Nordic Hydrology 4, 1973, 57–76
Published by Munksgaard, Copenhagen, Denmark

OPTIMAL POLICY FOR RESERVOIR MANAGEMENT

M. A. MUSPRATT

McGill University, Montreal, Canada

Development of a rational management philosophy for existing reservoirs is being stimulated by the rapid exploitation throughout the world of dam sites which are economically and ecologically viable. A nonlinear, stochastic mathematical programming algorithm is used to maximise profit from the operation of a reservoir over a finite time period, namely an annual hydrologic cycle, taking account of water used for domestic, industrial, irrigation, low-flow augmentation, and power demands, and of losses due to seepage and evaporation.

Temporal and spatial distribution of high quality water is becoming one of the major concerns of modern technology, and solution of this problem is a primary pacing factor controlling further industrialization – water is rapidly becoming a scarce resource. To provide a rationale for allocation of the limited volume of water available, a total systems approach must be pursued, and this must be sensitive to the conflict of demands by various users.

A piecemeal and local fragmentation and solution of water resource problems is no longer adequate – the integrated total purposive system expressed in canonical form provides the individual with a conceptual framework of reference with which he can determine the interacting variables defining the goals and facilitates forecasting, synthesis, optimization and manipulation, sensitivity analysis, decision making and resources allocation. Performance, reliability, cost and time are the criteria to be considered, but it should also be remembered that subsystem optimization does not necessarily contribute to overall system optimization; for example, optimal operation of one reservoir may cause water shortages at other reservoirs in the same hydrologic system.

With unexploited viable dam sites in developed countries becoming increasingly scarce, and with fresh water supply at a premium, technological knowledge is being marshalled to most efficiently monitor the use of existing water resources. Cost effectiveness of reservoir operation must be detailed not only quantitatively from technological and economic points of view, but also in terms of ecological and environmental disruption – the socio-behavioural aspects of water resource exploitation are being increasingly emphasized as the earth becomes more congested. These latter variables are not directly quantifiable, but can be accounted for indirectly in the operational system definition. As one example, augmentation of river flow downstream from a dam in the dry season does contribute to the flushing of accumulated pollution.

Deterministic and stochastic optimal allocation of water resources has been attempted in the past (Hall & Howell 1970, Reynolds 1971), but no completely general method for handling nonlinear systems embedded with arbitrary probability distributions appears to have yet been developed. A hypothetical system will now be defined and then subsequently developed to accommodate stochastic overtones. Profit optimization is the overall goal. Further extensions are indicated in Appendix A.

DETERMINISTIC SYSTEM

The problem chosen is to manage the water released from a hypothetical reservoir for domestic (X_1), industrial (X_2), irrigation (X_3), and low-flow augmentation (X_4) uses so as to maximise profit over a given period of time. Water may also be used for power generation (X_5), and losses are experienced due to seepage and evaporation (X_6). A transient hydrologic cycle is considered for the reservoir which has an initial stored-water volume of X_7. Ecological and environmental implications are not specifically articulated, and effects on migratory fish, profit from recreational use, etc. are neglected. See Fig. 1.

The objective is to maximise $\int_0^T \text{profit}\ (t).dt$, where profit is a function of time t, and T is the total period considered. T is taken as one year, and for convenience is discretised into two equal time periods – the dry season and the wet season. Superscripts D, W are used with $X_1, X_2, \ldots X_7$ to denote seasonal quantities.

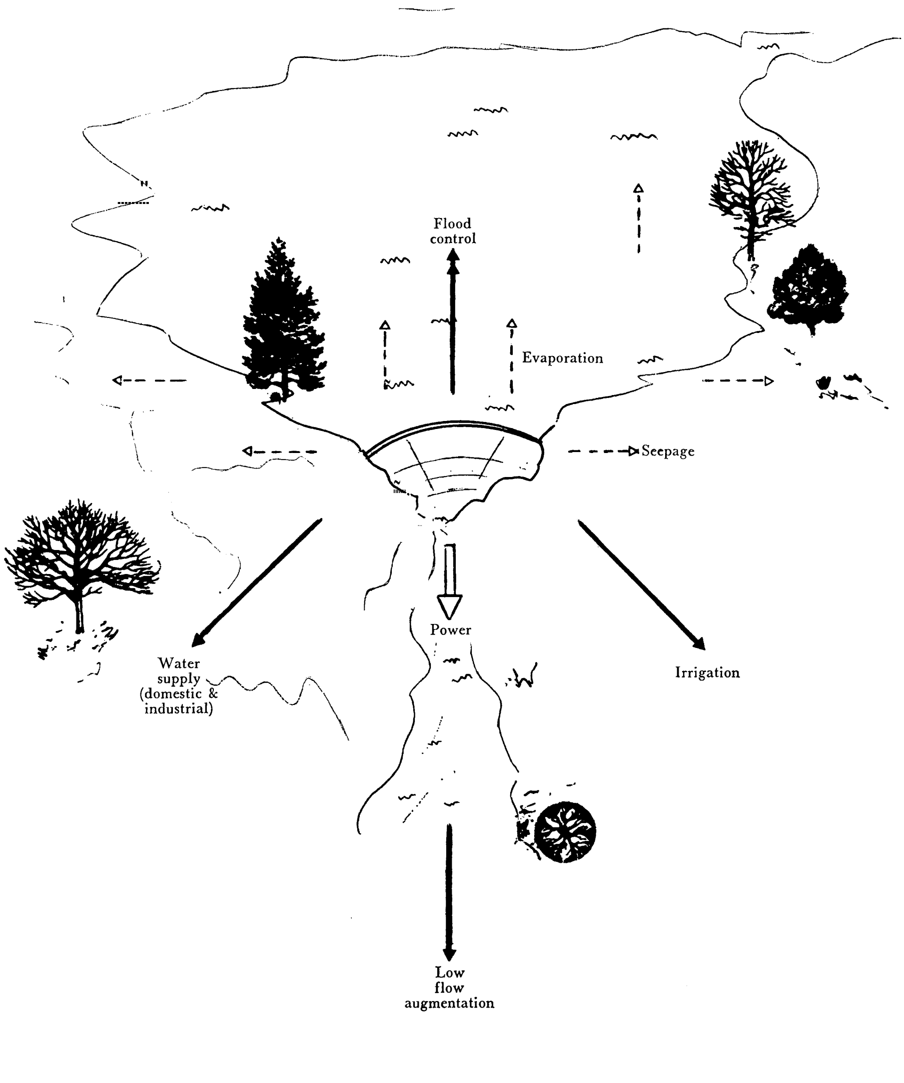


Fig. 1.
Reservoir system.

Dry Season

The profits accrued per unit volume of water put to various uses are assumed as follows:

$$\begin{bmatrix} X_1^D \\ X_2^D \\ X_3^D \\ X_4^D \\ X_5^D \\ X_6^D \\ X_7^D \end{bmatrix} \rightarrow \begin{matrix} \text{profit} \\ \text{per} \\ \text{unit} \\ \text{volume} \end{matrix} \begin{bmatrix} 10 \\ 8 \\ 4X_3^D \\ 2X_4^D \\ \dfrac{5.10^4}{(\text{volume used})^2} \\ -3X_6^D \\ 2 \end{bmatrix}$$

In view of the tremendous variability of precipitation, catchment, reservoir and demand characteristics and constraints, a greatly simplified and idealised hypothetical system is defined for the purpose of illustration. Generality is not lost, however, as even the most complex realistic system may be handled similarly, using the powerful mathematical tools described, by simply plugging in the relevant criteria, constraints, and probability spectra. As an example of realistic complexities, evaporation is one of the minor variables in reservoir management, but depends on free water surface area, air and water temperature, relative humidity, wind velocity, barometric pressure, and salinity of the water. (See U.S.D.I. 1965, Davis 1952.)

Water requirements for domestic and industrial use are relatively constant. Preemption for domestic use may be reflected in a higher profit coefficient.

The drier the season, the greater the demand for irrigation and downstream flow augmentation, and therefore the more valuable the water – a high seepage and evaporation rate also exists. Thus profit per unit is likely to be dependent on volume used, and a simple linear relationship is assumed.

All controlled release from the reservoir is assumed to contribute to power generation. The more water released, the greater the drawdown, and therefore the smaller the head available for power generation – the actual relationship depends on the shape and area of the reservoir, the slope of the banks, etc., and an inverse square law is assumed.

The objective function becomes:

$$\max. \left\{ \left[10, 8, 4X_3^D, 2X_4^D, \frac{5.10_4}{(X_5^D + X_6^D)}, - 3X_6^D, 2\right] \begin{bmatrix} X_1^D \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_7^D \end{bmatrix} \right\}$$

where $X_5^D = (X_1^D + X_2^D + X_3^D + X_4^D)$

Constraints on the optimization process are as follows:

(1) A tight high priority constraint on domestic use:

$$90 \leqq X_1^D \leqq 100$$

This may be bifurcated into upper and lower bound constraints to permit solution by mathematical programming.

$$X_1^D \leqq 100$$
$$-X_1^D \leqq \ 90$$

(2) Industrial and irrigation use:

$$200 \leqq X_2^D + X_3^D \leqq 250$$

(3) Minimum acceptable low-flow augmentation:

$$X_4^D \geqq 100$$

(4) A rough estimate of losses from evaporation and seepage is as follows:

$$X_6^D \leqq 50$$

(5) The volume of water stored in the reservoir has a lower bound for emergency use only, and an upper bound to provide enough freeboard in the event of a rare flood:

$$500 \leqq X_7^D \leqq 1500$$

(6) During the dry season, there is an inflow to the reservoir of 500 units:

$$X_1^D + X_2^D + X_3^D + X_4^D + X_6^D \equiv 500$$

(For the purpose of illustration, no change in storage is assumed during the dry season.)

Wet Season

Irrigation requirements, and seepage and evaporation losses are assumed negligible, while profit from downstream-flow augmentation is neglected. A lower preemptive coefficient is applied to the profit vector for domestic water use. The head available for power generation is relatively constant (little drawdown), while the volume of water stored in the reservoir is relatively less valuable.

The objective function becomes:

$$\max. \left\{ [6, 8, 5, 1] \begin{bmatrix} X_1^W \\ X_2^W \\ X_5^W \\ X_7^W \end{bmatrix} \right\}$$

where: $X_5^W = X_1^W + X_2^W + X_4^W$

constraints:

(1) Domestic and industrial demands are slightly smaller:

$$80 \leqq X_1^W \leqq 90$$
$$100 \leqq X_2^W \leqq 125$$

(2) Downstream flood control:

$$X_4^W \leqq 500$$

No lower bound is required on X_4 in the wet season.

(3) There is a large lower bound on reservoir storage volume to provide a water inventory for the next dry season:

$$1000 \leqq X_7^W \leqq 1500$$

(4) The total inflow is 1000 units, i.e.:

$$X_1^W + X_2^W + X_4^W - X_7^W + X_7^D = 1000.$$

MATHEMATICAL PROGRAMMING

The above constrained optimization problem is complex and virtually intractable if a closed-form analytical solution is required, unless unrealistic simplification of system definition is acceptable. A numerical approach is now described based on a powerful mathematical-programming algorithm, called the sequential, unconstrained minimisation technique (S.U.M.T.). See Appendix B.

S.U.M.T. is a stable and rapidly convergent method for optimization of non-linear systems in which both the objective and the constraints are continuously twice differentiable (analytically or numerically). An algorithm was obtained from the originators of this method (McCormick et al. 1968) and was modified by the author to accommodate equality constraints using techniques described by Fiacco & McCormick (1966). (The use of $w^2 \leqq 0 \vee w \leqq 0 \wedge w \geqq 0$ for $w = 0$ is not valid using S.U.M.T. because of feasibility violation.) In addition, efficiency of the method was greatly increased by problem-orienting the algorithm; this involved deleting parts of the comprehensive option set.

First and second variations of the objective function and the yield constraints were input requirements, as was initialisation of the optimization process. Feasibility need not generally be satisfied by the latter requirement, although a more judicious choice of the initialisation point enhances convergence characteristics.

By definition, all variable values are non-negative – the recycling of, say, industrial water is neglected. This constraint considerably limits the feasible domain and simplifies solution.

STOCHASTIC SYSTEM

Estimates of future costs and benefits over a period of time must be wrought with prediction uncertainties – forecasting of population and water demand is a risky business (Muspratt 1970). The optimal policy should also consider economic factors such as interest rates, inflation, economy of scale, learning and use generation, etc. In addition, purely physical factors over which the reservoir has no control are always present, e.g., variability of flood rains, temperature and evaporation, reservoir siltation, etc. In view of all the uncertainties, a stochastic system definition is mandatory.

For the purpose of illustration, the following stochastic variables were assumed:

(1) Domestic requirements depend on population growth and distribution, as well as environment, and psychological factors such as use generation for gardening, etc. Assume a negative lognomial distribution, the corresponding normal distribution (RN1) having a mean of 0.0 and standard deviation of 1.0.

(2) Industrial use varies with the amount of recycling, plant utilisation, etc. Assume uniform scatter (RN2), range ± 30.0.

(3) Irrigation requirements and low-flow augmentation vary with rainfall, temperature, and humidity characteristics of the previous season, etc., while the former also varies with type of crops grown. Superimposed normal distributions are assumed, with mean 0.0, and standard deviations 40.0 (RN3) and 30.0 (RN4) for irrigation and flow augmentation, respectively.

(4) Evaporation and seepage losses vary greatly with distribution of precipitation and humidity. Assume lognormal scatter (RN5), the corresponding normal distribution being RN3.

(5) Storm inflow into reservoir depends on storm distribution in catchment area, etc. Assume a Poisson distribution (RN6) with mean and standard deviation of 10.0.

(6) Profit coefficients vary with supply and demand. Assume a normal distribution with mean of 0.0 and standard deviation of 2.5 (RN7).

System definition

The overall system definition in canonical form becomes:

(1) Objective function:

$$\max.\ (1 + RN7)\left\{[10, 8, 4X_3^D,\ 2X_4^D, \frac{5.10^4}{(X_5^D + X_6^D)^2}, -3X_6^D,\ 2]\begin{bmatrix} X_1^D \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_7^D \end{bmatrix} + [6, 8, 5, 1]\begin{bmatrix} X_1^W \\ X_2^W \\ X_5^W \\ X_7^W \end{bmatrix}\right\}$$

(2) Inequality constraints:

$$90 \leqq (X_1^D + RN1) \leqq 100$$
$$80 \leqq (X_1^W + RN1) \leqq 90$$
$$200 \leqq (X_2^D + RN2) + (X_3^D + RN3) \leqq 250$$
$$100 \leqq (X_2^W + RN2) \leqq 125$$
$$X_4^D + RN4 \geqq 100$$
$$X_4^W + RN4 \leqq 500$$
$$X_6^D + RN5 \leqq 50$$
$$500 \leqq X_7^D \leqq 1500$$
$$1000 \leqq X_7^W \leqq 1500$$

(3) Equality constraints:

$$X_5^D + X_6^D = 500 + RN6$$
$$X_5^W + X_7^W - X_7^D = 1000 + RN6$$

Correlation

A realistic system definition must also include correlations.

Serial correlation in reservoir operation policy may occur due to seasonal fluctuations in temporal precipitation distributions, while cross correlation may occur in the dry season between the quantity of water required for irrigation and that required for low-flow augmentation.

As an example for a bivariate system, suppose that cross correlation (ϱ) between X_3^D and X_4^D occurs, and that serial correlations (ϱX_3^D, ϱX_4^D) are also apparent in demand volumes for these uses. The inclusion of these correlations in the Monte Carlo simulation may be effected by an expression after Fiering (1964, Muspratt 1971).

Assuming:

$$\varrho = 0.9$$

$$\varrho X_3^D = 0.3$$

$$\varrho X_4^D = 0.3$$

Standard deviations of volumes $= \delta X_3^D = \delta X_4^D = 20$

Mean volumes $= \mu X_3^D = \mu X_4^D = 100$

From Fiering (1964) and Muspratt (1971) it may be shown that for the j th interaction:

$$X_{4,j+1}^{D,c} = 4.4 + 0.9X_{3,j+1}^{D,c} + 0.056X_{4,j}^{D,c} + 8.2\,RN5_j$$

where $X_{3,j+1}^{Dc}$ is the normal stochastic volume variable previously generated for irrigation and $X_{4,j+1}^{Dc}$ is the stochastic capacity variable for low flow augmentation assuming both serial and cross correlations.

This expression is embedded directly in the stochastic system definition, together with an extra random variate generator for $RN5_j$.

Stochastic optimization

Optimization of the stochastic system described above may be best performed by invoking a stochastic mathematical programming algorithm.

In stochastic programming, the problem is to select a vector x which satisfies

$$Ax \geqq b \wedge x \geqq 0$$

and minimises cx where A is a mxn matrix, b is a m-vector, x and c are n-vectors, and A, b, c are non-deterministic. Solution methods have been formulated for conversion of linear programming problems, with various combinations of A, b and c probabilistic, into deterministic non-linear programming problems (Wessels 1967, Sengupta 1969). In many cases, the probabilistic aspect is restricted to Gaussian or uniform spectra. The general solution of the stochastic non-linear programming problem can be easily handled, at the expense of increased computational effort, by coupling Monte Carlo methods with the S.U.M.T. nonlinear programming technique described previously. The heuristic used involves discretisation of component probability distributions and optimization of various combinations of the discrete values according to some strategy.

The imbedding of random properties in otherwise invariant functional relationships is almost trivial computationally using the following three Monte Carlo steps:

(1) Generate sets of random deviates for all stochastic parameters in accordance with empirically determined or assumed density functions;

(2) Calculate the objective function value via normal deterministic mathematical relationships;

(3) Repeat the above steps until a large enough sample is obtained for the accuracy required.

As can be seen, the basic computational tool is the random number generator class of which the pseudo-random number generator has particular application for machine solution.

Standard algorithms are often supplied with computer software for generating random numbers with uniform distributions. Random numbers with other distributions generally may be generated by inverse transformations. Tests of auto-correlation of random numbers, especially those from a single string generator for multipurpose application, were considered but are not detailed here.

The solution method is to pulsate the feasible set input to the S.U.M.T. routine by perturbations (RN1, RN2, ... RN7) generated in the levels of the stochastic parameters according to the prespecified probability density functions. This is effectively a Monte Carlo sampling technique coupled with nonlinear mathematical programming.

The convolution of exogenous frequency spectra is conceptually and computationally of the stochastic programming genus, but is quite time-consuming unless the S.U.M.T. algorithm is drastically problem-oriented and variance reducing techniques subsequently described are included with the empirical sampling techniques (Muspratt 1973).

A convolved histogram of relative frequencies for maximum profit is shown in Fig. 2 for a sampling size of 100.

For the purpose of interpolation and a limited amount of extrapolation, a mathematical model describing the output results may be useful. It can be seen that the histogrammic values may be approximated by a normal distribution. Model parameters, viz. mean and standard deviation, may be estimated

Model for least squares fit after smoothing and exponential weighting:

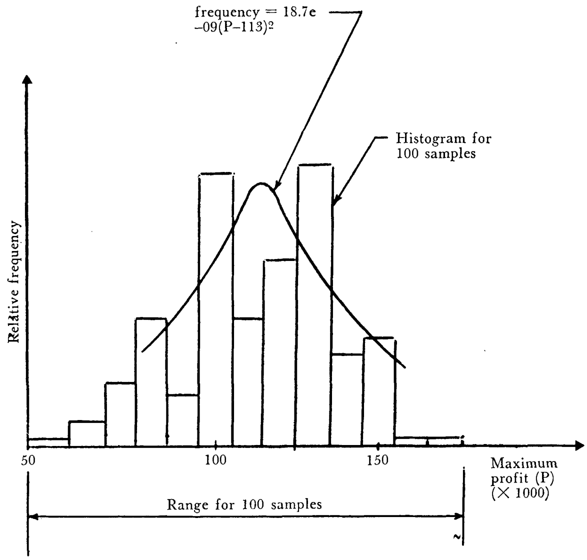


Fig. 2.
Distribution of maximum profit.

directly from test data and the model fit tested by a chi-square test if the distribution is approximately symmetric and mesokurtic. More accurate representation of posterior probability distributions with emphasis on the extreme range is achieved, however, by utilisation of a multiple regression analysis with heavy weighting in the tails to accentuate actual trends here. The regression model is non-linear in the coefficients, so some form of numerical optimization such as S.U.M.T. mathematical programming must be applied to minimise the sums of squares of the error terms – instead of normal matrix inversion as with the linear model. Prior quadratic smoothing using moving averages between triples, and exponential weighting increasing towards distribution tails were performed to increase accuracy of regression representation. See Fig. 2.

Confidence limits resulting from simulation tests easily may be estimated for single variate systems by binomial sampling laws, and multinomial systems can be handled in a cumbersome way. However, the determination of error in hybrid multivariate systems with various contributing density spectra is a trial and error process of observing convergence as the number of simulations increase (Muspratt 1973).

Closed-form analytical verification of the stochastic reservoir management problem is at present impossible. However, simplified deterministic problems invoking mathematical programming easily may be verified analytically (Muspratt 1971).

INTERPRETATION OF RESULTS

As can be seen from Fig. 2, the mean profit for any given year is likely to be about 110×10^3 units, and one accordingly can generate long-term plans and resource allocation for expansion and upgrading of the system performance, and for repaying debts, etc.

In a good year, profits may reach 150×10^3 units, while in a bad year they may be as low as 70×10^3 units – and contingency plans should be available for either eventuality. If the profit falls below 70×10^3 units, emergency measures may be required, e.g. allocating water resources differently and temporarily violating some pre-defined constraints. As an example, in a drought year, the low-flow augmentation constraint and other peripheral imperatives may be waived, and more water diverted for irrigation use.

The actual probability distribution is likely to be truncated. In the upper range, only so much water can be stored and sold, and so much power generated. In the lower range, a negative profit (loss) is delimited by the total loss if the dam bursts (this is actually a growing concern as dam sizes increase, because larger dams tend to generate seismic effects and increase the probability of earthquake damage). Extrapolation into the extreme tails of the distribution is invalid, however, as threshold phenomena are likely to be transcended, – and furthermore, human intervention is always possible.

APPENDIX A – EXTENSIONS

Further extensions to the proposed stochastic optimization procedure are shown in Fig. 3 and described below.

(1) Taxonomy

Precise, rational categorization of human experience is one of the foundations for a scientific understanding of the universe. Computers have made it possible

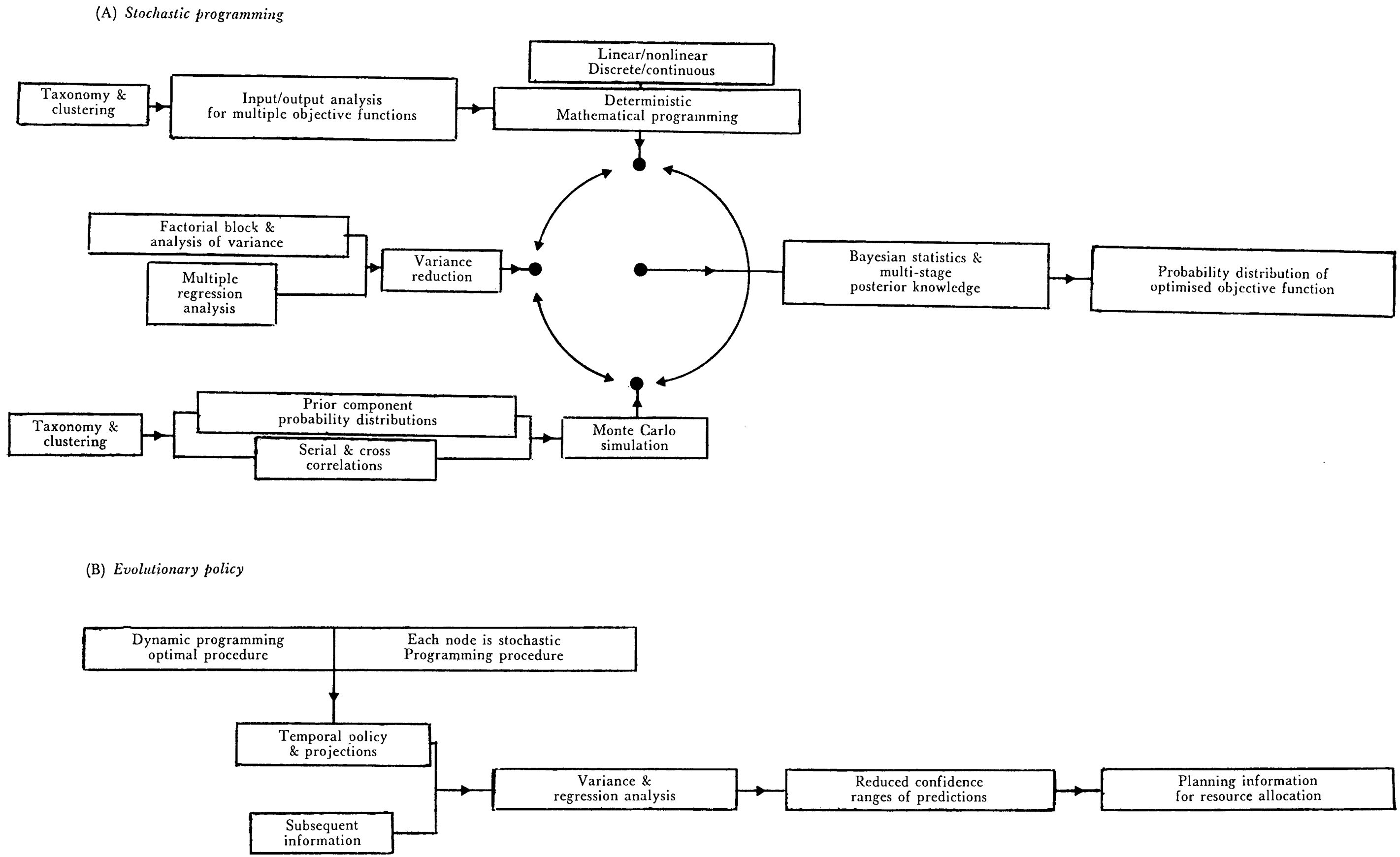
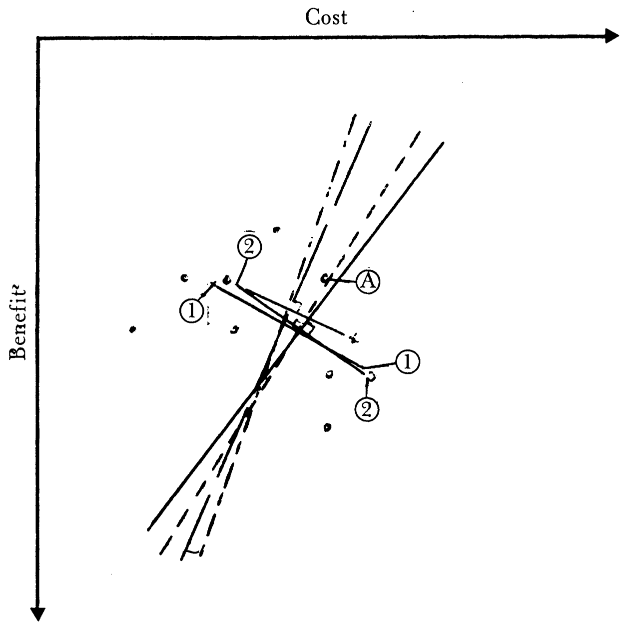


Fig. 3.
Stochastic optimization procedure.

to consider large numbers of characteristics in classifying many phenomena, and the impetus for the development of numerical classification is the ubiquitous use of automatic sensing and data recording devices with associated data banks. Facts and objects often occur in such variety and profusion that unless some system is created among them with minimum gain of entropy, they are unlikely to provide useful information or predictions.

As an example, cost/benefit ratios of dams depend on a multitude of factors such as reservoir and catchment characteristics, structure and mechanics of the



. data points
1 = center of gravity for first iteration, etc.
Hypersurfaces dividing clusters:
- - - - first guess division
– – – – division after one iteration
——— final division
=– –— final division with weighting of "A" = 2 units

Fig. 4.
Clustering of cost/benefit results for dams situated at various geographically suitable sites.

dam and foundation, potential market for water, flood abatement, etc., and these facts must be systematised into probability density functions of possible costs.

The "K means" algorithm is one of the simplest clustering procedures. The object is to minimise the mean square distance between samples in a given group. Approximate, arbitrary hyperplanes are used to separate clusters, together with centers of gravity of clusters so divided. New hyperplanes are drawn halfway between these centers of gravity, again centers of the new clusters calculated, and this process cycled to the desired accuracy. A two-dimensional rendition of this process is given in Fig. 4 for two clusters and two cycles – more cycles are not required as no reclassification occurs after these two initial cycles.

Weighting of data due to importance or accuracy of recording may be easily included. As an example, if point A in Fig. 4 is given double the weight of the other points, the final cluster subdivision is as indicated.

(2) Pattern recognition

A variation of the concepts described in (1) involves pattern recognition. As an example, automatic graphical records of domestic water demand must be deconvolved in terms of temperature, humidity, rainfall, etc.

Automatic graphical representation of results may be encoded into analog binary digits by electro-optical transducers (usually scanners). The machine is then used to automatically disect the response in terms of various stimuli.

This approach is particularly useful and enhances cost effectiveness if tedious, repetitive identification and classification is to be performed, of if the shades of difference are so fine and/or the characteristics to be compared to permit classification so numerous that manual procedures are no longer practicable. More consistent and objective results are obtained, and the output data are compatible directly with other data processing equipment for automatic compilation of statistical records, or with feedback for process control, etc.

The simplest form of pattern recognition system is shown in Fig. 5. This is an adaptive technique wherein a decision-logic network is trained to classify imagery. In the training process, the network accumulates statistical information from a variety of image samples of known classification and stores a repertoire of correct decisions.

(3) Input/output analysis

Where multiple criteria are to be optimized, an input/output analysis permits a single suitably weighted function of these criteria to be determined. For example, if profit is to be maximised concomitant with minimisation of ecological disruption, loss of consumer good-will because of delays, unfavourable reaction from competitors, etc., the input/output method facilitates solution.

This procedure involves trial and error monitoring of the optimization process at the man/machine interface and manipulation of the relative magnitudes of the weights to achieve a maximum or minimum response. Computer graphics (Muspratt 1970b) and, in particular, interactive graphics facilitate rapid evaluation of the masses of data spewed from modern high-speed computers.

(4) **Variance reduction**

Variance reducing techniques may be applied to reduce the number of simulations required and to increase the accuracy of simulation. For example, if a pilot factorial experimental design is performed on total cost with, say, three representative levels of RN1, RN , . . . RN7, a variance analysis would indicate significance of these factors and their interactions – insignificant stochastic factors could be held deterministic in the subsequent complete simulation.

Because of the computer time and cost involved in Monte Carlo simulation, and the necessity of complete reconvolution of the frequency distribution of the dependent variable if one of the independent variables is changed, functionalisation of the dependent variable in terms of the independent variables by multiple regression analysis is possible to minimise this problem. (See Muspratt (1973) for more details.)

The use of problem-oriented simulation languages such as SIMSCRIPT and GPSS may also reduce computer time requirements.

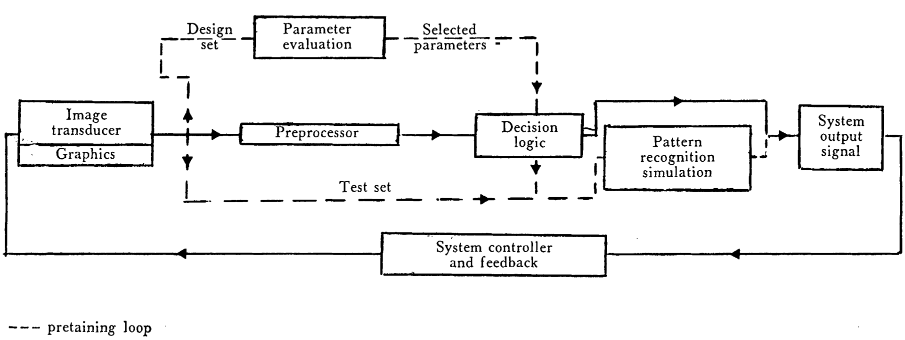


Fig. 5.
Automatic optimal pattern recognition.

(5) Posterior information

Posterior information may be included in the frequency distribution of the dependent variable by means of Bayes' theorem (Muspratt 1971). Thus intuitive knowledge based on experience or professional judgement (necessary as long as man remains in the decision-making loop), or knowledge derived from subsequent system operational observations may be evaluated.

(6) Dynamic programming

The whole stochastic optimization procedure may be nested in various stages of a dynamic programming optimal policy. Thus the effects of performance upgrading (e.g., two-stage dam construction) and inexact original system definition may be evaluated. (See Fig. 3.)

People invariably prefer current benefits and future costs to current costs and future benefits. Minimising this year's budget does not necessarily contribute to a minimum budget over, say, the next decade. This often results in acceptable initial performance but poor reliability with high maintenance, insurance and operational costs, and difficulty with future modification or performance upgrading. Thus, besides optimising the cost-benefit ratio at each development stage, an overall temporal optimization strategy should be enforced; for example, by applying Bellman's optimality principle of dynamic programming.

The more a system is optimised, the more sensitive it becomes to perturbation in levels of stochastic parameters used in original system definition. Thus evolutionary time-dependent changes, such as population growth (Muspratt 1970a), or inaccurate system definition, may generate deficiencies, and a sub-optimal allocation of resources with more fail-safe capability is often mandatory in view of the omnipresent role of environment in system adaptation and survival.

Because of the random nature of invention and conceptualisation, and the uncertainty of predicted variable values, an adaptive or evolutionary temporal policy monitored and updated in the light of subsequent information as it becomes available is desirable to maintain system equilibrium. The autoregressive nature of simple time series allows the dependent variable to be expressed in terms of its previous values (more recent values being weighted more heavily), together with a stochastic effect and confidence range – the confidence range can be successively narrowed as more information becomes available (Muspratt 1970a). In fact, real-time surveillance of evolving hydrologic systems is part of the forward move into the fourth generation computer philosophies and accompanying cybernetics. Automatic implementation of expedient decisions made at the man-machine interface permits more complete utilisation of existing facilities.

(7) Analogue machines

The use of analogue machines for effecting optimization by mathematical programming has been advocated (Jackson 1957), especially for non-linear systems. Analogue simulation is an attractive alternative because it is faster; it avoids approximations due to discretisation and digitalisation of probability distributions and so is inherently more accurate; it permits a high degree of interactive communication at the man-machine interface and so is useful if limits and density spectra are ill-defined; and its inability to handle logical expressions with facility is generally acceptable in Monte Carlo methods, provided the digital logic or other devices necessary for optimization are available. The enormous memory capacity of the digital machine is required only if storage of convolved variates for subsequent processing is required, e.g. selection of fractiles, etc., because assessing histogrammic characteristics using standard software such as merge-sort routines is facilitated by an enlarged memory capacity.

(8) Mathematical programming

Many mathematical programming algorithms are available (Muspratt 1970c), e.g., fast and efficient methods for linear programming based on the revised simplex method. Piecewise linearisation of nonlinear systems may be effected by first order Taylor's series expansions, with concomitant application of "move" limits to restrict possible large movement into infeasible regions because of system idealisation. The simplex algorithm requires a convex feasible set: non-convex sets may often be made convex by mapping into some hypothetical hyperspace. A concomitant dimensionality reduction is often feasible for reducing computer storage and precludes the use of cumbersome decomposition methods for large problems.

Consideration of the effects of individual floods, or of the demands of individual users, etc., can be accommodated by invoking integer programming algorithms – although convergence may be poor even for small problems.

Various nonlinear programming methods are available, such as quadratic and geometric programming, and more powerful multi-mode techniques. However, it has been the author's experience that linear and integer programming and S.U.M.T. provide all the constrained optimization capability generally required.

APPENDIX B – S.U.M.T.

The sequential unconstrained optimization process has been well documented elsewhere (McCormick et al. 1968).

In mathematical form, the problem is to find the N-dimensional vector x which will minimise a given function f(x) subject to L inequality constraints and M-L equality constraints.

$$g_i(x) \geqq 0 \qquad i = 1, 2, \ldots, L$$

$$g_i(x) = 0 \qquad i = L + 1, \ldots, M$$

A primal or P- function is defined by

$$P(x, r) = f(x) + r \sum_{i=1}^{L} \frac{1}{g_i(x)} + \frac{1}{r^2} \sum_{i=L+1}^{M} g_i^2(x)$$

where r is a positive parameter. The P- function is minimised according to a second order gradient method and it may be shown (McCormick et al. 1968) that a succession of such unconstrained minimisations for which the parameter r is decreased systematically will provide, under certain conditions, the required minimum of f(x).

S.U.M.T. generates dual feasible points as it solves the primal problem. These points give lower bounds for the solution of the primal problem, and are invaluable in providing a criterion for termination of the optimization. Finite Lagrange multipliers (dual variables) are generated and represent marginal values of the optimum.

Suffficient requirements for the application of this technique include the existence of the minimum, the continuity of f(x), and of its first two derivatives with respect to x throughout the domain, and the convexity of the P- function which is equivalent to the fact that the second derivative matrix $[\delta^2 P/\delta x_i \,.\, \delta x_j]$ must be positive definite. The last condition is sufficient but not necessary and the method may work even if it is not fulfilled.

ACKNOWLEDGEMENTS

The author would like to express his appreciation to Professor J. O. McCutcheon of McGill University for his encouragement, to the Emergency Measures Organisation of Canada for financial support, and to Lauie Robinson for help in preparation of the manuscript.

REFERENCES

Davis, C. (1952) *Handbook of applied hydraulics.* McGraw-Hill, N.Y., 1272 p.

Fiacco, A. & McCormick, G. (1966) Extensions of S.U.M.T. for nonlinear programming: equality constraints and extrapolation. *Management Sci. 12* (11), July, 816–828.

Fiering, M. (1964) Multivariate techniques for synthetic hydrology. *A.S.C.E. 90,* (Hy 5), Sept., 43–60.

Hall, W. & Howell, D. (1970) Optimal allocation of stochastic water supply. A.S.C.E., V.96, N.IR4, Dec., pp. 395–402.

Jackson, A. (1957) The analog computer solution of matrix and mathematical programming problems. Cornell University, Ithaca, March, 17 p.

McCormick, G., Mylander, W. & Fiacco, A. (1968) Computer program implementing the sequential unconstrained minimisation technique for nonlinear programming. Research Analysis Corporation, McLean, Va., Rept. No. RAC-TP-151, June.

Muspratt, M. (1970b) Graphic display. *Int. J. Comp. Math. 2,* (3), 259–268.

Muspratt, M. (1970c) Optimisation using mathematical programming. McGill University, Civil Eng. Rept. No. 70–1.

Muspratt, M. (1971) Optimal distribution of water to irrigation canals. *J. Hydrology 14* (1), Oct., 19–28.

Muspratt, M. (1973) Monte Carlo simulation and stochastic programming. To be published.

Muspratt, M. (1970a) Time dependent properties. *Civ. Eng. & Pub. Wks. Review 65,* (767), June, pp. 652–653.

Reynolds, J. (1971) Allocating water among alternative uses. A.S.C.E., V.97, N.IR1, Mar., pp. 85–92.

Sengupta, J. (1969) Safety-first rules under chance-constrained linear programming. *Operations Research 17* (1), 112–132.

U.S. Department of The Interior (1965) Design of small dams. U.S. Govt. Printing Office, Washington, 611 p.

Wessels, J. (1967) Stochastic programming. *Statistica Neerlandia, 21* (1), 39–53.

Received 12 January 1973.

Address:

M. A. Muspratt,
Australian Road Research Board,
60 Denmark Street,
Kew, Victoria,
Australia, 3101.