_q \{ \cos |\beta_q|^2 |\beta_q| \beta_{q-k}^\dagger \sigma_z n_q : \}
\[ - e^{-2i\varphi_q} \sin |\beta_q| n_q^\dagger \sigma_z \} + \text{h.c.}, \]
\[ H_2 = \sum_q \left( \frac{1}{2} (\varepsilon_{q-k} - \varepsilon_q + \mu) \sin 2 |\beta_q| + M_k |\alpha| (e^{-t(p_{q-k}^\dagger + \alpha^*/2}) \cos |\beta_q| \right) \]
\[ H_3 = \sum q \left( \beta_{q-k}^\dagger p_{q-k} + p_{q-k}^\dagger \beta_{q-k} \right) \]
The symbol $: :$ represents a normal product.

The stationary condition for $E_0$ with respect to $\beta_q$ and $\beta_q^*$ leads to the equations

$$\tan 2|\beta_q| = \frac{2M_k|\alpha|}{\varepsilon_q - \varepsilon_{q-k} - \mu} \cos \left( \varphi_q + \varphi_* + \frac{\pi}{2} \right)$$

and

$$\sin \left( \varphi_q + \varphi_* + \frac{\pi}{2} \right) = 0,$$

which make $H_2$ vanish. The phase $\varphi_q + \varphi_*$ is selected as $\pi/2$. Equation (3.11) agrees with the condition of diagonalization of the Hamiltonian in the mean field approximation. The minimum condition for $E_0$ with respect to $\alpha$ and $\alpha^*$ leads to $H_1 = 0$. Then

$$|\alpha| = -\frac{M_k}{2(\omega_k - \mu)} \sum_q \cos \left( \varphi_q + \varphi_* + \frac{\pi}{2} \right) \sin 2|\beta_q| \cdot N_q(n).$$

The Hamiltonian $H_1$ describes non-condensed pions. From the charge neutrality constraint $\partial E_0/\partial \mu = 0$ we have

$$N \cdot \sin^2 |\beta_q| = |\alpha|^2,$$

where the total baryon number $N$ is equal to $\sum N_q(n)$. In the static limit, we can treat $\sin |\beta_q|$ as $\theta$ ($q$-independent). We neglect the difference of the neutron and proton kinetic energies $\varepsilon_{q-k} - \varepsilon_q$. Thus, Eqs. (3.11)~(3.13) become respectively

$$\theta \left( 1 - \theta^2 \right)^{1/2} = \frac{M_k|\alpha|}{\mu},$$

$$|\alpha| = \frac{M_kN\theta (1 - \theta^2)^{1/2}}{\omega_k - \mu},$$

$$N\theta^2 = |\alpha|^2,$$

and the ground state energy is

$$E_0 = \sum_q \varepsilon_q N_q(n) + \sum_q (\mu\theta^2 - 2M_k|\alpha|\theta \cdot (1 - \theta^2)^{1/2}) N_q(n).$$
The quantity $\mu$ is determined from Eqs. (3·14), (3·15) and (3·16). Thus we obtain

$$\mu = \omega(k) = \frac{1}{2} \omega_k \left\{ 1 + \sqrt{1 - \frac{4M_k^2N}{\omega_k^2}} \right\} = \frac{1}{2} \omega_k \left\{ 1 + \sqrt{1 - \frac{\theta}{\rho_c}} \right\},$$

in the limit $\theta \to 0$ ($\rho \leq \rho_c$) (Appendix 1), and

$$\mu = \frac{5}{6} \omega_k \left\{ 1 - \sqrt{1 - \frac{12(2\omega_k^2 - M_k^2N)}{25\omega_k^2}} \right\} = \frac{5}{6} \omega_k \left\{ 1 - \frac{1}{5} \sqrt{1 + \frac{3}{\rho_c}} \right\}
\text{for } \theta \neq 0. \quad (\rho \geq \rho_c) \quad (3·19)$$

These coincide with each other at the critical density and become $\mu = \omega_k/2$. The former is equal to the excitation energy of pion in the normal state (Appendix 1), and is not the solution which we need. The value (3·19) of $\mu$ for $\theta \neq 0$ is the solution which we need. We can regard the quantities $\theta(1 - \theta^2)^{1/2}$ and/or $|\alpha|$ as the order parameters. From Eqs. (3·14) and (3·15), $|\alpha|$ is determined as a function of $\mu$, and we obtain

$$|\alpha|^2 = \frac{M_k^2N^2}{4(\omega_k - \mu)^2} - \frac{\mu^2}{4M_k^2}. \quad (3·20)$$

In order to eliminate the parameter $\theta$, we rewrite Eq. (3·17) as follows:

$$E_\theta = \sum_q \left\{ \varepsilon_q + \frac{1}{2} \mu \left( 1 - (1 - 2\theta)^2 - \frac{4M_k|\alpha|}{\mu} \theta (1 - \theta^2)^{1/2} \right) \right\} N_q(n). \quad (3·21)$$

Let us define $\gamma$ by

$$\gamma = 1 - 2\theta^2 + \frac{4M_k|\alpha|}{\mu} \theta (1 - \theta^2)^{1/2}. \quad (3·22)$$

Using Eq. (3·14) we obtain

$$\gamma = \sqrt{1 + \frac{4M_k^2|\alpha|^2}{\mu^2}} = \frac{M_k^2N}{\mu(\omega_k - \mu)}. \quad (3·23)$$

Then Eq. (3·21) becomes

$$E_\theta = \sum_q \left\{ \varepsilon_q + \frac{1}{2} \mu (1 - \gamma) \right\} N_q(n) + (\omega_k - \mu) |\alpha|^2
= \frac{3}{5} \varepsilon_F(n)N + \frac{1}{2} \mu (1 - \gamma) N + (\omega_k - \mu) |\alpha|^2. \quad (3·24)$$

By substituting Eq. (3·20) into Eq. (3·24), we have

$$E_\theta = \frac{3}{5} \varepsilon_F(n)N - \mu N \left( \frac{1 - \gamma}{4\gamma} \right)^{1/2}. \quad (3·25)$$

The condensation energy per baryon becomes, using the value of Eq. (3·19),
which is the energy decrease on account of our transformation and agrees the result by Sawyer and Scalapino.\(^5\)

\section{Critical conditions and discussion}

In this section, we investigate the critical condition in detail, and discuss the conclusions of Sawyer et al. and Migdal.

First of all, we define \(V_{q_1}\) and \(b_{k_1}\) in the same manner as \(U_{q_1}\) (Eq. (3.6)) by using the transformation function \(U:\)

\begin{equation}
\begin{aligned}
V_{q_1} &= U_{q_1} U_{q_1}^{-1}, \\
b_{k_1} &= U_{k_1} a_{k_1} U_{k_1}^{-1} = a_{k_1} - a_{k_1}^*. 
\end{aligned}
\end{equation}

We call hereafter \(U_{q_1}\) and \(V_{q_1}\) the "new neutron" and the "new proton", respectively. We fix the representation in such a way that the limit \((\Omega \to \infty)\) of the Hamiltonian \(\mathcal{H}\) (Eq. (3.8)) is diagonal with respect to \(u^*(\alpha), v^*(\beta)\) and \(a^*\). Where the condition for the off-diagonal terms to vanish is equal to Eq. (3.11). The Hamiltonian is rewritten as follows:

\begin{equation}
\begin{aligned}
\mathcal{H} &= \sum_q \left\{ \varepsilon_q (1 - \theta^2) + (\varepsilon_{q-k} + \tilde{\mu}) \theta^2 - 2M_k |\alpha| \cdot \theta (1 - \theta)^{1/2} \right\} u_{q_1}^* u_q \\
&\quad + \sum_q \left\{ (\varepsilon_{q-k} + \tilde{\mu}) (1 - \theta^2) + \varepsilon_q \theta^2 + 2M_k |\alpha| \cdot \theta (1 - \theta)^{1/2} \right\} v_{q-k}^* v_{q-k} \\
&\quad + (\omega_k - \tilde{\mu}) a_{k_1}^* a_k, 
\end{aligned}
\end{equation}

where we assume that \(|\alpha|/\Omega\) is finite for \(\Omega, |\alpha| (\infty N) \to \infty\). The other terms in \(\mathcal{H}\) tend to zero as \(\Omega \to \infty\). Therefore, \(u^*, v^*\) and \(b^*\) are the creation operators of the asymptotic fields in the field theory. The ground state is generally given as follows:

\begin{equation}
\begin{aligned}
|\Psi\rangle &= \prod_{q \in A_p} u_{q_1}^* \prod_{q \in A_p} v_{q_1}^* |\alpha\rangle \\
&= U_{\Phi_0},
\end{aligned}
\end{equation}

where \(|\Phi_0\rangle = \prod_{p \notin \mathcal{A}} n_{q_1}^* \prod_{p \notin \mathcal{A}} p_{q_1}^* |0\rangle\). From the view point of variation, this corresponds to a modification of the Hartree-Fock state (2.5). In this case the charge neutrality is guaranteed by Lagrange multiplier \(\tilde{\mu}\) as will be described later. The new neutron and the new proton states are equivalent to the coherent states of \((\rho \bar{n})\) pairs and \((n \bar{p})\) pairs respectively (see Eq. (3.6)). The state \(|\alpha\rangle\) is the coherent state of pions and satisfies the following equations:

\begin{equation}
\begin{aligned}
a_k^* |\alpha\rangle &= \alpha^* |\alpha\rangle \quad \text{or} \quad b_k |\alpha\rangle = 0.
\end{aligned}
\end{equation}

Neglecting the difference of the particle-hole energy, the expectation value is
\[ E_0 = \langle \Psi | \mathcal{H} | \Psi \rangle = \sum_{q \in \mathcal{Q}^{(n)}} \bar{E}_q(n) N_q(n) + \sum_{q \in \mathcal{Q}^{(p)}} \bar{E}_{q-k}(p) N_{q-k}(p) + (\omega_k - \bar{\mu}) |\alpha|^2, \quad (4.5) \]

where the single particle energies of "new neutron" and "new proton" are respectively given by
\[ \bar{E}_q(n) = \{ \varepsilon_q + \frac{1}{2} \bar{\mu}(1 - \gamma) \}, \]
\[ \bar{E}_{q-k}(p) = \{ \varepsilon_{q-k} + \frac{1}{2} \bar{\mu}(1 + \gamma) \}. \quad (4.6) \]

We put
\[ \sum_q N_q(n) = N - \bar{\nu}, \]
\[ \sum_q N_q(p) = \bar{\nu} \quad (4.7) \]

and
\[ |\alpha|^2 = \nu. \]

The quantity \( \bar{\nu} \) represents the number of new protons. Sawyer and Scalapino\(^b\) have used \( \bar{\nu} = 0 \). On the other hand, Migdal\(^b\) takes into account \( \bar{\nu}(\neq 0) \) as a fixed quantum number.

We shall consider the physical meaning of their situations. The equation \( \partial E_0 / \partial \nu = 0 \) gives the relation
\[ \bar{\mu} = \omega_k - \frac{(N - 2\bar{\nu}) M_k^2}{\gamma \bar{\mu}}. \quad (4.8) \]

From the charge neutrality condition \( \partial E_0 / \partial \bar{\mu} = 0 \) we have
\[ N - 2\bar{\nu} = (N - 2\nu) \gamma. \quad (4.9) \]

In the normal state (\( \nu = 0 \)), the quantity \( \gamma \) is reduced to unity. The new proton number \( \bar{\nu} \), therefore, vanishes in this state. On the other hand, if we put \( \bar{\nu} = 0 \), Eq. (4.9) becomes Eq. (3.16). The chemical potential \( \bar{\mu} \) is deduced by the use of Eqs. (4.8), (4.9) and (3.23), where \( \bar{\mu} \) is slightly different from Eq. (3.19) due to the introduction of \( \bar{\nu} \). It, however, will not be written explicitly in this paper. For \( \bar{\nu} = 0 \), it results in \( \mu \) of Eq. (3.19). In order to give a physical interpretation about the introduction of \( \bar{\nu} \), we consider the energy difference between the highest level of new neutron and the lowest level of new proton:
\[ \Delta = [\bar{E}_{q-k}(p)]_{\text{min}} - [\bar{E}_q(n)]_{\text{max}} = \gamma \cdot \bar{\mu} - \bar{\xi}_F^{(n)}, \quad (4.10) \]

where the highest level of new neutron is given by \([\bar{E}_q(n)]_{\text{max}} = \bar{\xi}_F^{(n)} + (1/2) \bar{\mu}(1 - \gamma)\). To clarify the description, the results are shown in Fig. 2.
(3.19)) of pions for the special cases $k^2 = 1.0$ and $0.4 \, m^2$, respectively. The energies ($= \omega_k/2$) at the critical densities $\rho_c$ for various pion momenta are shown by the dot-dashed curve. The solid curve OF shows the neutron Fermi energy.

1) For the pions of momenta $k (k^2 \geq 0.5 m^2)$ with the instability points on the curve XY, there is a density region satisfying $\Delta \geq 0$ (e.g., $A_2A_3$). In this region the new proton number $\nu$ is zero, and the ground state is obtained by filling new neutrons up to $q_F^{(n)}$ set by the total baryon density. Thus, $\tilde{\mu}$ and $\tilde{\varepsilon}_F^{(n)}$ of Eq. (4.10) are replaced by $\mu$ and $\varepsilon_F^{(n)}$, respectively. From the equilibrium condition the new neutron Fermi energy $\tilde{E}_F(n)$ is equal to the pion chemical potential

$$\varepsilon_F^{(n)} + \frac{1}{2} \mu (1 - \gamma) = \mu . \quad (4.11)$$

Thus we obtain

$$\varepsilon_F^{(n)} = \frac{1}{2} \mu (1 + \gamma) \leq \mu . \quad (4.12)$$

We see that the condition ($\Delta \geq 0$) is satisfied, and the condensed phase of $\pi^-$ and $p\bar{n}$-pair (quasi-boson) is formed at $\rho_c$. In this limited region the result of Sawyer and Scalapino holds. As the density increases, the density region in which $\Delta$ is negative appears (e.g., $A_3A_4$). Then the ground state is obtained by filling new neutrons and new protons up to $\tilde{q}_F^{(n)}$ and $\tilde{q}_F^{(p)}$ respectively. As the Fermi energies ($\tilde{E}_F(n),\tilde{E}_F(p)$) and the pion chemical potential are equal, we have

$$\varepsilon_F^{(n)} + \frac{1}{2} \tilde{\mu} (1 - \gamma) = \varepsilon_F^{(p)} + \frac{1}{2} \tilde{\mu} (1 + \gamma) = \tilde{\mu} . \quad (4.13)$$

Clearly

$$\tilde{\mu} + \tilde{\varepsilon}_F^{(n)} = - \tilde{\varepsilon}_F^{(p)} \leq 0 , \quad (4.14)$$

and for small $\nu$ and $\nu$ we have $\mu \leq \varepsilon_F^{(n)}$. The condition ($\Delta \leq 0$) is, therefore, satisfied. The relation $\mu \leq \varepsilon_F^{(n)}$ gives the condition for the formation of condensed phase of $\pi^-$, $p\bar{n}$-pair and $n\bar{p}$-pair.

2) For the pions of momenta $k (k^2 \leq 0.5 m^2)$ with the instability points on the curve YZ, the condition $\Delta \leq 0$ ($\mu \leq \varepsilon_F^{(n)}$) is always satisfied (e.g., $B_2B_3$). The new
protons should be, therefore, taken into account in the configuration. In this case, the condensate of \( \pi^- \), \( p\bar{n} \)-pair and \( n\bar{p} \)-pair appears at \( \rho_c \). Thus, the density region in which the conclusions of Sawyer and Scalapino are justified does not exist. In any case, the condensate arises above the critical densities determined by the double pole condition.

We consider the condensation energy. From Eqs. (3·23) and (4·8), \( \nu \) is written as follows:

\[
\nu = \frac{(N-2\bar{\nu})^2 M_{\pi}^2}{4(\omega_k - \bar{\mu})^2} - \frac{\bar{\mu}^2}{4M_{\pi}^2}.
\]

(4·15)

Using Eqs. (4·7) and (4·15) we obtain

\[
E_a = \frac{3}{5} \varepsilon_{\pi}^{(n)} N \left[ \left( \frac{1 - \bar{\nu}}{N} \right)^{5/2} + \left( \frac{\bar{\nu}}{N} \right)^{5/2} \right]
+ \frac{1}{2} \bar{\mu}(N-\bar{\nu})(1-\gamma) + \frac{1}{2} \bar{\mu}\bar{\nu}(1+\gamma) + \frac{1}{4} \bar{\mu}(N-2\bar{\nu}) \left( \frac{\gamma - \frac{1}{2}}{\gamma} \right).
\]

(4·16)

For small \( \nu \) and \( \bar{\nu} \), we have the condensation energy

\[
E_a - \frac{3}{5} \varepsilon_{\pi}^{(n)} N = -\frac{1}{4} \mu N \left( \frac{\gamma - 1}{\gamma} \right) + \left[ \frac{1 + \bar{\mu}}{2\gamma} - \varepsilon_{\pi}^{(n)} \right] \bar{\nu}
\approx -\frac{1}{4} \mu N \left( \frac{\gamma - 1}{\gamma} \right) + \left[ \mu - \varepsilon_{\pi}^{(n)} \right] \bar{\nu}. \quad (\leq 0)
\]

(4·17)

When \( \bar{\nu} = 0 \), the first term on the right-hand side gives Sawyer’s result (Eq. (3·26)).

Above the critical density \( \rho_c \), the new ground state (Eq. (3·6) or (4·3)) turns out to be that which is not connected with the normal state by the unitary transformation \( U \) in the limit \( \mathcal{Q} \to \infty \) (inequivalent representation). The inequivalent representations which are inherent in field theory\(^{10} \) play an essential role in explaining the condensed state. In our case this is closely related to the instability condition (Appendix 2).

In Ref. 6\(^{1} \) it is mentioned that the condition for the formation of \( \pi^- \) condensate is \( \mu \leq \varepsilon_{\pi}^{(\infty)} \), where the excitation energy \( \omega(k) \) (Eq. (3·18)) of \( \pi^- \) in the normal state is used for \( \mu \). In this case, the condition \( \omega(k) \leq \varepsilon_{\pi}^{(n)} \) is satisfied for the pions \( (k^2 \leq 0.5m_{\pi}^2) \) with the instability points on the curve YZ, and corresponds to the region CB\(_2 \) \( (\rho < \rho_c) \) for the special case \( k^2 = 0.4m_{\pi}^2 \). In this region the Hartree-Fock state is stable, and the order parameter \( \nu \) is zero \( (\therefore \gamma = 1, \bar{\nu} = 0) \). The condensate of \( \pi^- \) mesons, therefore, does not appear. The condensate appears in the region B\(_2\)B\(_3\) \( (\rho \geq \rho_c) \). In this case the chemical potential (Eq. (3·19) or \( \bar{\mu} \)) worked out in the condensed state \( (\rho \geq \rho_c) \) should be used for the calculation of the energy.

As the Hamiltonian (Eq. (2·3)) contains only the negative pion, the isospin symmetry of the pion fields is violated. Therefore the present model is unrealistic. A more realistic model for the pion, described by the Klein-Gordon field, should be used.
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Appendix 1

We calculate the excitation energy $\omega(k)$ of $\pi^-$ in the normal state. From Eq. (2.8) the dispersion equation is given as follows:

$$1 = -\frac{f^2k^3}{m_x^2Q} \frac{1}{\omega_k(\omega(k) - \omega_k)} \sum_{\varepsilon_q} \frac{1}{\omega(k) - \varepsilon_k(q)}, \quad (A\cdot1)$$

where

$$\varepsilon_k(q) = \varepsilon_q - \varepsilon_{q-k}.$$

Using the approximation

$$\omega(k) \gg [\varepsilon_k(q)]_{\text{max}} = kv_F - \varepsilon_k,$$  \quad (A\cdot2)

we have

$$\sum_{\varepsilon_q} \frac{1}{\omega(k) - \varepsilon_k(q)} = \sum_{\varepsilon_q} \left\{ \frac{1}{\omega(k)} + \frac{\varepsilon_k(q)}{\omega^2(k)} + \frac{\varepsilon_k(q)^2}{\omega^3(k)} + \cdots \right\}. \quad (A\cdot3)$$

Taking into account only the first term in the expansion, Eq. (A\cdot1) becomes

$$1 = -\frac{f^2k^3N}{m_x^2Q} \frac{1}{\omega_k(\omega(k) - \omega_k)} \omega \frac{1}{\omega(k) - \omega_k} = -\frac{f^2k^3N}{m_x^2Q} \frac{1}{\omega(k) - \omega_k} \omega. \quad (A\cdot4)$$

Thus, we obtain

$$\omega(k) = \frac{\omega_F}{2} \left[ 1 + \sqrt{1 - \frac{4f^2k^3N}{m_x^2\omega_k^3Q}} \right] = \frac{\omega_F}{2} \left[ 1 + \sqrt{1 - \frac{\rho}{\rho_c}} \right], \quad (A\cdot5)$$

where

$$\rho_c = \frac{m_x^2\omega_k^2}{4f^2k^3}.$$

Appendix 2

Here we shall show that $|\Phi_0\rangle$ and $|\Psi\rangle$ become inequivalent to each other in the limit $Q \to \infty$.

The inner product of $|\Phi_0\rangle$ and $|\Psi\rangle$ is given as follows:

$$\langle \Phi_0 | \Psi \rangle = \langle 0 | \alpha \rangle \langle \Phi_0 | U_N | \Phi_0 \rangle$$

$$= \langle 0 | U_z | 0 \rangle \langle \Phi_0 | U_x | \Phi_0 \rangle, \quad (B\cdot1)$$

where

$$\langle 0 | U_z | 0 \rangle = \langle 0 | \exp[\alpha a \dagger - \alpha^* a] | 0 \rangle = \exp(-\frac{1}{2} |\alpha|^2) \quad (B\cdot2)$$
\begin{align*}
\langle \phi_0 | U_X | \phi_0 \rangle &= \langle \phi_0 | \exp \left[ \sum \beta_q (\rho_{q-k}^* \sigma_{q-k} n_q - \sum \beta_q^* (n_q^* \sigma_{q-k}^* \rho_{q-k}) \right] | \phi_0 \rangle \\
&= \exp \left[ \sum \frac{\sigma_{q-k} \beta_q}{\beta_q^*} \right] \\
&= \exp \left[ \frac{\Omega}{(2\pi)^3} \int dq \log (1 - \theta^2)^{\Omega} \right] \\
&= \exp [\Omega \rho \log (1 - \theta^2)^{\Omega}]. \quad (B\cdot3)
\end{align*}

Using Eqs. (3·19) and (3·20) we have

\begin{align*}
|\alpha|^2 &= \frac{4 f^2 k^2}{m_c \omega_e} \frac{\omega_k^2 N_s^2}{16 (\omega_c - \mu)^2 \Omega} - \frac{m_e^2 \omega_k^3 \mu^2 \Omega}{4 f^2 k^2 \omega_k^2} \\
&= \frac{3}{2} \Omega \rho_c \left[ \frac{4}{3} \frac{\rho}{\rho_c} - \frac{4}{9} \left( 1 + \sqrt{1 + 3 \frac{\rho}{\rho_c}} \right) \right] \left[ \frac{4}{3} \frac{\rho}{\rho_c} + \frac{4}{9} \left( 1 + \sqrt{1 + 3 \frac{\rho}{\rho_c}} \right) \right] \\
&\times \left( 1 + \sqrt{1 + 3 \frac{\rho}{\rho_c}} \right)^{-1}, \quad (B\cdot4)
\end{align*}

and near the critical density

\begin{equation}
\simeq \Omega \rho_c \left( \frac{\rho}{\rho_c} - 1 \right). \quad (\rho \geq \rho_c) \quad (B\cdot5)
\end{equation}

In the limit \( \Omega \to \infty \) we obtain

\begin{equation}
\langle \phi_0 | T \rangle = \exp \left[ - \Omega \left( \frac{\rho}{\rho_c} - 1 \right) - 2 \rho \log (1 - \theta^2)^{\Omega} \right] \to 0. \quad (B\cdot6)
\end{equation}

Equation (B·5) shows that the instability arises at \( \rho_c \) (see Eq. (3·18)).

References

9) T. Kanai, preprint.