Equation of State in 1/n Expansion

— n-Vector Model in the Presence of Magnetic Field —

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The classical n-vector model in the presence of magnetic field is studied by 1/n expansion. A detailed discussion is given of the equation of state up to order 1/n. As an application, the critical temperature and a universal amplitude ratio are investigated.

§ 1. Introduction

Recently, critical phenomena described by the $\phi^4$ Wilson Hamiltonian have been considered by $\varepsilon$ expansion ($\varepsilon = 4 - d$, $d$ is space dimension) and by 1/n expansion ($n$ is the number of components of the $\phi$-field). These two different theories give rise to consistent results for critical exponents in the overlapping region ($\varepsilon \ll 1, n \gg 1$).

Concerning critical phenomena, a universal property is known that symmetry and space dimension determine the critical behavior. The n-vector model, which has been introduced by Stanley, is a model of spin system with $n$ components. It is a generalized version of Ising ($n = 1$), XY ($n = 2$) and Heisenberg ($n = 3$) models. This n-vector model has been investigated in 1/n expansion. Indeed these treatments lead to the same results as obtained by the $\phi^4$ theory.

However, the n-vector model has a distinct property of the constraint for the spin field $\sigma$; the norm of $\sigma$ is fixed to a certain value. Among field theories, nonlinear $\sigma$-model is known to have the same property. The linear $\sigma$-model, which has no constraint, corresponds to the $\phi^4$ theory. Quite recently, the nonlinear $\sigma$-model has been investigated by several authors for $d = 2 + \varepsilon$ dimension.

In this paper, we present a refined treatment of n-vector model by clarifying the renormalization procedure and give the expression for equation of state up to order 1/n. Recently, universal ratio of critical amplitudes has been discussed by $\varepsilon$ expansion. We also consider the same problem in 1/n expansion.

§ 2. n-vector model in a uniform magnetic field

We consider the n-vector model on the $d$-dimensional hypercubic lattice. Magnetic field $h$ is assumed to interact with all components of spin. The partition
function is given as

$$Z_h = \int \exp \left[ \frac{1}{2} \sum_{t, j} \sum_{m = 1}^{N} K_{ij} \sigma_i(m) \sigma_j(m) + h \sum_{j, m} \sigma_j(m) \right] \prod [\delta[n - \sum \sigma_j^2(m)] \prod \delta_k(m).$$

(2.1)

Introducing $t_j$-fields ($j = 1, \cdots, N$), which are auxiliary fields and play the role of composite fields, we derive the following expression:

$$Z_h = \frac{1}{(2\pi t)^N} \int_{a-t}^{a+t} \prod dt_j \exp \left( n \sum t_j + \ln f(h; t_1, t_2, \cdots, t_N) \right),$$

(2.2)

where

$$f(h; t_1, \cdots, t_N) = \int \exp \left\{ \frac{1}{2} \sum_{t, j} K_{ij} \sigma_i \sigma_j + h \sum_{j, t} \sigma_j - \sum_{j, t} \sigma_j^2 \right\} \prod \delta_k.$$  

(2.3)

By applying the same procedure as in Refs. 3) - 5), the saddle point equation is given by

$$1 - \langle \sigma_j^2 \rangle_h = 0$$

(2.4)

with

$$\langle \cdots \rangle_h = \int \cdots \exp \left\{ \frac{1}{2} \sum_{t, j} K_{ij} \sigma_i \sigma_j - t \sum \sigma_j^2 + h \sum \sigma_j \right\} \prod \delta_k / f_0(h, t),$$

(2.5)

$$f_0(h, t) = \int \exp \left\{ \frac{1}{2} \sum_{t, j} K_{ij} \sigma_i \sigma_j - t \sum \sigma_j^2 + h \sum \sigma_j \right\} \prod \delta_k.$$  

(2.6)

The logarithm of $f_0(h, t)$ in (2.6) is easily calculated to be

$$\ln f_0(h, t) = \frac{N}{2} \ln \pi + \frac{Nh^2}{4[t - K(0)/2]} - \frac{1}{2} \sum_{q} \ln \left[ \frac{t - K(q)}{2} \right].$$

(2.7)

From (2.5) and (2.7), the saddle point equation becomes

$$\frac{1}{2N} \sum \frac{1}{t - K(q)/2} + \frac{h^2}{4[t - K(0)/2]} = 1.$$  

(2.8)

This result has been already obtained previously. 5) - 6) In deriving the term of order $1/n$, however, as remarked recently by Pesch and Selke, 19 a previous treatment 5) contains some misleading procedure even though a final result is correct. In the following, we present correct expressions for partition function in the presence of magnetic field.

To consider the $1/n$ expansion, we put

$$t_j = t + ix_j$$

(2.9)

and

$$f(h; t_1, \cdots, t_N)/f_0(h, t) = \langle e^{-ix_j \sigma_j^2} \rangle_h = G.$$  

(2.10)

The partition function of (2.2) is written as
\[ Z_n = \exp \left\{ n \left[ Nt + \ln f_n(h, t) \right] \right\} \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} d\mathbf{x} \exp \left\{ n \left[ i \sum x_j + \ln G \right] \right\}. \] (2.11)

In the presence of magnetic field \( h \), the linear terms of \( \sigma_j \) appear in the calculation. To eliminate these linear terms, we transform
\[ \sigma_j \rightarrow \sigma_j + ha_j. \] (2.12)
Putting this into the exponent of (2.3) and taking the coefficient of \( \sigma_j \) zero, we get
\[ 2a_j t_j - \sum_k K_{jk} a_k = 1. \] (2.13)
Thus, (2.3) becomes
\[ f(h; t_1, \ldots, t_N) = \exp \left( \frac{h^2}{2} \sum_j a_j \right) \int \exp \left[ \frac{1}{2} \sum_{j,k} K_{jk} \sigma_j \sigma_k - \sum_j t_j \sigma_j^2 \right] d\sigma_k. \] (2.14)
Also, by denoting \( a_j \) in the case \( x_1 = x_2 = \ldots = x_N = 0 \) by \( a_j(0) \), \( f_o(h, t) \) is expressed as
\[ f_o(h, t) = \exp \left( \frac{h^2}{2} \sum_j a_j(0) \right) \int \exp \left[ \frac{1}{2} \sum_{j,k} K_{jk} \sigma_j \sigma_k - t \sum_j \sigma_j^2 \right] d\sigma_k. \] (2.15)
From (2.9), (2.13) is rewritten as
\[ \sum_k A_{jk} a_k = 1 - 2i x_j a_j \] (2.16)
with
\[ A_{jk} = 2t \delta_{jk} - K_{jk}. \] (2.17)
We denote \( g(k, l) \) as
\[ g(k, l) = (A^{-1})_{kl} = \frac{1}{N} \sum_q e^{i q \cdot (r_j - r_i)} \] (2.18)
From (2.16) and (2.18), it follows that
\[ a_j = \sum_l g(j, l) - 2i \sum_l g(j, l) x_i a_i. \] (2.19)
Therefore, by iteration we have
\[ \sum_{j_1} a_{j_1} - \sum_{j_1} a_{j_1}(0) = -2i \sum_{j_1, j_2, j_3} g(j_1, j_2) x_{j_3} g(j_2, j_3) \]
\[ + (-2i)^2 \sum_{j_1, j_2, j_3, j_4} g(j_1, j_2) g(j_2, j_3) g(j_3, j_4) x_{j_4} x_{j_5} \]
\[ + \ldots, \] (2.20)
where
Here, the second equality defines a variable $s$. Putting (2.20) in the expression for $G$, (2.10), we have

$$G = \exp \left\{ \frac{\hbar^2}{2K^2 s^2} \left[ -2i \sum_{j}^{\beta} x_j + (-2i)^\alpha \sum_{j \neq i}^{\beta} g(j_i, j_k) x_j x_j, \right. \right.$$

$$+ \left. \left. (-2i)^\beta \sum_{j}^{\beta} g(j_i, j_l) x_j x_j x_l + \cdots \right]\right\} G_{s}, \quad (2.22)$$

where we denote $G$ without magnetic field by $G_{s}$. By the use of the result obtained previously,\cite{2, 21} from (2.22) the partition function is written as

$$Z_h = \exp \left\{ n \left[ Nt + \ln f_{h}(h, t) \right] \right\} - \frac{Z_h'}{(2\pi)^{n/2}} \quad (2.23)$$

with

$$Z_h' = \sum_{\alpha} \prod \int dy_j \exp \left[ \sum_{m=2}^{\infty} \frac{(-i)^{m-1}}{m^2} \sum_{j}^{\beta} y_{j, \ldots, j} g(j_i, j_k) \right.$$

$$\left. \cdots g(j_m, j_i) + \frac{\hbar^2}{K^2 s^2} \sum_{m=2}^{\infty} \frac{(-i)^{m-1}}{n^{m/2}} \sum_{j_m=j_i} y_{j, \ldots, j} \right]. \quad (2.24)$$

§ 3. Calculation up to order $1/n$

In a previous section, we have obtained the general expression for the partition function. Up to order $1/n$, we have from (2.24),

$$Z_{h'} = \sum_{\alpha} \int dy_j \exp \left[ -\frac{2\hbar^2}{K^2 s^2} \sum_{j} y_{j, y_{k}} g(j, k) \right. - \sum_{j} y_{j, y_{k}} g^2(j, k) \right]. \quad (3.1)$$

Denoting $\nu_{h}(q)$ by

$$g^2(j, k) + \frac{2\hbar^2}{K^2 s^2} g(j, k) = N^{-1} \sum_{q} \nu_{h}(q) e^{i q \cdot (rj-rk)}, \quad (3.2)$$

we have

$$\ln Z_{h'} = \frac{N}{2} \ln \pi - \frac{1}{2} \sum_{q} \ln \nu_{h}(q). \quad (3.3)$$

From (2.23), the logarithm of the partition function is given as

$$\frac{1}{nN} \ln Z_h = t + \frac{1}{N} \ln f_{h}(h, t) - \ln \frac{2\pi}{n} + \frac{n}{2n} + \frac{1}{nN} \ln Z_{h'}. \quad (3.4)$$

Using the previous notation $\nu(q)$ in Refs. 3)~6), $\nu_{h}(q)$ is written as
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\[ \nu_h(q) = \nu(q) + \frac{2h^2}{K^2 s^3} g(q). \]  

(3.5)

The magnetization up to order 1/n is given as

\[ \langle \sigma \rangle = M = \frac{\hbar}{K_s} - \frac{1}{2nN} \frac{\partial}{\partial h} \sum_{q} \ln \nu_h(q). \]  

(3.6)

We notice that the quantity \( s \) depends upon \( h \), through the following saddle point equation:

\[ \frac{1}{2N} \sum_{q} \frac{1}{t - K(q)/2} + \frac{h^2}{K^2 s^3} = 1. \]  

(3.7)

Noting that the following equations hold,

\[ \frac{ds}{dt} = \frac{2}{K}, \]  

(3.8)

\[ \frac{\partial s}{\partial h} = \frac{2h}{K^2 s^3} \left[ \frac{2h^2}{K^2 s^3} + \frac{1}{N} \sum_{q} g^2(q) \right], \]  

(3.9)

\[ \frac{\partial \nu(q)}{\partial h} = \frac{\partial}{\partial h} \left\{ \frac{1}{N} \sum_{q} g(k) g(q-k) \right\} \]  

\[ = -2Kq(q) \frac{\partial s}{\partial h}, \]  

(3.10)

we have

\[ \frac{\partial \nu_h(q)}{\partial h} = 2K \frac{\partial s}{\partial h} \left[ J(q) - \frac{h^2}{K^2 s^3} g^2(q) \right] \]  

(3.11)

with

\[ J(q) = g(q) \frac{1}{N} \sum_{k} g^2(k) - \frac{1}{N} \sum_{k} g^2(k) g(k-q). \]

From (3.6) and (3.11), the expression for magnetization in the presence of magnetic field is obtained as

\[ M = \frac{h}{K_s} - \frac{K}{nN} \frac{\partial s}{\partial h} \sum_{q} J(q) - \frac{h^2}{K^2 s^3} g^2(q) + O\left(\frac{1}{n^2}\right). \]  

(3.12)

This expression agrees with the previous result.\(^5\)

§ 4. Equation of state

We introduce a quantity \( r \) defined by

\[ M = \frac{h}{Kr}. \]  

(4.1)
From (3.12), it is seen that $s$ is related to $r$ as
\[
\frac{1}{r} = \frac{1}{s} - K^2 \frac{2}{nN} \frac{h^2}{K^2 s^2} \frac{1}{K^2 s^3} + \frac{1}{N} \sum \limits_{q} g^2(q) \left[ \sum \limits_{q} \frac{J(q) - (h^2/K^2 s^2)g^2(q)}{\nu(q) + (2h^2/k^2 s^2)g(q)} + O\left( \frac{1}{n^2} \right) \right].
\]

Expressing $s$ by $r$, we have
\[
s = r + \frac{2}{nNK} \left\{ \sum \limits_{q} \frac{1}{r + q^2} \frac{M^2}{K^2} \left[ \frac{1}{K^2 r^3} \sum \limits_{q} \frac{J(q) + M^2 g^2(q)}{\nu(q) + 2M^2 g(q)} + O\left( \frac{1}{n^2} \right) \right] \right\}.
\]

The saddle point equation (3.7) becomes
\[
K = \frac{1}{N} \sum \limits_{q} \frac{1}{r + q^2} + KM - \frac{2}{nN} \sum \limits_{q} \frac{J(q) + M^2 g^2(q)}{\nu(q) + 2M^2 g(q)} \bigg|_{s=r} + O\left( \frac{1}{n^2} \right).
\]

We take for convenience the following notations:
\[
J(q, r) \to \frac{1}{K^2} J(q, r), \quad \nu(q, r) \to \frac{1}{K^2} \nu(q, r),
\]
\[
\sqrt{KM} = \dot{M}.
\]

Then we have from (4.4)
\[
K = \frac{1}{N} \sum \limits_{q} \frac{1}{r + q^2} + M^2 - \frac{2}{nN} \sum \limits_{q} \frac{J(q, r) + M^2 (r + q^2)}{\nu(q, r) + 2M^2 (r + q^2)}.
\]

Equation (4.6) is simply represented by diagrams [Fig. 1]:
\[
K = \Sigma_{A}(r) + \Sigma_{A}^{'}(a) + \Sigma_{B}(r, a) + \Sigma_{B}^{'}(r, a).
\]

The solid line represents $(r + q^2)^{-1}$ and the cross represents the magnetic field $h$.
\[
\Sigma_{A} = \frac{1}{N} \sum \limits_{k} g(k),
\]
\[
\Sigma_{A}^{'} = \frac{h^2}{r^2} = \frac{a}{2},
\]
\[
\Sigma_{B} = -\frac{1}{N} \sum \limits_{q} g^2(q) \left[ \Sigma_{C}(q) - \Sigma_{C}(0) \right],
\]
\[
\Sigma_{B}^{'} = -\frac{1}{N} \sum \limits_{q} g^2(q) \left[ \Sigma_{C}^{'}(q) - \Sigma_{C}^{'}(0) \right],
\]
\[
\Sigma_{C}(q) = \frac{2}{Nn} \sum \limits_{v} \frac{g(q - k)}{\nu_{k}(k)}.
\]
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\[ \Sigma_{e'}(q) = \frac{2}{n
\nu_k(q)} \frac{\hbar^2}{r^2} = \frac{a}{n
\nu_k(q)}, \quad (4\cdot13) \]

with

\[ \nu_k(q) = \nu(q, r) + \frac{2a}{r + q^2}. \quad (4\cdot14) \]

The critical point is defined as the singular point where the susceptibility diverges, i.e., $r = 0$. We have from (4·6) the expression for critical temperature by putting $r = 0$ and $M = 0$:

\[ K_c = \frac{1}{N} \sum_q \frac{1}{q^2} + \frac{2}{nN} \sum_q J(q, 0) + O\left(\frac{1}{n^2}\right). \quad (4\cdot15) \]

This expression has been discussed previously.\(^{0,20,21}\) Subtracting (4·15) from (4·6), we obtain by putting $K_c = K = t$

\[ t = \frac{1}{N} \sum_q \left( \frac{1}{q^2} - \frac{1}{r + q^2} \right) - M^2 + \frac{2}{nN} \sum_q \nu(q, 0) - \frac{2}{nN} \sum_q P(q, r, M), \quad (4\cdot16) \]

where we have used new notations $Q$ and $P$:

\[ Q(q, r) = \frac{J(q, r)}{\nu(q, r)}, \quad (4\cdot17) \]

\[ P(q, r, M) = \frac{J(q, r) - M^2 q^2(q, r)}{\nu(q, r) + 2M^2 q^2(q, r)}. \quad (4\cdot18) \]

Note that $P(q, r, 0) = Q(q, r)$.

We consider (4·16) in three cases i) $T > T_c$, ii) $T = T_c$, iii) $T < T_c$. In the region $T > T_c$, putting $M = 0$ we obtain

\[ t = \frac{1}{N} \sum_q \left( \frac{1}{q^2} - \frac{1}{r + q^2} \right) + \frac{2}{nN} \sum_q Q(q, 0) - \frac{2}{nN} \sum_q P(q, r, 0) \]

\[ = (Fr)^{1/r}, \quad (4\cdot19) \]

where $F$ is the critical amplitude of susceptibility, and $r$ is the critical exponent. At the critical point $T = T_c$, putting $t = 0$ we have

\[ 0 = (Fr)^{1/r} + \frac{2}{nN} \sum_q Q(q, r) - \frac{2}{nN} \sum_q P(q, r, M) - M^2. \quad (4\cdot20) \]

In deriving this equation, we have used (4·19). From (4·20), we get

\[ (Fr)^{1/r} = M^2 + \frac{2}{nN} \sum_q P(q, r, M) - \frac{2}{nN} \sum_q Q(q, r), \quad (4\cdot21) \]

and noting the definition $r = H/M$, we obtain at $T = T_c$
\[
\frac{H}{M} = \frac{1}{T} \left[ M^2 + \frac{2}{nN} \sum P(q, r, M) - \frac{2}{nN} \sum Q(q, r) \right] \left( \frac{H}{M^3} \right)^{1/3}.
\]

Below the critical point \( T<T_c \), we put \( r=0 \). Equation (4.16) becomes

\[
-t = M^2 - \frac{2}{nN} \sum Q(q, 0) + \frac{2}{nN} \sum P(q, 0, M)
= (M/B)^{1/3}.
\]

Equation (4.22)

The quantities \( D \) and \( B \) in (4.22) and (4.23) are critical amplitudes.

Thus, we have obtained the expression for critical amplitudes \( \Gamma', D \) and \( B \). Combination of these amplitudes gives a universal quantity \( R_x \). The \( R_x \) is defined as

\[
R_x = \Gamma' DB^{1/3} = \Gamma DB^{1/3}.
\]

Using the results (4.21) ∼ (4.23), \( R_x \) up to order \( 1/n \) is expressed as

\[
R_x = \left[ 1 + \frac{2}{n N M^2} \sum \left[ \Psi(q, r, M) - \Psi(q, 0, M) \right] \right]^{1/3},
\]

where

\[
\Psi(q, r, M) = P(q, r, M) - Q(q, r).
\]

In (4.25), \( r \) is determined from the condition

\[
\frac{1}{N} \sum \left( \frac{1}{q^2} - \frac{1}{r + q^2} \right) = M^2.
\]

The equation of state is known to be written in the scaled form:

\[
H = DM^3 f \left( \frac{t}{\sqrt[3]{M/B}} \right).
\]

The quantity \( M^{1/3} \) is obtained from (4.23). From (4.16), we find

\[
t = (\Gamma r)^{1/3} - M^2 + \frac{2}{nN} \sum Q(q, r) - \frac{2}{nN} \sum P(q, r, M).
\]

Deviding this equation by \((M/B)^{1/3}\), we obtain

\[
x = \frac{t}{(M/B)^{1/3}} = \frac{(\Gamma r)^{1/3}}{(M/B)^{1/3}} \frac{M^2 + (2/nN) \sum \Psi(k, r, M)}{M^2 + (2/nN) \sum \Psi(k, 0, M)}.
\]

Noting that

\[
\frac{(\Gamma r)^{1/3}}{(M/B)^{1/3}} = \left[ \frac{\Gamma r}{(M/B)^{1/3}} \right]^{1/3} = \left( \frac{\Gamma B^{1/3} H}{M^3} \right)^{1/3},
\]

we obtain

\[
\left( \frac{\Gamma B^{1/3} H}{M^3} \right)^{1/3} = 1 + x + \frac{2}{n M^2 N} \sum \Psi(k, r, M) - \Psi(k, 0, M).
\]
Therefore, the scaling function is written as

\[
Df(x) = \left[ 1 + x + \frac{1}{n} g(x) \right]^r \quad (4.32)
\]

with

\[
g(x) = \frac{2}{nM^3N} \sum \{ \Psi(k, r, M) - \Psi(k, 0, M) \}. \quad (4.33)
\]

At the critical point \( T = T_c \), we have

\[
\frac{H}{M^8} = D. \quad (4.34)
\]

Consequently, the following equation holds:

\[
D = \frac{1}{\Gamma B^{r-1}} \left( 1 + \frac{1}{n} g(0) \right)^r. \quad (4.35)
\]

The universal ratio \( R_x \) is then written as

\[
R_x = D \Gamma B^{r-1} = \left( 1 + \frac{1}{n} g(0) \right)^r. \quad (4.36)
\]

From (4.32) and (4.36), we have

\[
f(x) = \left[ 1 + x + \frac{1}{n} g(x) \right]^r / \left[ 1 + \frac{1}{n} g(0) \right]^r = \left[ 1 + x + \frac{1}{n} g(x) - x + \frac{1}{n} g(0) \right]^r. \quad (4.37)
\]

We have derived the expression for the scaling function (4.37). However, (4.33) does not explicitly represent \( x \) dependence. To see the situation, we represent \( \Psi(k, r, M) \) as

\[
\frac{2}{n} \Psi(k, r, M) = \frac{1}{n} \sum \frac{J(k, r)}{r + k^2 + 2M^2/\nu(k, r)} + \frac{1}{n} \sum \frac{1}{r + k^2} \quad (4.38)
\]

The derivation of this equation is given in the Appendix. In the large \( n \)-limit, we have exactly the equation of state for spherical model

\[
H = M^8 \Delta_0 (1 + x)^r_0 \quad (4.39)
\]

with

\[
\Delta_0^{-1/r_0} = \Delta_0 = \frac{\Gamma [2 - (d/2)]}{2^{d-2} \pi^{d/2} [(d/2) - 1]}. \quad (4.40)
\]

Therefore, we have
Performing the change of variable $k^2 \rightarrow r k^2$, we obtain

$$J(k, r) = r^{(d-2)/2} J(k, 1)$$

and

$$\nu(k, r) = r^{(d-1)/d} \nu(k, 1).$$

Putting these expressions into (4.38) and by the help of (4.41), $g(x)$ is written as

$$g(x) = K_d \int \frac{-4k^2 J(k, 1) \nu^{-1}(k, 1) - (x + 1) \nu^{-1}(k, 1) - 2 k^{d-1} dk}{k^2 (1 + k^2)}$$

$$+ \frac{K_d}{c_0} (x + 1) \int k^{d-1} \frac{dk}{1 + k^2}$$

$$- K_d \int \frac{-4k^2 J(k, 0) \nu^{-1}(k, 0) - 2 x^{d-1} dk}{k^2 (k^2 \nu(k, 0) + 2c_0/(x + 1))}.$$  

(4.43)

This expression agrees with the result obtained by Brézin and Wallace.\(^\text{10}\)

In the limit $x \rightarrow \infty$, $g(x)$ is finite. Therefore,

$$\lim_{x \rightarrow \infty} f(x) = \frac{1}{n^{(d+1)/d}}.$$  

(4.44)

This quantity is equal to $R_x^{-1}$.\(^\text{10}\) This procedure is confirmed since $R_x$ is given by (4.36). In three dimension, the $g(0)$ is calculated to be $g(0) = -0.956$. Therefore, $R_x$ is given by

$$R_x \approx \left(1 - \frac{0.956}{n}\right)^{1/n} \approx 1 - \frac{1.912}{n}.$$  

(4.45)

By $\varepsilon$ expansion,\(^\text{10}\) $R_x$ is estimated as 1.61 ($n=1$) and 1.33 ($n=3$). The series expansion\(^\text{10}\) gives 1.75 ($n=1$) and 1.23 ($n=3$). Up to the first order term of $1/n$ expansion, $R_x$ is less than 1.

§ 5. Discussion

We have derived the expression for equation of state in $1/n$ expansion for the $n$-vector model. We have represented the quantity $s$ by the inverse susceptibility $r$. This procedure can be applied to other problems, for example, to the study of energy. For the case of energy, we have the same result as obtained from $\phi^4$ theory.\(^\text{12}\)

Even though the numerical value $R_x$ up to order $1/n$ is not so good, the

\(^{10}\) There is a misprint in the expression derived by Brézin and Wallace. See the correction by Wallace and Zia [Ref. 22].
value of \( f(x) \) for small \( x \) and near \( x = -1 \) is fairly good. As pointed out by Nelson\(^{20} \), the \( \varepsilon \) expansion\(^{20} \) gives spurious behavior near \( x = -1 \) for \( f(x) \) in its original form. It is interesting to perform \( \varepsilon \) expansion for the expression of (4-37). However, this will be discussed in another paper.

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Appendix

Derivation of (4·38)

We define \( A \) and \( B \) in the following form:

\[
A = \frac{2M^2}{r + k^2}, \quad B = \frac{M^2}{(r + k^2)^2}.
\]

The \( \mathcal{F}(k, r, M) \) of (4·26) is written as

\[
\mathcal{F}(k, r, M) = \frac{J(k, r) - M^2/(r + k^2)^2 J(k, r)}{\nu(k, r) + 2M^2/(r + k^2) \nu(k, r)}
\]

\[
= \frac{J(k, r) - B J(k, r)}{\nu(k, r) + A \nu(k, r)}
\]

\[
= \frac{A J(k, r) \nu^{-1}(k, r) - B}{1 + A \nu^{-1}(k, r)} \nu(k, r) + A
\]

\[
= \frac{2M^2 J(k, r) \nu^{-1}(k, r) + 1}{(r + k^2) \nu(k, r) + 2M^2 \nu(k, r) + 1} \frac{1}{2} \frac{1}{2} \frac{1}{r + k^2 + 2M^2 \nu(k, r)} - \frac{1}{2} \frac{1}{r + k^2}.
\]

Thus, (4·38) is derived.

References


