Universality of Mean Charged Hadron Multiplicities in Particle and Nuclear Collisions

Fujio TAKAGI

Department of Physics, Tohoku University, Sendai 980

(Received July 21, 1976)

We propose that mean charged hadron multiplicities in various particle collision processes including hadron-nucleus collisions are described in terms of a universal function $F(<M_x^p>)$, where $<M_x^p>$ is the mean square invariant mass of the final state subsystem $X$ which consists of all the produced hadrons other than the leading particles. Leading particle effects in general and nuclear mass effects in hadron-nucleus collisions are most important factors which determine $<M_x^p>$ at a given incident energy. It is found that in proton-emulsion interaction the leading particle effects depend strongly on the number of heavily ionizing prongs. Within the framework of our model, the observed values of mean shower particle multiplicities in proton-emulsion interaction indicate that $F(<M_x^p>)$ obeys a power-law.

§ 1. Introduction

It has been revealed experimentally that the mean charged hadron multiplicities $<n_{ch}>$ in various particle collision processes show a rather universal dependence on energy if an appropriate energy variable is chosen$^{1~9}$ and the leading particle effects are correctly taken into account.$^{10,11}$ It has recently been suggested that such a universality may also hold in hadron-nucleus collisions.$^{12,13}$ However, the leading particle effects are not explicitly taken into account in Refs. 12) and 13). The purpose of this paper is to propose a unified and quantitative way of describing the energy dependence of $<n_{ch}>$ in various processes including hadron-nucleus collisions taking account explicitly of the leading particle effects. General formalism is given in § 2. It is applied to various processes in § 3. Comparison with experimental data is made in § 4. Concluding remarks are given in § 5.

§ 2. General formalism

We consider a semi-inclusive process

$$a + b \rightarrow a' + b' + X,$$

(2·1)

where $a'(b')$ is the leading particle stemming from the incident particle $a(b)$, and $a$ and $a'$ ($b$ and $b'$) are assumed to belong to the same isospin multiplet, so that $m_a = m_{a'}$ and $m_b = m_{b'}$. The system $X$ consists of any number of hadrons. We propose that the mean charged hadron multiplicity $<n_{ch}>$ (mean shower particle multiplicity $<n_s>$ in case of hadron-nucleus collisions) in the process (2·1) is described by a universal function $F(<M_x^p>)$;
Here, \(\langle M^2 \rangle\) is the mean square invariant mass of the system \(X\) and \(\gamma_{ab}\) is the contribution to \(\langle n_{ab} \rangle\) from the leading particles \(a'\) and \(b'\). We neglect the effect of different mean total charge \(\langle Q_X \rangle\) of the system \(X\) on \(\langle n_{ab} \rangle\) for \(|\langle Q_X \rangle| \leq 2\).

A simple argument to justify this approximation is given in Appendix A.

An absolute upper bound for \(\sqrt{\langle M^2 \rangle}\) is given by the mean energy \(E_r\) which is released by the incident particles to produce the system \(X\) in the (effective) center of mass system ((e), c.m.s.):

\[
\sqrt{\langle M^2 \rangle} \leq E_r = \sqrt{\langle k^2 \rangle} + \langle M^2 \rangle,
\]

where \(\langle k^2 \rangle\) is the mean square momentum of the system \(X\) in (e), c.m.s. The term "effective" center of mass system (e.c.m.s.) is used for hadron-nucleus collisions, in which \(a\) is a hadron while \(b\) is a portion of a nucleus with \(m_b = A_s m_N\) (\(m_N\) is the nucleon mass) and \(A_s\) being the effective mass number of the nucleus.\(^{10-10}\) An upper bound for \(E_r\) is given by \(E_{av}\) which is the energy kinematically available for particle production in (e)cm. system:

\[
E_r \leq E_{av} = (m_a^2 + m_b^2 + 2m_b E_{lab})^{1/2} - m_a - m_b
\]

where \(E_{lab} = \sqrt{p_{lab}^2 + m_a^2}\) is the lab. energy of the particle \(a\) and \(\sqrt{s}\) is the total energy in (e) c.m.s. The inelasticity \(\epsilon\) is defined as

\[
0 < \epsilon = E_r / E_{av} \leq 1,
\]

so that we have

\[
\sqrt{\langle M^2 \rangle} \leq E_r = \epsilon E_{av} \leq E_{av}.
\]

Since \(F(x)\) is experimentally a monotonically increasing function of \(x\), we have

\[
F(\langle M^2 \rangle) \leq F(\epsilon^2) \leq F(E_{av}^2).
\]

When there is no leading particle coming from the particle \(a\), the process to be considered is

\[
a + b \rightarrow X + b',
\]

where \(b'\) is the leading particle coming from \(b\), and we have

\[
\langle n_{ch} \rangle = F(\langle M^2 \rangle) + \gamma_b,
\]

where \(\gamma_b\) is the contribution from the leading particle \(b'\) to \(\langle n_{ch} \rangle\). In this case, \(E_{av}\) is given by

\[
E_{av} = \sqrt{s} - m_b.
\]

The inequality (2.6) is applied also to the process (2.8).

Finally when there is no leading particle at all, the relevant process turns out to be
and we have
\[ \langle n_{\text{ch}} \rangle = F(\langle M_x^2 \rangle) \quad (2.12) \]

with
\[ \langle M_x^2 \rangle = M_x^2 = E_r^2 = E_{av}^2 = s. \quad (2.13) \]

§ 3. Application

In the following, the universal function \( F(\langle M_x^2 \rangle) \) is determined except for an unknown ratio \( \sqrt{\langle M_x^2 \rangle}/E_{av} \) by fitting to \( \langle n_{\text{ch}} \rangle \) in \( p-p \) collisions. The ratio \( \sqrt{\langle M_x^2 \rangle}/E_{av} \) in \( p-p \) collisions is determined by fitting to \( \langle n_{\text{ch}} \rangle \) in \( \bar{p}-p \) annihilation at \( p_{\text{lab}} = 100 \text{ GeV}/c \). Then the energy-dependence of \( \langle n_{\text{ch}} \rangle \) in other various processes is analyzed in terms of \( F(\langle M_x^2 \rangle) \).

3a. \( p-p \) collisions

The relevant process is
\[ p+p \rightarrow N+N+X, \quad (3.1) \]
where \( N \) is the leading nucleon. Applying (2.2) to this process, we have
\[ \langle n_{\text{ch}} \rangle_{pp} = F(\langle M_x^2 \rangle) + 1.4, \quad (3.2) \]
where \( \gamma_{pp} \) has been estimated to be 1.4.\(^{*1} \) We first consider the following two typical fits to \( \langle n_{\text{ch}} \rangle_{pp} \):\(^{15,10} \)
\[
\langle n_{\text{ch}} \rangle_{pp} = \bar{F}(E_{av}) + 1.4 \\
= \\
\begin{cases} 
0.348 + 1.883E_{av}^{4/3} & \text{for } E_{av} \geq 1.0 \text{ GeV (Case P),} \\
-4.02 + 3.98 \ln E_{av} & \text{for } E_{av} \geq 20 \text{ GeV (Case L),} 
\end{cases} \quad (3.3a, 3.3b)
\]
where (3.3a) is also used in Case L for \( E_{av} \leq 20 \text{ GeV}. \)\(^{*2} \) An appreciable difference between the two cases appears when \( E_{av} \geq 200 \text{ GeV} \) where of course no data on \( \langle n_{\text{ch}} \rangle_{pp} \) are available yet. Comparing (3.2) with (3.3), we have
\[ F(\langle M_x^2 \rangle) = \bar{F}(\sqrt{1 + \rho} \sqrt{\langle M_x^2 \rangle}/s), \quad (3.4) \]
where
\[ \rho = \langle k_x^2 \rangle/\langle M_x^2 \rangle. \quad (3.5) \]

3b. \( \bar{p}p \) annihilation

Since there is no leading particle in \( \bar{p}p \) annihilation, we have

\(^{*1} \) This estimate seems quite reasonable because the mean proton multiplicity and the mean antiproton multiplicity in \( p-p \) collisions at CERN ISR are 1.62~1.63 and 0.085~0.144, respectively. See A. M. Rossi et al., Nucl. Phys. B84 (1975), 269.

\(^{*2} \) That is, Case L = Case P for \( E_{av} < 20 \text{ GeV}. \) The reason why we do not use the parametrization (3.3b) for \( E_{av} < 20 \text{ GeV} \) is that it does not give a good fit to \( \langle n_{\text{ch}} \rangle_{pp} \) there.
by using (2.12) and (2.13). By adjusting $F(s)$ to the observed value of $\langle n_{\text{ch}} \rangle_{p\bar{p} \text{annih}}$ at $p_{\text{lab}}=100 \text{ GeV/c}$,\textsuperscript{19} we get a reasonable result

$$\epsilon/\sqrt{1+p} = 0.368.$$ \quad (3.7)

That is, 36.8% of $E_{\text{av}}$ is on the average converted into the mass of the system $X$. A typical choice $\epsilon=0.5$ yields $p=0.846$. Though the factor $\epsilon/\sqrt{1+p}$ has been adjusted only at $p_{\text{lab}}=100 \text{ GeV/c}$ in $p\bar{p}$ annihilation (which is now equivalent to $p_{\text{lab}}=821 \text{ GeV/c}$ in $pp$ collisions), it will be assumed to be independent of $p_{\text{lab}}$ hereafter in this paper because we do not know any reliable method to incorporate a possible weak-dependence of $\epsilon$ on $p_{\text{lab}}$. We then have

$$F(\langle M_{X'}^2 \rangle) = \left\{ \begin{array}{ll} -1.052 + 2.99 \langle M_{X'}^2 \rangle^{0.42} \\ -1.44 + 1.99 \ln \langle M_{X'}^2 \rangle \end{array} \right\} \text{ for Case } \begin{cases} P \\ L \end{cases}. \quad (3.8)$$

3c. $e^+e^- \text{ annihilation into hadrons}$

The situation in $e^+e^-\rightarrow\text{hadrons}$ is very similar to that in $p\bar{p}$ annihilation, so that we have

$$\langle n_{\text{ch}} \rangle_{e^+e^- \text{annih}} = F(\langle M_{X'}^2 \rangle) = F(s). \quad (3.9)$$

3d. $\gamma+p\rightarrow\text{hadrons}$

It may be reasonable to assume that there is no leading particle arising from the photon in photoabsorption processes.\textsuperscript{4)\footnote{An alternative possibility is discussed in Appendix C.}} Accordingly, the process to be considered is

$$\gamma + P \rightarrow X + N, \quad (3.10)$$

where $N$ is the leading nucleon arising from the incident proton. The mean charged hadron multiplicity $\langle n_{\text{ch}} \rangle$ is given by

$$\langle n_{\text{ch}} \rangle_p = F(\langle M_{X'}^2 \rangle) + 0.7, \quad (3.11)$$

where the leading particle contribution $\gamma_p$ has been taken to be 0.7. Using (2.7) and (2.10), we have

$$\langle n_{\text{ch}} \rangle_{\gamma p} \leq F(\sqrt{s} - m_N)^3 + 0.7. \quad (3.12)$$

3e. $p+p\rightarrow p+\text{anything}$

We now proceed to discuss $\langle n_{\text{ch}} \rangle$ of the system $X'$ in the inclusive reaction $p+p\rightarrow p+X'$, where the detected final state protons are restricted to those in the target fragmentation region. Therefore, we are led to consider the process

$$R + P \rightarrow X' + X + N, \quad (3.13)$$

where $R$ is an off-mass shell, neutral object something like Reggeons or the Pom-
Universality of Mean Charged Hadron Multiplicities

eran and \( N \) is the leading nucleon. The situation is very similar to that in \( \gamma_{\text{virtual}} + p \rightarrow \text{hadrons} \) discussed in the preceding subsection, so that we have

\[
\langle n_{\text{ch}} \rangle_{\text{pp}} = F(\langle M_x^z \rangle) + 0.7 \leq F(\sqrt{s} - m_N) + 0.7, \tag{3.14}
\]

where \( \sqrt{s} = M_X \).

3f. \( \nu p \) collisions

Experimental results on \( \langle n_{\text{ch}} \rangle \) of the hadronic system \( X' \) in the inclusive process \( \nu + p \rightarrow \mu^+ + X' \) have recently been reported.\(^7\) For this process, we have to consider

\[
J^+ + p \rightarrow X' \rightarrow X + N, \tag{3.15}
\]

where \( J^+ \) is the positively charged, weak current and \( N \) is the leading nucleon. The situation is again similar to that in virtual photon proton collisions (except for the total mean charge of \( X \)), so that we have

\[
\langle n_{\text{ch}} \rangle_{\text{pp}} = F(\langle M_X^z \rangle) + 0.7 \leq F(\sqrt{s} - m_N) + 0.7, \tag{3.16}
\]

where \( \sqrt{s} = M_X \).

3g. hadron-nucleus collisions

For multihadron (mainly pions) production in hadron \( h \)-nucleus \( A \) (the mass number \( A \)) collisions, it is assumed that on the average the target nucleus behaves like "a big hadron" with a mean effective mass number \( \langle A_\circ \rangle.\(^{12,13}\) The process to be considered is

\[
h + A_\circ \rightarrow h' + A'_\circ + X, \tag{3.17}
\]

where \( A_\circ \) is the effective target nucleus of which mass is \( A_\circ m_X \) (the residual \( A - A_\circ \) nucleons in the nucleus pass through as spectators), and \( h' \) and \( A'_\circ \) are the leading particles stemming from \( h \) and \( A_\circ \), respectively. Shower particles (lightly ionizing, relativistic charged particles) are contained in \( X \) and \( h' \), while the recoiled nucleus or its fragments \( A'_\circ \) and the spectator nucleons are in most cases counted as heavily ionizing prongs. In emulsion experiments, one observes the number \( N_h \) of the heavily ionizing prongs as well as the number \( n_s \) of the shower particles. The relevant quantities to be studied here are \( \langle A_\circ \rangle, \langle n_s \rangle \) and the associated quantities \( \langle A_\circ(N_h) \rangle \) and \( \langle n_s(N_h) \rangle \). Applying (2.2) to the process (3.17), we have

\[
\langle n_s \rangle = F(\langle M_x^z \rangle) + \gamma_{hA} \tag{3.18a}
\]

and

\[
\langle n_s(N_h) \rangle = F(\langle M_x^z(N_h) \rangle) + \gamma_{hA} \tag{3.18b}
\]

The correction factor \( \gamma_{hA} \) is estimated to be 0.7 when \( h = p \) and \( \pi^- \) because the leading nucleus \( A'_\circ \) seldom contributes to \( \langle n_s \rangle \). An upper bound for \( \langle M_x^z(N_h) \rangle^{1/2} \) and \( \langle M_x^z(N_h) \rangle^{1/2} \) is given by
\[ \left( \frac{\langle M_x^2 \rangle^{1/2}}{\langle M_x^2(N_h) \rangle^{1/2}} \right) \leq E_{av}(p_{lab}, x) \]
\[ = (m_h^2 + m_N^2 x^2 + 2m_N E_{lab} x)^{1/2} - m_h - m_N x \]  
(3.19)

with \( x = \left[ \frac{\langle A_e \rangle}{\langle A_e(N_h) \rangle} \right] \), where \( m_h \) is the mass of the hadron \( h \). Since this bound is fairly loose, we would like to obtain a better upper bound on a basis of some physical consideration.

In addition to the inelasticity \( \epsilon = E_{av} / E_{av} \), it is convenient to define the projectile inelasticity \( \epsilon_h \) and the target inelasticity \( \epsilon_A \) as follows:

\[ E_r = \epsilon_h (\sqrt{p^2 + m_h^2} - m_h) + \epsilon_A (\sqrt{p^2 + x^2 m_N^2} - m_N) \]  
(3.20)

where \( p \) is the e.c.m. momentum in the initial state, and \( x = \langle A_e \rangle / \langle A_e(N_h) \rangle \).

The first (second) term on the right-hand side of (3.20) represents the energy released by \( h(A_e) \) in e.c.m.s.:

\[ \epsilon_h = (\sqrt{p^2 + m_h^2} - k_h^2 + m_h) / (\sqrt{p^2 + m_h^2} - m_h) \]  
(3.21a)

\[ \epsilon_A = (\sqrt{p^2 + x^2 m_N^2} - k_A^2 + x^2 m_N^2) / (\sqrt{p^2 + x^2 m_N^2} - m_N) \]  
(3.21b)

where \( k_h(k_A) \) is the e.c.m. momentum of \( h(A_e) \). Now we consider the dependence of \( \epsilon_h \) and \( \epsilon_A \) on \( p_{lab} \) and \( x \). Useful information is obtained by considering the case where \( p_{lab} \gg m_N \). In such a case, the invariant momentum transfer squared \( t_A \) on the target side becomes \( -x^2 m_N^2 \epsilon_A^2 / (1 - \epsilon_A) \). (See Appendix B.) Therefore, \( \epsilon_A \) has to satisfy the bound \( \epsilon_A \leq \text{constant} \times x \) for large \( x \) in order to keep \( |t_A| \) small enough. From this result, we can reasonably suppose that \( \epsilon_A \) is a monotonically decreasing function of \( x \) for a fixed but sufficiently large \( p_{lab} \). Hence an upper bound for \( \epsilon_A \) is given by \( \epsilon \) in \( h-A_e \) collisions with \( \langle A_e \rangle = 1 \) which is approximately equivalent to \( h \)-proton collisions. We thus have

\[ \epsilon_A \leq \epsilon_{pp} = 0.368 \sqrt{1 + \rho} \]  
(3.22)

from (3.7). On the other hand, the projectile \( h \) will lose more energy as \( x \) increases with \( p_{lab} \) fixed. Therefore, \( \epsilon_h \) will be a monotonically increasing function of \( x \) for a fixed \( p_{lab} \). Possible maximum value of \( \epsilon_h \) is unity. Thus we have the following upper bound for \( \langle M_x^2 \rangle_{hA}^{1/2} \) and \( \langle M_x^2(N_h) \rangle_{hA}^{1/2} \):

\[ \left[ \frac{\langle M_x^2 \rangle_{hA}^{1/2}}{\langle M_x^2(N_h) \rangle_{hA}^{1/2}} \right] \leq E_r \leq \sqrt{p^2 + m_h^2} - m_h + \epsilon_{pp} \left( \sqrt{p^2 + x^2 m_N^2} - m_N \right) = E_{r \text{ upper}}. \]

where \( x = \left[ \frac{\langle A_e \rangle}{\langle A_e(N_h) \rangle} \right] \).

The above upper bound is better than (3.19) because \( \epsilon_{pp} < 1 \).

Using experimental values of \( \langle n_e \rangle \) and \( \langle n_e(N_h) \rangle \) as inputs in (3.18), we can determine \( \langle M_x^2 \rangle \) and \( \langle M_x^2(N_h) \rangle \). Then, by using (3.23), we can obtain a lower bound for \( \langle A_e \rangle \) and \( \langle A_e(N_h) \rangle \). However, for the process with \( N_h \) fixed, we
assume, following Berlad et al.,\(^{12}\) that \(\langle A_e(N_h)\rangle\) is given by the mean number of knocked-out and evaporated nucleons
\[
\langle A_e(N_h)\rangle = AN_p/Z
\]  
(3.24)
except for \(N_p=0\), where \(N_p\) (which should be proportional to \(N_h\)) is the number of knocked-out and evaporated protons and \(Z\) is the atomic number of the target nucleus \(A\). By inserting (3.24) into the expression for \(E_r^{\text{upper}}\) in (3.23), we can determine the upper bound \(E_r^{\text{upper}}(N_h)\) for \(\langle M_s^z(N_h)\rangle^{1/2}\) and hence the upper bound for \(\langle n_e(N_h)\rangle\). Finally, when both \(\langle n_e(N_h)\rangle\) and \(\langle A_e(N_h)\rangle\) are given, we can determine \(\langle M_s^c(N_h)\rangle\) and \(E_{av}(p_{lab}, \langle A_e(N_h)\rangle)\) by using (3.18b) and (3.19), respectively and hence the mass conversion rate (MCR) \(\sqrt{\langle M_s^c(N_h)\rangle}/E_{av}(p_{lab}, \langle A_e(N_h)\rangle)\).

\section{4. Comparison with experiments}

According to the results obtained in 3a\textasciitilde{}3c, \(\langle n_{ch}\rangle_{pp}, \langle n_{ch}\rangle_{pp\text{ annih} + 1.4}^{\text{Ref. 16}}\) and \(\langle n_{ch}\rangle_{e^+e^-\text{hadrons} + 1.4}^{\text{Ref. 10}}\) should lie on a universal curve when they are plotted against \(\langle M_s^z\rangle\). This is indeed the case as is shown in Fig. 1. The power-law curve \(F(\langle M_s^z\rangle) + 1.4 = 0.348 + 2.99\langle M_s^z\rangle^{0.232}\) fits well to the universal curve. Data on \(\langle n_{ch}\rangle_{p+p}\) and \(\langle n_{ch}\rangle_{K^-K^+\to\text{pions}}\) are also shown in Fig. 1, where \(K^-\) is a hypercharge carrying, positively charged object something like \(K^*\) or \(K_{N}\) Regge trajectory and it is assumed that \(\sqrt{\langle M_s^z\rangle} = 0.368 E_{av}\) in \(\pi^-p\) collisions while \(\sqrt{\langle M_s^z\rangle} = \sqrt{s}\) in \(K^- + K_{p}^{+}\to\text{pions}\). They also lie well on the universal curve. This result for \(\langle n_{ch}\rangle_{p+p}\) suggests that the leading particle effects in \(\pi^-p\) collisions are not much different from those in \(p+p\) collisions.

The results of 3d\textasciitilde{}3f predict that \(\langle n_{ch}\rangle_{p+p}, \langle n_{ch}\rangle_{p+p\text{ annih} + 1.4}^{\text{Ref. 18}}\) and \(\langle n_{ch}\rangle_{p+p\text{ annih} + 1.4}^{\text{Ref. 17}}\) are bounded by

\[0.348 + 2.99 (M_s^{1/2})\]

Fig. 1. Test of universality of mean charged hadron multiplicities as functions of \(\langle M_s^z\rangle\), where \(\sqrt{\langle M_s^z\rangle} = 0.368 E_{av}\) for \(p+p\) and \(\pi^-p\) collisions, and \(\sqrt{\langle M_s^z\rangle} = \sqrt{s}\) for \(e^+e^-\) annihilation, \(p+p\) annihilation and \(K^-K^+\to\text{pions}\).
\[ F(\sqrt{s} - m_N)^3 + 0.7. \] This prediction is entirely consistent with data as is shown in Fig. 2. Furthermore \( \langle n_{ch}\rangle_{\gamma p} \) and \( \langle n_{ch}\rangle_{Rp} \) lie roughly on a universal curve except for ISR data on \( \langle n_{ch}\rangle_{Rp} \). We can also determine MCR \( \sqrt{\langle M_x^2 \rangle} / E_{av} \) by using (3.8) (Case P), (3.11), (3.14) and (3.16) together with the data. MCR is found to be 0.52 \pm 0.05 being almost independent of \( s \) for \( Rp \) and \( \gamma p \) data except for \( Rp \) data obtained at ISR. The ISR results on \( \langle n_{ch}\rangle_{Rp} \) are consistently higher than the previous data by about unity, and MCR \( \sqrt{\langle M_x^2 \rangle} / E_{av} \) determined from the ISR data decreases monotonically from 0.96 to 0.66 when \( \sqrt{s} \) increases from 2 GeV to 13 GeV. For \( J^*p \) collisions, MCR is found to decrease monotonically from 0.92 to 0.51 as \( \sqrt{s} \) increases from 2 GeV to 14 GeV. It is reasonable that MCRs for these processes are greater than that for \( pp \) collisions (= 0.368).

There is an alternative upper bound for \( \langle n_{ch}\rangle_{\gamma p} \):

\[ \langle n_{ch}\rangle_{\gamma p} \leq F(\sqrt{s} - m_v - m_N)^3 + 2.4, \tag{4.1} \]

where \( m_v \) is the average mass of the vector mesons which couple with the photon and is taken to be 0.78 GeV. This upper bound is obtained from an alternative assignment for the leading particle effect on photon side (see Appendix C for derivation) and is also satisfied experimentally. In fact, (4.1) is numerically less
Universality of Mean Charged Hadron Multiplicities

restrictive than (3·12).

For hadron-nucleus collisions, we can obtain lower bound for \( \langle A_e \rangle \) by using experimental values of \( \langle n_{hA}^{20} \rangle \) with Eqs. (3·18a) and (3·23). The results are shown in Fig. 3, where the absolute upper bound \( A_e = A \) and \( \langle \nu \rangle \) for \( \pi^{-}A^{20} \) and \( p-A^{20} \) collisions are also shown for the sake of comparison. Here \( \langle \nu \rangle \) is the mean number of hadron-nucleon inelastic collisions which the incident hadron experiences in traversing a nucleus. The inelasticity \( \epsilon_{pp} \) in (3·23) has been taken to be a fair value 0.5. There are some remarkable features seen in Fig. 3.

(i) The lower bound for \( \langle A_e \rangle \) in \( p-AgBr \) collisions violates an obvious upper bound \( \langle A_e \rangle \leq A \) at 3000 GeV/c in Case L.

(ii) All the other results are consistent with the upper bound \( \langle A_e \rangle \leq A \) in both Cases P and L.

(iii) The possibility that \( \langle A_e \rangle = \langle \nu \rangle \) is probably ruled out because the lower bounds for \( \langle A_e \rangle \) are somewhat larger than \( \langle \nu \rangle \) at large \( A \).

For hadron-nucleus collisions with \( N_h \) or \( N_p \) fixed, we have given explicit expressions for \( \langle A_e(N_h) \rangle \) or \( \langle A_e(N_p) \rangle \) by noting (3·24) and have calculated the upper bound for \( R_{hA}(N_h) \) or \( R_{hA}(N_p) \) by using (3·18b), (3·23) with \( \epsilon_{pp} = 0.5 \) instead of calculating the lower bound for \( \langle A_e(N_h) \rangle \) by using experimental values of \( \langle n_{hA} \rangle \). The expressions for \( \langle A_e(N_h) \rangle \) and \( \langle A_e(N_p) \rangle \) are

\[
\begin{align*}
\langle A_e(N_h) \rangle &= 0.108N_h^2 + 1.892N_h + 1.0 \quad \text{for } 1 \leq N_h \leq 7 \\
\langle A_e(N_h) \rangle &= 2.76N_h + 1.0 \quad \text{for } N_h \geq 8
\end{align*}
\]

(4·2)

with \( \langle A_e(N_h = 0) \rangle = 1 \) for \( p \)-emulsion interactions, and

\[
\langle A_e(N_p) \rangle = 2N_p \quad \text{for } N_p \geq 1
\]

(4·3)

with \( \langle A_e(N_p = 0) \rangle = 1 \) for \( \pi^{-}Ne \) collisions. (4·2) has been obtained by taking

Fig. 4. Upper bound for \( R_{hA}(N_p) \). Solid (dashed) line is for Case P (Case L).

Fig. 5. Upper bound for \( R_{p-emulsion}(N_h) \). Solid (dashed) line is for Case P (Case L).
into account the fact that hydrogen target contributes only to $N_h=0$ and 1 events, and CNO target only to events with $N_h\leq8$. The results are shown in Figs. 4 and 5. The data are taken from Refs. 27) and 28). The upper bound for $R_{\pi N_p}(N_p)$ is satisfied experimentally in both Cases P and L, while that for $R_{p-\text{emulsion}}(N_h)$ is violated for $N_h\geq12$ in Case L. In Case P, the upper bound for $R_{p-\text{emulsion}}(N_h)$ appears to be almost saturated for $N_h\geq30$.

By using experimental values of $\langle n_p(N_h) \rangle$ and the expression (4.2) for $\langle A_e(N_h) \rangle$, we can calculate both $\langle M_\chi^2(N_h) \rangle$ and $E_{av}(p_{lab}, \langle A_e(N_h) \rangle)$ and hence MCR $\sqrt{\langle M_\chi^2(N_h) \rangle/E_{av}(p_{lab}, \langle A_e(N_h) \rangle)}$. The results are shown in Fig. 6. MCR exceeds unity for $N_h\geq12$ in Case L, which is a reflection of the violation of the upper bound for $R_{p-\text{emulsion}}(N_h)$ demonstrated in Fig. 5. On the other hand, the results for Case P are very reasonable and MCR appears to increase monotonically as $N_h$ increases with $p_{lab}$ fixed. From the viewpoint of our model, an approximate linear dependence of $\langle n_p(N_h) \rangle_{p-\text{emulsion}}$ on $N_h$ is due to combined effects of increasing $E_{av}(p_{lab}, \langle A_e(N_h) \rangle)$ and increasing MCR (a kind of inelasticity) $\sqrt{\langle M_\chi^2(N_h) \rangle/E_{av}(p_{lab}, \langle A_e(N_h) \rangle)}$ with $N_h$.

§ 5. Concluding remarks

The results of § 4 are summarized with some concluding remarks.

(i) The mean charged hadron multiplicities in various collision processes can be described in terms of a universal function $F(\langle M_\chi^2 \rangle)$. This fact suggests that the primitive hadronic matter produced in high energy collisions has a universal nature which is rather independent of the type of the incident particles. However, the leading particle effect (e.g., the mass conversion rate) should depend strongly
on the initial channel in order that the universality of the mean multiplicity holds, and hence this point should be confirmed experimentally. (See also the remark (v) given below.)

(ii) Among the factors which determine $\langle M_X^2 \rangle$ at a given incident energy, the leading particle effects are most important in general, while the nuclear mass effects are equally crucial in hadron-nucleus collisions.

(iii) Logarithm fit to $\langle n_{ch} \rangle$ (Case L) has conventionally been adopted for various processes by most authors. However, the energy ranges over which $\langle n_{ch} \rangle$ have been measured are so restricted that the power-law fit is equally acceptable at present. Our results for hadron-nucleus collisions rather indicate that $\langle n_{ch} \rangle$ will obey a power law (Case P). The Case P might have already been ruled out by some cosmic ray data, but it is also the fact that many events with very high multiplicities have been observed in other cosmic ray experiments. We consider that the problem deserves further study from both experimental and theoretical sides.

(iv) In hadron-nucleus collisions, the leading particle effects depend on $N_h$; the larger the $N_h$, the less the leading particle effects. Therefore, it may be interesting to measure the $N_h$-dependence of the energy-momentum distribution in e.c.m.s. of the leading particles from both projectile and target. The actual identification of e.c.m.s. is not so easy that it may be more convenient to measure the $N_h$-dependence of the lab. angular distribution of the heavily ionizing prongs.

(v) In most experiments, $\langle n_{ch} \rangle$ have not been measured in conjunction with $M_X$, so that we have had to introduce another parameter $\sqrt{\langle M_X^2 \rangle}/E_{av}$. It is therefore desirable to measure $\langle n_{ch} \rangle$ of the processes of the type (2·1) and (2·8) at a fixed $M_X^2$ for a direct test of our model.

We have proposed that $\langle n_{ch} \rangle$ will be described by a universal function $F(\langle M_X^2 \rangle)$. There are at least two alternatives as for the choice of the appropriate energy variable. One is $E_{av}$, and another is $E_r$. However, from both theoretical and phenomenological points of view, $M_X$ (or $\langle M_X^2 \rangle$) appears best among these.

The leading particle effects cannot be directly incorporated if $E_{av}$ is used as the universal energy variable. In fact, the difference $E_{av}(\bar{p}p\text{ annih}) - E_{av}(pp) = 2m_N$ alone cannot explain the large difference $\langle n_{ch} \rangle_{\bar{p}p\text{ annih}} - \langle n_{ch} \rangle_{pp} = (9.06 \pm 0.56) - 6.3$ at 100 GeV/c. The authors of Refs. 6) and 7) have shown that $\langle n_{ch} \rangle_{pp}$ and $\langle n_{ch} \rangle_{e^+e^-\text{hadrons}}$ lie approximately on a single curve at low energies when they are plotted against $E_{av}$. This is, however, a numerical accident from our point of view. Both the leading particle effects and the charge correction factor $r_{pp}$ are not taken into account in their analyses. The leading particle effects and the charge correction contribute in an opposite sign to the difference $\langle n_{ch} \rangle_{\bar{p}p\text{ annih}} - \langle n_{ch} \rangle_{pp}$, so that neglecting both effects could result in an accidental coincidence between $\langle n_{ch} \rangle_{\bar{p}p\text{ annih}}$ (or $\langle n_{ch} \rangle_{e^+e^-\text{hadrons}}$) and $\langle n_{ch} \rangle_{pp}$ as functions of $E_{av}$ for a limited range of $E_{av}$. The variable $E_r$ is apparently better than $E_{av}$ because the leading particle effects can be taken into account through the inelasticity $E_r/E_{av}$.
However, if $E_r$ is adopted, it turns out that, from analysis of $pp$ collisions and $\bar{p}p$ annihilation, $E_r/E_{av} = 0.368$ (referred to (3.7)) which seems to be slightly too small. A preliminary analysis of the $N_h$-dependence of the rapidity distribution of shower particles produced in proton-emulsion interactions also prefers $M_x$ to $E_r$.

Though we believe that $M_x^2$ (or $\langle M_x^2 \rangle$) is better than $E_{av}$ or $E_r$ to describe the universality of the mean charged hadron multiplicities, a possibility that, in addition to $M_x^2$, there is another degree of freedom which controls $\langle n_{ch}\rangle$ is not ruled out. For example, an excited hadronic matter with the same $M_x$ but different space-time structure would decay into different number of hadrons. Or there might even be a single variable (say total entropy or something like that) which is better than $M_x$. Such possibilities may be best tested by comparing $\langle n_{ch}\rangle_{e-e\rightarrow\text{hadrons}}$ with $\langle n_{ch}\rangle_{pp \text{ annih}}$ at high energies. The possibility that $F(x) \propto \ln x$ (Case L) might again be favoured if one of these possibilities is the case.

Finally, we would like to mention a relation between our model and a well-known fact that there are two components (diffractive and non-diffractive ones) in multihadron production processes. Our model will be better applied to the non-diffractive component than to the diffractive one because separation of the leading particles and the excited hadronic matter is clearer in the former.

Acknowledgements

The author would like to thank Professor G. Takeda and Doctor N. Sakai for useful discussions.

Appendix A

—Correction Due to Different Total Charge—

Consider a system $X$ which consists of $l\pi^+$'s, $m\pi^0$'s and $n\pi^-$'s for simplicity, and denote the system as $(l, m, n)$. The charged pion multiplicity of $X$ is

$$n_{ch}(X) = l + n \quad . \quad (A\cdot1)$$

There are $m$ ways of adding one unit charge (without changing the number of pions) to yield the state $(l+1, m-1, n)$ and $n$ ways to lead to the state $(l, m+1, n-1)$. Therefore, the expectation value $\langle n_{ch}\rangle$ of the state $X$ with one unit charge added may be given by

$$\langle n_{ch}(X^+) \rangle = \frac{(l+n+1)m + (l+n-1)n}{m+n} = n_{ch}(X) + \frac{m-n}{m+n} \quad . \quad (A\cdot2)$$

It is expected that $|\langle (m-n)/(m+n) \rangle| \ll 1$ if $X$ is neutral and $M_x$ (and hence $\langle l+m+n \rangle$) is not too small. Repeating the same argument, we obtain
Universality of Mean Charged Hadron Multiplicities

\[ \langle n_{ch}(X^{++}) \rangle = n_{ch}(X) + \mathcal{D}^{++} \]  \hspace{1cm} (A·3)

for \( X \) with two unit charges added, where

\[ \mathcal{D}^{++} = \frac{[2n(n-1) - 2(m+n) \langle n(n-1) - m(m-1) \rangle]}{(m+n)^2(m+n-1)}. \hspace{1cm} (A·4) \]

It is again expected that \( |\langle \mathcal{D}^{++} \rangle| \ll 1 \) if the starting system \( X \) is neutral and \( l+m+n \geq 3 \). In fact, we have

\[ |\mathcal{D}^{++}| = |(n-1)/2n(2n-1)| \ll 1 \quad \text{for} \quad n \geq 1, \hspace{1cm} (A·5) \]

when \( m = n \). From these results, we can safely assume that the correction due to different total charge \( Q_x \) of the system \( X \) for \( \langle n_{ch}(X) \rangle \) is negligibly small compared to unity at least when \( |Q_x| \leq 2 \) and \( \langle M_x^2 \rangle \) (and hence \( \langle n_{ch} \rangle_x \) itself) is not too small.

**Appendix B**

--- Momentum Transfer of Target Nucleus at Very High Energies ---

Denoting the four momenta of \( A_e \) and \( A_e' \) in the process \( h + A_e \rightarrow h' + A_e' + X \) as \( p \) and \( k_{A_e} \), respectively, we study an asymptotic property of the invariant momentum transfer squared \( t_1 \) defined as

\[ t_1 = (\sqrt{p^2 + m_{X}^2} - \sqrt{k_{A_e}^2 + m_{X}^2})^2 - (p - k_{A_e})^2, \hspace{1cm} (B·1) \]

where \( |p| = \bar{p} \) and \( |k_{A_e}| = k_{A_e} \), and \( x m_{X} \) is the mass of the "effective" nucleus \( A_e \). We consider the case when \( \bar{p}k_{A_e}/p k_{A_e} = 1 \) (vanishing recoil angle in e.c.m.s.). That is, we are calculating the so-called \( t \)-minimum on the target side. The initial c.m. momentum \( p \) is expressed in terms of \( p_{lab} \) as

\[ p = x m_{X} p_{lab} / \left( m_{X}^2 + x^2 m_{X}^2 + 2 x m_{X} E_{lab} \right)^{1/2}, \hspace{1cm} (B·2) \]

from which we have

\[ p = \left( x m_{X} p_{lab} / 2 \right)^{1/2} \left\{ 1 - \frac{m_{X}^2 + x^2 m_{X}^2}{4 x m_{X} p_{lab}} + O \left( \frac{p_{lab}^2}{p_{lab}} \right) \right\}, \hspace{1cm} (B·3) \]

for \( p_{lab} \gg x m_{X} \). The inelasticity \( \epsilon_A \) of the target is defined as

\[ \epsilon_A = \left( \sqrt{p^2 + x^2 m_{X}^2} - \sqrt{k_{A_e}^2 + x^2 m_{X}^2} \right) / \left( \sqrt{p^2 + x^2 m_{X}^2} - x m_{X} \right). \hspace{1cm} (B·4) = (3·21b) \]

Then a straightforward calculation yields

\[ -t_{A_e} = x^2 m_{X}^2 \{ \epsilon_A^2 / (1 - \epsilon_A) + O((x m_{X} / p_{lab})^{1/2}) \} \hspace{1cm} (B·5) \]

for \( p_{lab} \gg x m_{X} \). For example, the first term on the r.h.s. of (B·5) becomes 17.7 GeV$^2$ and 23.3 GeV$^2$ when \( (x, \epsilon_A) = (20, 0.2) \) and \( (100, 0.05) \), respectively.
Appendix C

--- Alternative Assignment of Leading Particles in $\gamma + p \rightarrow$ Hadrons ---

Being due to a special property of photon, there may be a leading particle stemming from the incident photon. That is, instead of $(3 \cdot 10)$, we may have to consider the process

$$\gamma + p \rightarrow V + N + X,$$

where $N$ is the leading nucleon coming from the incident proton while $V$ is the leading vector meson arising from the photon. If this is the case, we have

$$\langle n_{ch}\rangle_{\gamma p} = F\left(\langle M_V^2\rangle\right) + \gamma_{\text{FX}},$$

where $\gamma_{\text{FX}}$ is estimated to be 2.4 by assuming that $V$ is dominated by $\rho$ and $\omega$ meson. An upper bound for $\sqrt{\langle M_V^2\rangle}$ is given by $\sqrt{s} - m_V - m_N$, where $m_V$ is the mean vector meson mass. Hence, we have

$$\langle n_{ch}\rangle_{\gamma p} \leq F\left(\sqrt{s} - m_V - m_N\right) + 2.4.$$

References

16) Data compiled by Ganguli and Malhotra are used.
19) Data compiled by Czyzewski and Rybicki are used.
25) W. Busza, private communication.