Slowly-Varying Amplitude of the Taylor Vortices near the Instability Point. II
—Mode-Coupling-Theoretical Approach—

Hideo YAHATA
Department of Materials Science
Hiroshima University, Hiroshima 730
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The Taylor instability in a rotating fluid between two coaxial cylinders is studied. Nonlinear analysis of temporal development of the flow velocity amplitude near the instability point is made on the basis of mode-coupling theory. Results of the numerical calculations on the mean torque and the local flow velocity are presented and compared with those of experimental observations and previous theoretical calculations.

§ 1. Introduction

Many investigations, both theoretical and experimental, have been made on the Taylor instability in a Couette flow between rotating coaxial cylinders. In a previous paper which is referred to as I hereafter, we have studied the nonlinear development of the flow velocity of the Taylor vortices near the instability point. We have first expanded the flow velocity \( u \) and pressure \( p \) in powers of a smallness parameter \( \varepsilon \) which characterizes the deviation of the Taylor vortex system from its state at the instability point. Substituting these expressions in the governing differential equations, we have determined each expansion coefficient of \( u \) and \( p \) with the aid of the solvability condition that the solutions of the differential equations of each order do not become secular. Finally numerical calculations have been performed on the basis of the Galerkin method.

There exists another method for making a nonlinear analysis of the development of the amplitude of the unstable mode slightly above the threshold. The method is due to Eckhaus who studied a class of nonlinear stability problems with quadratic nonlinearities by expanding the field in terms of the eigenfunctions of the corresponding linear stability problem. After that several authors employed essentially the same procedure in their investigations on the Taylor and Bénard problems. The purpose of the present paper is to study the same problem as presented in I in the light of this mode-coupling theoretical approach.

§ 2. Basic equations and method of solution

We consider the flow of a viscous incompressible fluid confined between two...
concentric cylinders of infinite length. Let $r$, $\theta$, $z$ denote cylindrical coordinates, and $R_1$, $R_2$ denote the radii of the inner and outer cylinders, respectively. We consider the situation where the inner cylinder rotates at the rate of the angular velocity $\Omega$, while the outer one is kept at rest. The velocity $u(u_r,u_\theta,u_z)$ and pressure $p$ of axisymmetric disturbance to the stationary laminar Couette flow are shown to obey the dimensionless equations of the forms

$$
\begin{aligned}
\frac{\partial u_r}{\partial t} - (\partial \partial_{\theta} + \partial^2_z) u_r + \beta \left(1 - \frac{1}{r^2}\right) u_\theta + \frac{p}{\rho} = -u_r \partial u_r - u_\theta \partial u_\theta + \frac{u_\theta^2}{r}, \\
\frac{\partial u_\theta}{\partial t} - \beta u_r - (\partial \partial_{\theta} + \partial^2_z) u_\theta = -u_r \partial u_\theta - u_\theta \partial u_\theta, \\
\frac{\partial u_z}{\partial t} - (\partial \partial_{\theta} + \partial^2_z) u_z + \partial u_\theta + \frac{p}{\rho} = -u_r \partial u_z - u_\theta \partial u_z, \\
\partial u_r + \partial u_\theta = 0,
\end{aligned}
$$

with $\partial = \partial / \partial r$, $\partial_{\theta} = (\partial / \partial r) + (1/r)$, $\beta = 2R_2 \Omega / \nu (1 - \gamma^2) = 2\eta Re / (1 - \gamma)^2 (1 + \gamma)$, where $R_2$, $\nu / R_2$ are chosen as quantities of reference for length and velocity, while $\rho$ and $\nu$ denote respectively the density and kinematic viscosity. The boundary conditions are $u = 0$ for $r = \gamma$ and 1 with $\gamma = R_1 / R_2$.

We first expand $u$ and $p$ in the form

$$
\begin{aligned}
&u_r(r,z,t) = \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij}(t) e^{il\alpha z} u_{1j}(r), \\
&u_\theta(r,z,t) = \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij}(t) e^{il\alpha z} v_{1j}(r), \\
&u_z(r,z,t) = \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij}(t) e^{il\alpha z} (-i) w_{1j}(r), \\
&p(r,z,t) / \rho = \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij}(t) e^{il\alpha z} \Pi_{1j}(r),
\end{aligned}
$$

where $\alpha_{ij}(t) = \alpha_{-ij}(t)$, and $u_{1j}$, $v_{1j}$, $w_{1j}$, $\Pi_{1j}$ are the eigenfunctions of the eigenvalue equations of the linear stability problem

$$
\begin{aligned}
(\lambda_{1j} + \partial \partial_{\theta} - l^2 \alpha^2) u_{1j} - \beta \left(1 - \frac{1}{r^2}\right) v_{1j} - \partial \Pi_{1j} &= 0, \\
\beta u_{1j} + (\lambda_{1j} + \partial \partial_{\theta} - l^2 \alpha^2) v_{1j} &= 0, \\
(\lambda_{1j} + \partial \partial_{\theta} - l^2 \alpha^2) w_{1j} + \lambda a \Pi_{1j} &= 0, \\
\partial \Pi_{1j} &= 0,
\end{aligned}
$$

with $u_{1j} = v_{1j} = w_{1j} = 0$ at $r = \gamma$ and 1, while $a$ denotes the axial wavenumber of the Taylor vortex and the linear damping rate $\lambda_{1j}$ is the eigenvalue which is determined as a function of the parameters $l$, $a$ and $\beta$. We also define the eigenvalue
equations adjoint to Eqs. (3), and denote the components of their solutions by \( \tilde{u}_{ij}, \tilde{v}_{ij}, \tilde{w}_{ij}, \tilde{\Pi}_{ij} \).

In order to find the numerical solutions to Eqs. (3), we employ the direct numerical procedure made use of by several authors in their investigations on similar hydrodynamic stability problems and convert this two-point boundary-value problem to an initial-value problem expressed in terms of a system of first-order differential equations. The procedure is described in the articles cited above to which reference should be made for a detailed exposition.

At the instability point where the Taylor vortex flow appears, \( \lambda_{ij} = 0 \) for \( l = 1 \) and \( j = 1 \). With a view to finding the neutral stability relation \( \beta = \beta(a) \) corresponding to this instability, we solve Eqs. (3) for \( l = 1 \) and \( j = 1 \) by setting \( \lambda_{ij} = 0 \). In the numerical procedure mentioned above the initial-value problem must be solved so that its solution may satisfy the terminal conditions at the outer boundary \( u_{ii}(1) = v_{ii}(1) = w_{ii}(1) = 0 \) which are expressed in the form that a certain determinant vanishes: \( \mathcal{A}(\beta, a) = 0 \). On the other hand the condition which determines the most critical wave number \( a = a_c \) reduces to

\[
\mathcal{A}_c(\beta_c, a_c) = \int \left[ 2 \alpha_c (\tilde{u}_{ii} u_{ii} + \tilde{v}_{ii} v_{ii} + 2 \tilde{w}_{ii} w_{ii}) 
+ (\tilde{\partial}_r u_{ii} + \tilde{\Pi}_{ii}) w_{ii} + \tilde{\omega}_{ii} (\tilde{\partial}_r u_{ii} - \Pi_{ii}) \right] dr = 0,
\]

(4)

where \( \beta_c = \beta(a_c) \) is the critical value of the dimensionless inner angular velocity. In order to find the critical values \( \beta_c \) and \( a_c \) we have integrated a system of first order differential equations by means of the Runge-Kutta-Gill method so as to minimize \( \mathcal{A}(\beta, a)^2 = \mathcal{A}(\beta, a)^2 + \mathcal{A}_c(\beta, a)^2 \) with the aid of nonlinear optimization techniques due to Nelder-Mead and Davies-Swann-Campey. By the use of this procedure with 64 steps of integration we have the results \( \beta_c = 181.830, a_c = 6.32105 \) for \( \eta = 0.5 \), while \( \beta_c = 365.648, a_c = 8.10998 \) for \( \eta = 0.6122 \). The same procedure with 128 steps gives the results \( \beta_c = 181.830, a_c = 6.32125 \) for \( \eta = 0.5 \) and \( \beta_c = 365.648, a_c = 8.10987 \) for \( \eta = 0.6122 \). These indicate that the relative truncation errors are at most less than one per cent of the true values.

Since the order of magnitude of \( \alpha_{ij}(t) \) and \( \lambda_{ij} \) for the wave number \( a_c \) near \( \beta_c \) is given by

\[
\alpha_{ij}(t) = O(\varepsilon |l-1|+1), \quad \lambda_{ij} = \begin{cases}
O(\varepsilon) & \text{for } l=1, j=1, \\
O(1) & \text{otherwise},
\end{cases}
\]

(5)

where \( \varepsilon = (\beta - \beta_c) / \beta_c \), we have in the Markovian approximation the equation which describes slowly-varying process of the amplitude \( \alpha_{ii}(t) \) with the wave number equal to \( a_c \) and the characteristic time scale of order \( \varepsilon^{-2} \) in the form

\[
\frac{\partial}{\partial t} \alpha_{ii}(t) = c_0 \varepsilon^2 \alpha_{ii}(t) - c_1 |\alpha_{ii}(t)|^2 \alpha_{ii}(t),
\]

(6)

where
Slowly-Varying Amplitude of the Taylor Vortices

\[ c_0 = \frac{\beta_c}{N_{ji}} \int_0^1 \left( \tilde{v}_{ji} u_{ji} - \tilde{u}_{ji} \left( 1 - \frac{1}{r^2} \right) v_{ji} \right) rdr, \]  
\[ c_2 = -\sum_{j=1}^{m} \left( \frac{1}{\lambda_{kj}} \left( CV_{ji;1,1} + CV_{ji;2,1} \right) CV_{ji;1,1} \right) \]
\[ + \frac{1}{\lambda_{kj}} \left( CV_{ji;1,1,0} + CV_{ji;0,1,0} \right) \left( CV_{ji;1,1,0} + CV_{ji;0,1,0} \right), \]
\[ CV_{ji;1,0,0} = \frac{\delta_{i,m+\nu}}{N_{ij}} \int_0^1 rdr \left[ \tilde{u}_{ij} \left( u_{mj,0} \partial u_{nj,0} + n a w_{mj,0} u_{nj,1} \frac{1}{r} v_{mj,0} v_{nj,0} \right) \right. \]
\[ + \tilde{v}_{ij} \left( u_{mj,0} \partial v_{nj,0} + n a w_{mj,0} v_{nj,1} + \tilde{v}_{ij} \left( u_{mj,0} \partial w_{nj,0} + n a w_{mj,0} w_{nj,0} \right) \right), \]
and the orthonormality of the eigenfunctions is defined by
\[ \int_0^1 \left( \bar{u}_{ij} \right) u_{ij} \left( 1 + \eta \right) = \bar{u}_{ij} \left( 1 + \frac{\eta}{2} \right) = 1. \]

Substituting the solutions of the first order differential equations in Eq. (7), we have \( c_0 = 107.5 \) for \( \eta = 0.5 \) and \( c_0 = 176.6 \) for \( \eta = 0.6122 \). The eigenvalues \( \lambda_{ij} \) and their corresponding eigenfunctions for \( l=0 \) and \( 2 \) have been calculated by setting \( \beta = \beta_c \) and \( a = a_c \) in the first order differential equations, and the results are given in Tables I and II to order \( j=9 \) for \( \eta = 0.5 \) and \( \eta = 0.6122 \) respectively.\(^*\)

With the aid of these eigenvalues and eigenfunctions we obtain the numerical values of \( c_2 \) in the different approximations for \( \eta = 0.5 \) and \( \eta = 0.6122 \) which are given in Tables III and IV respectively, where the order of approximation \( N \) is the number of terms of the series (8) made use of in the calculations.

### Table I. Eigenvalues \( \lambda_{ij} \) of Eqs. (3) for \( l=0 \) and 2 at \( \beta = \beta_c \), \( a = a_c \), \( \eta = 0.5 \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>( \lambda_{0j} )</th>
<th>( \lambda_{2j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40.8725</td>
<td>69.4734</td>
</tr>
<tr>
<td>2</td>
<td>159.383</td>
<td>254.886</td>
</tr>
<tr>
<td>3</td>
<td>356.794</td>
<td>346.429</td>
</tr>
<tr>
<td>4</td>
<td>633.162</td>
<td>(490.874</td>
</tr>
<tr>
<td>5</td>
<td>988.515</td>
<td>±21.9977</td>
</tr>
<tr>
<td>6</td>
<td>1422.897</td>
<td>738.369</td>
</tr>
<tr>
<td>7</td>
<td>1936.382</td>
<td>771.945</td>
</tr>
<tr>
<td>8</td>
<td>2529.090</td>
<td>1068.172</td>
</tr>
<tr>
<td>9</td>
<td>3301.196</td>
<td>1139.420</td>
</tr>
</tbody>
</table>

### Table II. Eigenvalues \( \lambda_{ij} \) of Eqs. (3) for \( l=0 \) and 2 at \( \beta = \beta_c \), \( a = a_c \), \( \eta = 0.6122 \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>( \lambda_{0j} )</th>
<th>( \lambda_{2j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>66.8082</td>
<td>114.594</td>
</tr>
<tr>
<td>2</td>
<td>263.723</td>
<td>421.084</td>
</tr>
<tr>
<td>3</td>
<td>591.870</td>
<td>574.873</td>
</tr>
<tr>
<td>4</td>
<td>1051.284</td>
<td>(811.862</td>
</tr>
<tr>
<td>5</td>
<td>1642.003</td>
<td>±32.9925i</td>
</tr>
<tr>
<td>6</td>
<td>2364.097</td>
<td>1217.513</td>
</tr>
<tr>
<td>7</td>
<td>3217.689</td>
<td>1284.608</td>
</tr>
<tr>
<td>8</td>
<td>4202.974</td>
<td>1770.033</td>
</tr>
<tr>
<td>9</td>
<td>5320.240</td>
<td>1892.097</td>
</tr>
</tbody>
</table>

\(^*\) The same procedure with double steps also gives the same numerical values except for the small variations of the last figures listed in these tables.
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Table III. The values of $c_2$ and $g$ defined respectively by Eqs. (8) and (12) for $\eta=0.5$ in the different approximations.

<table>
<thead>
<tr>
<th>N</th>
<th>$c_2$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.5510</td>
<td>0.9788</td>
</tr>
<tr>
<td>6</td>
<td>0.5512</td>
<td>0.9383</td>
</tr>
<tr>
<td>7</td>
<td>0.5514</td>
<td>0.9481</td>
</tr>
<tr>
<td>8</td>
<td>0.5519</td>
<td>0.9353</td>
</tr>
<tr>
<td>9</td>
<td>0.5519</td>
<td>0.9389</td>
</tr>
</tbody>
</table>

The values of $c_2$ depend on the normalization of eigenfunctions for $l=1$, $j=1$. If we normalize the eigenfunctions $u_{1l}^{(i)}(r)$, $\tilde{u}_{1l}^{(i)}(r)$ given in I in the same way as in Eq. (10), the values of $c_2$ obtained in I give the results $c_2=0.5550$ for $\eta=0.5$ and $c_2=0.5062$ for $\eta=0.6122$.

§ 3. Numerical results

In this section we compare the numerical results of the present calculations with the experimental data on the average torque and the velocity amplitude.

The mean torque $G$ exerted on the outer cylinder at rest for $Re \geq (Re)_c$ is expressed in the form

$$G = \frac{4\pi \eta \rho v^2 h}{(1-\eta^2)(1+\eta)} (1+g) Re + \text{const}$$

$$= \eta^2 Re + \text{const},$$

(11)

where

$$g = \frac{c_9}{c_9 \beta_c} \sum_{j=1}^{\infty} \frac{C_{0j;11} + C_{0j;111}}{\lambda_{0j}} (1).$$

(12)

In Table III we give the values of $g$ for $\eta=0.5$ calculated numerically in the different approximations. In Table V we list the values of $g^-$ together with those of $\beta_c$ and $\alpha_c$ for $\eta=0.5$ obtained by experimental observations and various theoretical calculations.

The local flow velocity $u_r(r,z)$ at $r=(1+\eta)/2$ slightly above $\beta_c$ is written in the form

$$u_r(r,z) = A_1 \cos \alpha_c z + A_2 \cos 2\alpha_c z + \cdots,$$

(13)

where

$$A_1 = 2 \sqrt{\frac{c_9}{c_2}} u_{11} \left(1+\eta\right),$$

(14)

We have the relation $c_2^{\eta}=c_1^{\eta}/u_{11}^{(1)}((1+\eta)/2)^2$, where $c_2^{\eta}$, $c_1^{\eta}$ represent the parameters $c_2$ which appear in Eq. (5-17) of I and Eq. (6) of the present paper, respectively. We note that $u_{11}^{(1)}(0.75)=3.025$ for $\eta=0.5$ and $u_{11}^{(1)}(0.8061)=0.9017$ for $\eta=0.6122$. 
In the present calculation \( u_\xi ((1 + \eta)/2) \) is set equal to unity, and the values of \( A_2 \) in the different approximations are given in Table IV. In Table VI we summarize the numerical values of \( \beta_c \) and \( a_c \) obtained by the light scattering measurements and our numerical calculations.

Table V. The values of \( \beta_c, \alpha_c \) for \( \eta = 0.5 \) given by Donnelly's experiment and various calculations.

<table>
<thead>
<tr>
<th>( \beta_c )</th>
<th>( (Re)_c )</th>
<th>( \alpha_c )</th>
<th>( g )</th>
<th>( g' )</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>181.047</td>
<td>67.893</td>
<td>6</td>
<td>0.9316</td>
<td>2.07</td>
<td>a</td>
</tr>
<tr>
<td>181.830</td>
<td>68.186</td>
<td>6.32</td>
<td>0.9316</td>
<td>2.044</td>
<td>b</td>
</tr>
<tr>
<td>182.187</td>
<td>68.32</td>
<td>6.32</td>
<td>0.8850</td>
<td>1.994</td>
<td>c</td>
</tr>
<tr>
<td>181.901</td>
<td>68.213</td>
<td>6.32</td>
<td>0.9286</td>
<td>2.041</td>
<td>d</td>
</tr>
<tr>
<td>181.830</td>
<td>68.186</td>
<td>6.32</td>
<td>0.9389</td>
<td>2.051</td>
<td>e</td>
</tr>
</tbody>
</table>

a) Donnelly's experiment
b) Davey

Table VI. Values of \( A_1, \alpha_c \) together with \( \beta_c \) and \( a_c \) for \( \eta = 0.6122 \), where \( \alpha_c/R_2, \nu A_1/R_2, \nu A_2/R_2 \) are expressed in the ordinary c.g.s. unit.

<table>
<thead>
<tr>
<th>( \beta_c )</th>
<th>( \alpha_c/R_2 )</th>
<th>( \nu A_1/R_2 )</th>
<th>( \nu A_2/R_2 )</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>370.4</td>
<td>3.20</td>
<td>0.145±0.013</td>
<td>0.063±0.005</td>
<td>Gollub-Freilich</td>
</tr>
<tr>
<td>365.781</td>
<td>3.19</td>
<td>0.126</td>
<td>0.0951</td>
<td>Galerkin method</td>
</tr>
<tr>
<td>(8.11)</td>
<td>(37.34)</td>
<td>(28.30)</td>
<td></td>
<td>present</td>
</tr>
<tr>
<td>365.648</td>
<td>3.19</td>
<td>0.126</td>
<td>0.0955</td>
<td>calculations</td>
</tr>
<tr>
<td>(8.11)</td>
<td>(37.44)</td>
<td>(28.40)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) The second term of Eq. (13) is said not to be proportional to \( \hat{t}^8 \) in Ref. 11.

\[ A_2 = - \frac{2c_0}{c_2} \sum_{\lambda=1}^{\infty} \frac{1}{\lambda^2} C_{\lambda_2,11,11} u_{\lambda_2} \left( \frac{1 + \eta}{2} \right). \]  

(15)

\[ \]  

§ 4. \textbf{Concluding remarks}

In this paper we have considered the Taylor instability with the aid of mode-coupling theory. Results of the present numerical calculations are shown to be in good agreement with those given in I on the basis of the Galerkin method. In the case \( \eta = 0.5 \) results of our calculations both agree considerably well with those of various previous observations and calculations. However in the case \( \eta = 0.6122 \) there still remain some numerical discrepancies between the experimental results on the local flow velocity by means of light scattering experiments and our numerical results obtained in I and the present work. One reason for these
discrepancies is thought to lie in the fact that in our calculations the cylinders are assumed to be infinitely long in the axial direction.

References

6) J. A. Nelder and R. Mead, the Computer J. 5 (1965), 308.